Fault Hamiltonicity of Meshes with Two Wraparound Edges*

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Abstract. We consider the fault hamiltonian properties of $m \times n$ meshes with two wraparound edges in the first row and the last row, denoted by $M_2(m,n)$, $m \geq 2$, $n \geq 3$. $M_2(m,n)$ is a spanning subgraph of $P_m \times C_n$ which has interesting fault hamiltonian properties. We show that $M_2(m,n)$ with odd n is hamiltonian-connected and 1-fault hamiltonian. For even n, $M_2(m,n)$, which is bipartite, with a single faulty element is shown to be 1-fault strongly hamiltonian-laceable. In previous works[1, 2], it was shown that $P_m \times C_n$ also has these hamiltonian properties. Our result shows that two additional wraparound edges are sufficient for an $m \times n$ mesh to have such properties rather than m wraparound edges. As an application of fault-hamiltonicity of $M_2(m,n)$, we show that the n-dimensional hypercube is strongly hamiltonian laceable if there are at most $n-2$ faulty elements and at most one faulty vertex.

1 Introduction

Meshes represent the communication structures of many applications in scientific computations as well as the topologies of many large-scale interconnection networks. One of the central issues in parallel processing is embedding of linear arrays and rings into a faulty interconnection network. The embedding is closely related to a hamiltonian problem in graph theory.

An interconnection network is often modeled as an undirected graph, in which vertices and edges correspond to nodes and links, respectively. A graph G is called *hamiltonian-connected* if there exists a hamiltonian path joining every pair of vertices in G. We consider the hamiltonian properties of a graph in the presence of faulty elements(vertices and/or edges). A graph G is called *k-fault hamiltonian*

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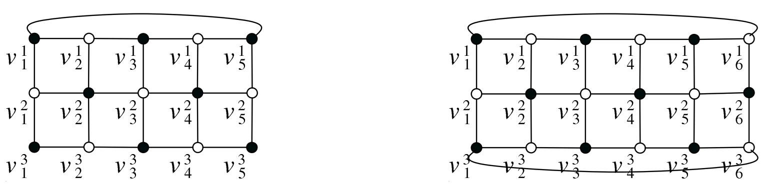
(resp. *k-fault hamiltonian-connected*) if $G - F$ has a hamiltonian cycle (resp. a hamiltonian path joining every pair of vertices) for any set F of faulty elements such that $|F| \leq k$. Apparently, a bipartite graph is not hamiltonian-connected. In [3], the concept of hamiltonian laceability for hamiltonian bipartite graphs was introduced. Bipartition sets of a bipartite graph are often represented as sets of black and white vertices. A bipartite graph G is *hamiltonian-laceable* if there is a hamiltonian path joining every pair of vertices with different colors. In [4], this concept was extended into strongly hamiltonian laceability. A hamiltonian laceable graph G with N vertices is *strong* if there is a path of length $N-2$ joining every pair of vertices with the same color.

For any faulty set F such that $|F| \leq k$, a bipartite graph G which has an L^{opt}-path joining every pair of fault-free vertices is called *k-fault strongly hamiltonian laceable*[2]. An L^{opt}-path is defined as follows. Let G be a bipartite graph and let B and W be the sets of black and white vertices in G, respectively. Denote by F_v and F_e the sets of faulty vertices and edges in G, respectively. Let $F = F_v \cup F_e$, $f_v = |F_v|$, $f_e = |F_e|$, and $f = |F|$. The numbers of fault-free black and white vertices are denoted by n_b and n_w, respectively. When $n_b = n_w$, a fault-free path of length $2n_b - 1$ joining a pair of vertices with different colors is called an L^{opt}-*path*. For a pair of vertices with the same color, a fault-free path of length $2n_b - 2$ between them is called an L^{opt}-*path*. When $n_b > n_w$, fault-free paths of length $2n_w$ for a pair of black vertices, of length $2n_w - 1$ for a pair of vertices with different colors, and of length $2n_w - 2$ for a pair of white vertices, are called L^{opt}-*paths*. Similary, an L^{opt}-path can be defined when $n_w > n_b$. A fault-free cycle of length $2{\cdot}\min\{n_b, n_w\}$ is called an L^{opt}-*cycle*. The lengths of an L^{opt}-path and an L^{opt}-cycle are the largest possible.

Fault hamiltonicity of various interconnection networks has been investigated. In [5] and [6], linear-time algorithms that find hamiltonian paths in $m \times n$ meshes were developed . In [7] and [8], the fault hamiltonian properties of $m \times n$ torus and $P_m \times C_n$ were considered, where $P_m \times C_n$ is a graph obtained by product of a path P_m with m vertices and a cycle C_n with n vertices. $P_m \times C_n$ forms an $m \times n$ mesh with a wraparound edge in each row. Futhermore, it was shown that $P_m \times C_n$ is hamiltonian-connected and 1-fault hamiltonian if it is not bipartite[1]; otherwise, $P_m \times C_n$ is 1-fault strongly hamiltonian laceable[2].

In this paper, we consider the hamiltonian properties of $m \times n$ mesh ($m \geq 2, n \geq 3$) with two wraparound edges in the first row and the last row. We denote the graph by $M_2(m, n)$. We show that $M_2(m, n)$ with odd n is hamiltonian-connected and 1-fault hamiltonian. For a graph G to be hamiltonian-connected, G should be non-bipartite and $\delta(G) \geq 3$, where $\delta(G)$ is the minimum degree of G. For a graph G to be k-fault hamiltonian, it is necessary that $k \leq \delta(G) - 2$. Thus, $M_2(m, n)$ with odd n satisfies the above condition by adding two(minimum) edges to an $m \times n$ mesh. Futhermore, for n even, we show that $M_2(m, n)$, which is bipartite, with a single faulty element is strongly hamiltonian laceable. In previous works[1, 2], it was shown that $P_m \times C_n$ also has these hamiltonian properties. Our result shows that two additional wraparound edges are sufficient for an $m \times n$ mesh to have such properties rather than m wraparound edges.



(a) $M_1(3,5)$ (b) $M_2(3,6)$

Fig. 1. Examples of $M_1(m,n)$ and $M_2(m,n)$

For some m and n, $M_2(m,n)$ is a spanning subgraph of many interconnection networks such as tori, hypercubes, k-ary n-cubes, double loop networks, and recursive circulants. Thus, our results can be applied to discover the fault hamiltonicity of such interconnection networks. It was shown in [9] that n-dimensional hypercube Q_n with $n-2$ faulty edges is strongly hamiltonian laceable. By applying fault hamiltonicity of $M_2(m,n)$, we show that Q_n with at most $n-2$ faulty elements and at most one faulty vertex is strongly hamiltonian laceable.

2 Preliminaries

Let $M(m,n) = (V,E)$ be an $m \times n$ mesh, where the vertex set V is $\{v_j^i | 1 \le i \le m, 1 \le j \le n\}$ and the edge set E is $\{(v_j^i, v_{j+1}^i) | 1 \le i \le m, 1 \le j < n\} \cup \{(v_j^i, v_j^{i+1}) | 1 \le i < m, 1 \le j \le n\}$. We propose a graph which has two wraparound edges in the first row and the last row in $M(m,n)$.

Definition 1. *Let $M(m,n) = (V,E)$. $M_2(m,n)$ is defined as (V_{M_2}, E_{M_2}), where the vertex set $V_{M_2} = V$ and the edge set $E_{M_2} = E \cup \{(v_1^1, v_n^1), (v_1^m, v_n^m)\}$.*

The vertices of $M_2(m,n)$ are colored with black and white as follows: v_j^i is called a *black* vertex if $i+j$ is even; otherwise it is a *white* vertex. We denote by $R(i)$ and $C(j)$ the vertices in row i and column j, respectively. That is, $R(i) = \{v_j^i | 1 \le j \le n\}$ and $C(j) = \{v_j^i | 1 \le i \le m\}$. We let $R(i:j) = \cup_{i \le k \le j} R(k)$ if $i \le j$; otherwise $R(i:j) = \emptyset$. Similarly, $C(i:j) = \cup_{i \le k \le j} C(k)$ if $i \le j$; otherwise $C(i:j) = \emptyset$.

We denote by $H[s,t|X]$ a hamiltonian path from s to t in the subgraph $G\langle X \rangle$ induced by a vertex set X, if any. A path is represented as a sequence of vertices. If X is an empty set, $H[s,t|X]$ is an empty sequence. We denote by $v_j^i \to v_{j'}^i$ a path $(v_j^i, v_{j+1}^i, \cdots, v_{j'-1}^i, v_{j'}^i)$ if $j < j'$; otherwise, $(v_j^i, v_{j-1}^i, \cdots, v_{j'+1}^i, v_{j'}^i)$. Similary, $v_j^i \to v_j^{i'}$ a path from v_j^i to $v_j^{i'}$ in the subgraph $G\langle C(j) \rangle$. We employ three lemmas on the hamiltonian properties of $M(m,n)$ and $P_m \times C_n$. We call a vertex in a mesh a *corner vertex* if it is of degree two.

Lemma 1. *[10]* (a) *If mn is even, then $M(m,n)$ has a hamiltonian path from any corner vertex v to any other vertex with color different from v.* (b) *If mn*

is odd, then $M(m,n)$ has a hamiltonian path from any corner vertex v to any other vertex with the same color as v.

Lemma 2. *[5] Let two vertices s, t have different color each other.* (a) *If $m, n \geq 4$ and mn is even, then $M(m,n)$ has a hamiltonian path joining s and t.* (b) *If $m = 2$, $n \geq 3$, and $s, t \notin C(k)(2 \leq k \leq n-1)$, then $M(m,n)$ has a hamiltonian path joining s and t.*

Lemma 3. (a) *For $m \geq 2, n \geq 3$ odd, $P_m \times C_n$ is hamiltonian-connected and 1-fault hamiltonian[1].* (b) *For $m \geq 2, n \geq 4$ even, $P_m \times C_n$ is 1-fault strongly hamiltonian-laceable[2].*

Let P and Q be two vertex-disjoint paths $(a_1, a_2, \cdots, a_k)$ and $(b_1, b_2, \cdots, b_l)$ in a graph G, respectively, such that (a_i, b_1) and (a_{i+1}, b_l) are edges in G. If we replace (a_i, a_{i+1}) with (a_i, b_1) and (a_{i+1}, b_l), then P and Q are merged into a single path $(a_1, a_2, \cdots, a_i, b_1, b_2, \cdots, b_l, a_{i+1}, \cdots, a_k)$. We call such a replacement a *merge* of P and Q w.r.t. (a_i, b_1) and (a_{i+1}, b_l). If P is a closed path(that is, a cycle), the merge operation results in a single cycle. We denote by $V(P)$ the set of vertices on a path P.

To show the fault hamiltonicity of $M_2(m,n)$, we first show some hamiltonian properties of $M_1(m,n)$ which has a single wraparound edge on the first row in $M(m,n)$. An $M_1(m,n)$ has two corner vertices v_1^m and v_n^m.

Lemma 4. *For $m \geq 2$, $n \geq 3$ odd, $M_1(m,n)$ has a hamiltonian path between any corner vertex s and any other vertex t.*

Proof. The proof is by induction on m. Without loss of generality, we assume that $s = v_1^m$. First, we observe that the lemma holds for $m = 2$. For $t \in B$, there exists a hamiltonian path by Lemma 1; otherwise we can construct a hamiltonian path $P = (s, v_1^1, H[v_n^1, t|C(2:n)])$. By Lemma 1, $H[v_n^1, t|C(2:n)]$ exists.

For $m \geq 3$, we assume that the lemma is true for every $k < m$. The proof is divided into two cases.

Case 1: $t \in R(1:m-1)$. When $t \neq v_n^{m-1}$, we can construct a hamiltonian path $P = (s, v_2^m \to v_n^m, H[v_n^{m-1}, t|R(1:m-1)])$. By induction hypothesis, $H[v_n^{m-1}, t|R(1:m-1)]$ exists. When $t = v_n^{m-1}$, $P = (s, v_1^{m-1} \to v_1^1, H[s', t|C(2:n)])$, where s' is v_n^1 if m is odd; s' is v_2^1 if m is even.

Case 2: $t \in R(m)$. Let $t = v_i^m$. By induction hypothesis, there exists a hamiltonian path P' joining v_{i-1}^{m-1} and v_n^{m-1} in $G\langle R(1:m-1)\rangle$. We construct a hamiltonian path $P = (s, v_2^m \to v_{i-1}^m, P', v_n^m, v_{n-1}^m \to t)$. □

In the following lemmas, we just summarize the results and omit the proofs.

Lemma 5. *For $m \geq 2, n \geq 4$ even, $M_1(m,n)$ is strongly hamiltonian laceable.*

Lemma 6. *For $m \geq 2, n \geq 4$ even, $M_1(m,n)$ with a single faulty vertex v_f has an L^{opt}-path joining a corner vertex s and any other vertex t if s has a different color from v_f and at most one of v_f and t is adjacent to s.*

3 Hamiltonian properties of $M_2(m,n)$

3.1 $M_2(m,n)$ with odd n

When n is odd, $M_2(m,n)$ is not bipartite. We show that $M_2(m,n)$ is hamiltonian-connected and 1-fault hamiltonian.

Theorem 1. *For $m \geq 2, n \geq 3$ odd, $M_2(m,n)$ is hamiltonian-connected.*

Proof. The proof is by induction on m. $M_2(2,n)$ is isomorphic to $P_2 \times C_n$. Thus, the theorem is true by Lemma 3 when $m = 2$. For $m \geq 3$, we assume that the theorem is true for every $k < m$. Let $s = v_i^x$, $t = v_j^y$. We show that $M_2(m,n)$ has a hamiltonian path P between s and t. The proof is divided into two cases.

Case 1: $s,t \in R(1:m-1)$. If we assume that a virtual edge (v_1^{m-1}, v_n^{m-1}) exists, then there exists a hamiltonian path P' joining s and t in $G\langle R(1:m-1)\rangle$ by induction hypothesis. If P' passes through the edge (v_1^{m-1}, v_n^{m-1}), then we can construct a hamiltonian path P by replacing (v_1^{m-1}, v_n^{m-1}) with a path $(v_1^{m-1}, v_1^m \to v_n^m, v_n^{m-1})$; otherwise we choose an edge (u,v) in $G\langle R(m-1)\rangle$ such that P' includes it. Let u' and v' be the vertices in $R(m)$ adjacent to u and v, respectively. Since $\langle R(m)\rangle$ forms a cycle, it has a hamiltonian path P'' joining u' and v'. By a merge of P' and P'' w.r.t. (u,u') and (v,v'), we have a hamiltonian path.

Case 2: $s \in R(1:m-1)$ and $t \in R(m)$. When $s \in R(2:m-1)$, this case is symmetric to Case 1. Thus, we only consider the case that $s \in R(1)$.

Case 2.1: $m = 3$. If either s or t is on the first column or the last column, then there exist a hamiltonian path by Lemma 4. Otherwise(that is, $s,t \in C(2:n-1)$), by Lemma 1 and Lemma 2, we can construct a hamiltonian path P as follows:

(i) $s,t \in W$, $P = (s, v_{i+1}^1 \to v_n^1, v_1^1 \to v_{i-1}^1, H[v_{i-1}^2, v_1^3 | R(2:3)] \cap C(1:j-1)], H[v_n^3, t | R(2:3) \cap C(j:n)])$.
(ii) $s \in B$ and $t \in W$, $P = (H[s, v_1^3 | C(1:i)], H[v_n^3, t | C(i+1:n)])$.
(iii) $s,t \in B$,

$$P = \begin{cases} (H[s, v_{i+1}^1 | C(1:i+1)], H[v_{i+2}^1, t | C(i+2:n)]) & \text{if } i \neq j \\ (s, H[v_{i-1}^1, v_{i-1}^2 | C(1:i-1)], v_i^2, H[v_{i+1}^2, v_{i+1}^3 | C(i+1:n)], t) & \text{if } i = j \end{cases}$$

Case 2.2: $m \geq 4$. By Lemma 4, there exist two paths $P' = (H[s, v_1^{m-2} | R(1:m-2)])$ and $P'' = (H[v_1^{n-1}, t | R(m-1:m)])$. (P', P'') forms a hamiltonian path. □

Theorem 2. *For $m \geq 2, n \geq 3$ odd, $M_2(m,n)$ is 1-fault hamiltonian.*

Proof. We prove by induction on m. Due to Lemma 3, the theorem holds for $m = 2$. For $m \geq 3$, we assume that the theorem is true for every $k < m$, and we consider $M_2(m,n)$. Without loss of generality, we assume that the faulty element is contained in $G\langle R(1:m-1)\rangle$.

If we assume that a virtual edge (v_1^{m-1}, v_n^{m-1}) exists, then there exists a fault-free hamiltonian cycle C' in $G\langle R(1:m-1)\rangle$ by induction hypothesis. If

C' passes through (v_1^{m-1}, v_n^{m-1}), then we can construct a hamiltonian cycle C by replacing (v_1^{m-1}, v_n^{m-1}) with a path $(v_1^{m-1}, v_1^m \to v_n^m, v_n^{m-1})$; If C' does not pass through (v_1^{m-1}, v_n^{m-1}), we choose an edge (u, v) in $G\langle R(m-1)\rangle$ such that C' includes it. Let u' and v' be the vertices in $R(m)$ adjacent to u and v, respectively. Since $G\langle R(m)\rangle$ forms a cycle, it has a hamiltonian path P' joining u' and v'. By a merge of $C' - (u, v)$ and P' w.r.t. (u, u') and (v, v'), we have a fault-free hamiltonian cycle C. □

3.2 $M_2(m, n)$ with even n

When n is even, $M_2(m, n)$ is bipartite. First, we show that $M_2(m, n)$ with a single faulty vertex is strongly hamiltonian-laceable.

Lemma 7. *For $n \geq 4$ even, $M_2(3, n)$ with a single faulty vertex is strongly hamiltonian laceable.*

Proof. L^{opt}-paths can be constructed for all cases: i) $s, t \in R(1:2)$, ii) $s \in R(1:2), t \in R(3)$, iii) $s, t \in R(3)$. The details are omitted. □

Lemma 8. *For $m \geq 2, n \geq 4$ even, $M_2(m, n)$ with a single faulty vertex is strongly hamiltonian laceable.*

Proof. The proof is by induction on m. For $m = 2$ and $m = 3$, the lemma is true by Lemma 3 and Lemma 7, respectively. For $m \geq 4$, we assume that the lemma holds for every $k < m$, and we consider $M_2(m, n)$. Let $s = v_i^x$, $t = v_j^y$. Without loss of generality, we assume that a faulty vertex $v_f \in W$ and $v_f \in R(1:m-2)$.

Case 1: $s, t \in R(1:m-1)$. Similar to Case 1 in Theorem 1, we can constructed an L^{opt}-path.

Case 2: $s \in R(1:m-1)$ and $t \in R(m)$.

Case 2.1: $s \in R(1:m-2)$. We choose a black vertex s' which is one of the two vertices v_1^{m-2} and v_n^{m-2}. If $s = s'$ or s' is adjacent to both v_f and s, then let s' be the black vertex in $R(m-2) \cap C(2:n-1)$. There exists an L^{opt}-path P' joining s and s' in $G\langle R(1:m-2)\rangle$ as follows. When either s or s' is v_n^{m-2}(resp. v_1^{m-2}) and m is even(resp. odd), P' exists by Lemma 6. Otherwise P' can be constructed as follows:

$$P' = \begin{cases} (H[s, v_{n-1}^1 | R(1:m-3) \cap C(n-1:n)], \\ \ H[v_{n-2}^1, s' | R(1:m-2) \cap C(1:n-2)]) & \text{if } m \text{ is even} \\ (H[s, v_1^1 | R(1:m-3) \cap C(1:2)], \\ \ H[v_n^1, s' | R(1:m-2) \cap C(3:n)]) & \text{if } m \text{ is odd} \end{cases}$$

Let t' be the vertex in $R(m-1)$ adjacent to s'. By Lemma 5, there exists an L^{opt}-path P'' joining t' and t in $G\langle R(m-1:m)\rangle$. P' and P'' form an L^{opt}-path.

Case 2.2: $s \in R(m-1)$. If v_f is in $R(2:m-2)$, then this case is symmetric to Case 1. Thus, we only consider the case that $v_f \in R(1)$. We choose two vertices u and v as follows. When $s, t \in B$, let $u = v_2^{m-2}$ and $v = v_n^{m-2}$ if m is even; otherwise $u = v_1^{m-2}$ and $v = v_{n-1}^{m-2}$. When at least one of s or t is white, $u = v_1^{m-2}$ and $v = v_n^{m-2}$. Let u' and v' be the vertices in $R(m-1)$ adjacent to u and v, respectively. By Lemma 6, there exists an L^{opt}-path P' joining u and v in $G\langle R(1:m-2)\rangle$. Let P'' and P''' be two vertex-disjoint paths in $G\langle R(m-1:m)\rangle$ such that $V(P'')\cup V(P''') = R(m-1:m)$ and they are joining s and u', v' and t, or s and v', u' and t, respectively. We can construct an L^{opt}-path $P=(P'',P',P''')$. P'' and P''' can be constructed as follows:

Without loss of generality, we assume that m is even.

Case 2.2.1 $i \le j$. (i) $s,t \in B$,

$$P'' = \begin{cases} (s,u') & \text{if } t \in C(2) \\ (H[s,u'|C(1:i)\cap R(m-1:m)]) & \text{if } t\in C(3:n) \end{cases}$$

$$P''' = \begin{cases} (H[v',v_n^m|C(3:n)\cap R(m-1:m)],v_1^m,t) & \text{if } t\in C(2) \\ (H[v',t|C(i+1:n)\cap R(m-1:n)]) & \text{if } t \in C(3:n) \end{cases}$$

(ii) $s\in W$ and $t \in B$, $P''=(H[s,u'|C(1:i)\cap R(m-1:m)])$ and $P'''=(H[v',t|C(i+1:n)\cap R(m-1:m)])$.

(iii) $s\in B$ and $t\in W$, $P''=(s,v_{i-1}^{m-1}\to u')$ and $P'''=(H[v',v_n^m|C(j+1:n)\cap R(m-1:m)],v_1^m\to v_i^m,H[v_{i+1}^m,t|C(i+1:j)\cap R(m-1:m)])$.

(iv) $s,t\in W$. $P''=(H[s,v'|C(1:j-1)\cap R(m-1:m)])$ and $P'''=(H[v',v_{j+1}^y|C(j+1:n)\cap R(m-1:m)],t)$.

Case 2.2.2 $i>j$. Similar to Case 2.2.1, P'' and P''' can be constructed. The details of P'' and P''' are omitted.

Case 3: $s,t\in R(m)$. If v_f is in $R(2:m-2)$, then this case is symmetric to Case 1. Thus, we only consider the case that $v_f \in R(1)$. The same way as Case 2.2, we can construct an L^{opt}-path P except the case that $s\in B$ and $t\in W$. We only show the case that $s \in B$ and $t\in W$. An L^{opt}-path P' in $G\langle R(1:m-2)\rangle$ can be obtained by using the same way as Case 2.2, and two vertex-disjoint paths P'' and P''' in $G\langle R(m-1:m)\rangle$ can be constructed as follows: $P''=(H[s,v_{j-1}^{m-1}|C(i:j-1)\cap R(m-1:m)],v_j^{m-1}\to v')$ and $P'''=(H[u',v_1^m|C(1:i-1)\cap R(m-1:m)],v_n^m\to t)$. (P'',P',P''') forms an L^{opt}-path. □

Lemma 9. *For $m\ge 2, n\ge 4$ even, $M_2(m,n)$ with a single faulty edge is hamiltonian laceable.*

Proof. We prove by induction on m. Due to Lemma 3, the lemma holds for $m=2$. For $m\ge 3$, we assume that the lemma is true for every $k<m$, and we consider $M_2(m,n)$. There exists a hamiltonian path by Lemma 2, if e_f is one of (v_1^1,v_n^1) and (v_1^m,v_n^m). Let $s=v_i^x$ and $t=v_j^y$. Without loss of generality, we assume that the faulty edge $e_f\in G\langle R(1:m-1)\rangle$.

Case 1: $s,t\in R(1:m-1)$. Similary to Case 1 in Theorem 1, we can construct an L^{opt} path.

Case 2: $s\in R(1:m-1)$ and $t\in R(m)$.

Case 2.1: $e_f=(v_k^z,v_k^{z+1})$. When $x>z$, there exists a hamiltonian path P' joining s and t in $G\langle R(z+1:m)\rangle$ by Lemma 5. We choose an edge (u,v) in $G\langle R(z+1)\rangle$ such that P' includes it and neither u nor v is v_k^{z+1}. Let u' and v' be the vertices in $R(z)$ adjacent to u and v, respectively. By Lemma 5, there exists a hamiltonian path P'' joining u and v in $G\langle R(1:z)\rangle$. By a merge of P'

and P'' w.r.t. (u, u') and (v, v'), we have a hamiltonian path P. When $x \le z$, we choose a vertex s' in $R(z)$ such that s' has a different color from s and is not v_k^z. Let t' be the vertex in $G\langle R(z+1)\rangle$ adjacent to s'. By Lemma 5, we can construct a hamiltonian path $P = (H[s, s'|R(1:z)], H[t', t|R(z+1:m)])$.

Case 2.2: $e_f = (v_k^z, v_{k+1}^z)$. Without loss of generality, P can be constructed according to the three cases. (i) $i \le k < j$, (ii) $i \le j \le k$, and (iii) $k < i \le j$. We only show the case that $i \le k < j$, $s \in B$, and $t \in W$. Proofs of other cases are omitted.

a) For $k = 1$, If $x > z$, then $s \in B$ and $s \in C(1)$.

$$P = (s \to v_1^1, H[v_n^1, v_{j-1}^x|C(2:n) \cap R(1:x)], H[v_{j-1}^{x+1}, t|R(x+1:m)])$$

If $x \le z$,

$$P = \begin{cases} (s \to v_1^m, v_2^m \to v_2^x, H[v_2^{x-1}, v_3^{x-1}|R(1:x-1)], \\ \ H[v_3^x, t|C(3:n) \cap R(x:m)]) & \text{if } j = n \\ (s \to v_1^m, v_n^m \to v_n^x, H[v_n^{x-1}, v_{n-1}^{x-1}|R(1:x-1)], \\ \ H[v_{n-1}^x, t|C(2:n-1) \cap R(x:m)]) & \text{if } j \ne n \end{cases}$$

b) For $1 < k < n-1$,

$$P = \begin{cases} (H[s, v_k^m|C(1:k)], H[v_{k+1}^m, t|C(k+1:n)]) & \text{if } j = n \\ (H[s, v_1^m|C(1:k)], H[v_n^m, t|C(k+1:n)]) & \text{if } j \ne n \end{cases}$$

Case 3: $s, t \in R(m)$.

Case 3.1: $e_f = (v_k^z, v_k^{z+1})$. By Lemma 5, there exists a hamiltonian path P' joining s and t in $G\langle R(z+1:m)\rangle$. We choose an edge (u, v) in $G\langle R(z+1)\rangle$ such that P' includes it and neither u nor v is v_k^{z+1}. Let u' and v' be the vertices in $R(z)$ adjacent to u and v, respectively. In $G\langle R(1:z)\rangle$, there exists a hamiltonian path P'' joining u' and v' by Lemma 5. By a merge of P' and P'' w.r.t. (u, u') and (v, v'), we have a hamiltonian path.

Case 3.2: $e_f = (v_k^z, v_{k+1}^z)$. Without loss of generality, we assume that $i < j$.

Case 3.2.1: $i < j \le k$. When $s \in B$ and $t \in W$,

$$P = \begin{cases} (H[s, v_1^m|C(1:j-1) \cap R(2:m)], H[v_n^m, v_n^1|C(k+1:n)], \\ \ v_1^1 \to v_{j-1}^1, H[v_j^1, t|C(j:k)]) & \text{if } m \text{ is even} \\ (H[s, v_{j-1}^2|C(1:j-1) \cap R(2:m)], \\ \ H[v_j^2, v_j^1|C(j:k) \cap R(1:m-1)], v_{j-1}^1 \to v_1^1, \\ \ H[v_n^1, v_{k+1}^m|C(k+1:n)], v_k^m \to t) & \text{if } m \text{ is odd} \end{cases}$$

When $s \in W$ and $t \in B$,

$$P = \begin{cases} (s \to v_1^m, H[v_n^m, v_n^1|C(k+1:n)], \\ \ H[v_1^1, v_1^{m-2}|C(1:k) \cap R(1:m-2)], v_1^{m-1} \to v_i^{m-1}, \\ \ H[v_{i+1}^{m-1}, t|C(i+1:k) \cap R(m-1:m)]) & \text{if } m \text{ is even} \\ (H[s, v_{j-1}^{m-1}|C(1:j-1) \cap R(m-1:m)], v_j^{m-1} \to v_k^{m-1}, \\ \ H[v_k^{m-2}, v_1^1|C(1:k) \cap R(1:m-2)], \\ \ H[v_n^1, v_{k+1}^m|C(k+1:n)], v_k^m \to t) & \text{if } m \text{ is odd} \end{cases}$$

Case 3.2.2: $i \le k < j$. We can construct a hamiltonian path P as follows. When m is even, let $s' = v_1^m$ and $t' = v_n^m$ if $s \in B$; $s' = v_1^1$ and $t' = v_n^1$ if $s \in W$.

When m is odd and $s \in B$, let $s' = v_k^m$ and $t' = v_{k+1}^m$ if k is even; $s' = v_1^1$ and $t' = v_n^1$ if k is odd. $P = (H[s, s'|C(1:k)], H[t', t|C(k+1:n)])$. When m is odd and $s \in W$, $P = (H[s, v_k^m|C(i:k)], v_{k+1}^m \to v_{k+1}^1 \to v_n^1, H[v_1^1, v_n^1|C(1:i-1)], H[v_n^m, t|C(k+2:n) \cap R(2:m)])$. □

Theorem 3. *For $m \geq 2, n \geq 4$ even, $M_2(m, n)$ is 1-fault strongly hamiltonian laceable.*

Proof. By Lemma 8, $M_2(m, n)$ with a single faulty vertex is strongly hamiltonian laceable. By Lemma 9, $M_2(m, n)$ with a single faulty edge has a hamiltonian path between any two vertices with different colors. It remains to show that there is an L^{opt}-path (of length $mn - 2$) joining every pair of vertices s and t with the same color. Let (u, v) be the faulty edge. Without loss of generality, we assume that $u \in B$ and $v \in W$. When s and t are black, we can find an L^{opt}-path P between s and t regarding v as a faulty vertex by using Lemma 8. P does not pass through (u, v) as well as v, and the length of P is $mn - 2$. Thus, P is a desired L^{opt}-path. In a similar way, we can construct an L^{opt}-path for a pair of white vertices. □

4 Fault hamiltonicity of Hypercubes

An n-dimensional hypercube, denoted by Q_n, consists of 2^n vertices that can be represented in the form of binary strings, $b_n b_{n-1} ... b_1$. Two vertices are adjacent if and only if their labels differ in exactly one bit. An edge is referred to as an i-dimension edge if the vertices it connects differ in bit position i. A k-dimensional subcube in Q_n is represented by a string of n symbols over set $\{0, 1, *\}$, where $*$ is a *don't care* symbol, such that there are exactly k $*$'s in the string.

Lemma 10. *For $f_e \leq n - 3$ and $1 \leq r \leq n - f_e - 2$, Q_n with f_e faulty edges has $M_2(2^r, 2^{n-r})$ as a spanning subgraph.*

Proof. Let $D = \{1, 2, ..., n\}$ be the set of dimensions in Q_n, $D_f = \{f_1, f_2, ..., f_i\}$ be the set of dimensions which contain faulty edges, and $D_s = D - D_f = \{s_1, s_2, ..., s_j\}$. Since $f_e \leq n - 3$, we have $|D_s| \geq 3$.

If we replace $b_{s_1}, b_{s_2}, ..., b_{s_r}$ bits of each vertex label in Q_n by '$*$', then Q_n can be partitioned into 2^{n-r} r-subcubes and theses subcubes form Q_{n-r} by replacing each subcube as a vertex. We denote such a graph by a condensation graph $Q_{n-r}^{\mathcal{C}}$ of Q_n. Let u, v be the vertices in Q_n, and $\mathcal{C}_u$, $\mathcal{C}_v$ be the components containing u and v, respectively. We assume that an edge $(\mathcal{C}_u, \mathcal{C}_v)$ of $Q_{n-r}^{\mathcal{C}}$ is faulty if (u, v) is faulty. For all $n \geq 2$, Q_n with $n - 2$ faulty edges has a hamiltonian cycle[11]. Since $f_e \leq n - r - 2$, there exists a hamiltonian cycle in $Q_{n-r}^{\mathcal{C}}$. Let $C = (x_1, x_2, ..., x_d, ..., x_{2^{n-r}}, x_1)$ be a hamiltonian cycle in $Q_{n-r}^{\mathcal{C}}$, where $1 \leq d \leq 2^{n-r}$. Each vertex of Q_n can be mapped to v_d^k of $M_2(m, n)$ as follows:

In $b_n b_{n-1} ... b_1$ of each vertex label of Q_n,

(i) $b_{s_1} b_{s_2} ... b_{s_r}$ is the k-th sequence of r-bit Gray code, and

(ii) $(n - r)$-bits (except $b_{s_1}, b_{s_2}, ..., b_{s_r}$ bits) represent the label of the d-th vertex in C.

Thus, Q_n has fault-free $M_2(2^r, 2^{n-r})$ as a spanning subgraph. □

By applying fault-hamiltonicity of $M_2(m, n)$ to a hypercube, we have the following theorem.

Theorem 4. *Q_n with $f \leq n-2$ and $f_v \leq 1$ is strongly hamiltonian laceable.*

5 Conclusion

In this paper, we considered the fault hamiltonian properties of $m \times n$ meshes with two wraparound edges in the first row and the last row. We showed that $M_2(m, n)$ with odd n is hamiltonian-connected and 1-fault hamiltonian. $M_2(m, n)$ has these hamiltonian properties by adding minimum edges to an $m \times n$ mesh. For $n \geq 4$ even, we showed that $M_2(m, n)$ is 1-fault strongly hamiltonian laceable. By applying fault hamiltonicity of $M_2(m, n)$ to a hypercube, we obtained that Q_n with at most $n-2$ faulty elements and at most one faulty vertex is strongly hamiltonian laceable. Also, our results can be applied to the fault hamiltonian properties of other interconnection networks which has $M_2(m, n)$ as a spanning subgraph.

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