Panconnectivity and Pancyclicity of Hypercube-Like Interconnection Networks with Faulty Elements*

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Abstract. In this paper, we deal with the graph $G_0 \oplus G_1$ obtained from merging two graphs G_0 and G_1 with n vertices each by n pairwise nonadjacent edges joining vertices in G_0 and vertices in G_1. The main problems studied are how fault-panconnectivity and fault-pancyclicity of G_0 and G_1 are translated into fault-panconnectivity and fault-pancyclicity of $G_0 \oplus G_1$, respectively. Applying our results to a subclass of hypercube-like interconnection networks called *restricted HL-graphs*, we show that in a restricted HL-graph G of degree $m(\geq 3)$, each pair of vertices are joined by a path in $G \backslash F$ of every length from $2m-3$ to $|V(G \backslash F)|-1$ for any set F of faulty elements (vertices and/or edges) with $|F| \leq m-3$, and there exists a cycle of every length from 4 to $|V(G \backslash F)|$ for any fault set F with $|F| \leq m-2$.

1 Introduction

Linear arrays and rings are two of the most important computational structures in interconnection networks. So, embedding of linear arrays and rings into a faulty interconnection network is one of the important issues in parallel processing[9, 13, 15]. An interconnection network is often modeled as a graph, in which vertices and edges correspond to nodes and communication links, respectively. Thus, the embedding problem can be modeled as finding fault-free paths and cycles in the graph with some faulty vertices and/or edges. In the embedding problem, if the longest path or cycle is required the problem is closely related to well-known hamiltonian problems in graph theory. In the rest of this paper, we will use standard terminology in graphs (see ref. [3]).

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Definition 1. *A graph G is called* f-fault hamiltonian *(resp.* f-fault hamiltonian-connected*) if there exists a hamiltonian cycle (resp. if each pair of vertices are joined by a hamiltonian path) in $G\backslash F$ for any set F of faulty elements with $|F| \le f$.*

On the other hand, if the paths joining each pair of vertices of every length shorter than or equal to a hamiltonian path are required the problem is concerned with panconnectivity of the graph. If the cycles of arbitrary size (up to a hamiltonian cycle) are required the problem is concerned with pancyclicity of the graph.

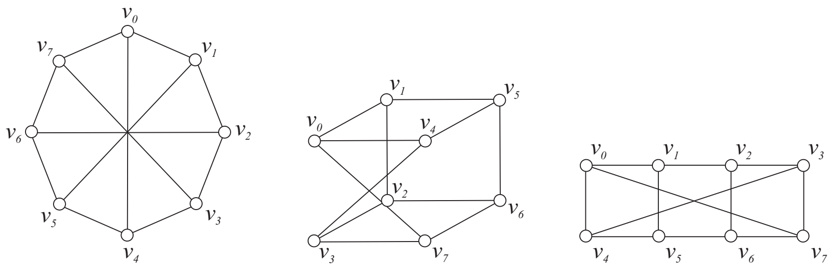
Definition 2. *A graph G is called* f-fault q-panconnected *if each pair of fault-free vertices are joined by a path in $G\backslash F$ of every length from q to $|V(G\backslash F)|-1$ inclusive for any set F of faulty elements with $|F| \le f$.*

Definition 3. *A graph G is called* f-fault pancyclic *(resp.* f-fault almost pancyclic*) if $G\backslash F$ contains a cycle of every length from 3 to $|V(G\backslash F)|$ (resp. 4 to $|V(G\backslash F)|$) inclusive for any set F of faulty elements with $|F| \le f$.*

Pancyclicity of various interconnection networks was investigated in the literature. Recursive circulant $G(2^m, 4)$ of degree m was shown to be 0-fault almost pancyclic in [2] and then $m-2$-fault almost pancyclic in [12]. Möbius cube of degree m is 0-fault almost pancyclic[5] and $m-2$-fault almost pancyclic[8]. Crossed cube and twisted cube of degree m were also shown to be $m-2$-fault almost pancyclic in [17] and in [18]. Edge-pancyclicity of some fault-free interconnection networks such as recursive circulants, crossed cubes, twisted cubes was studied in [1], [7], and [6]. The work on panconnectivity of interconnection networks has a relative paucity and some results can be found in [4, 10]. As the authors know, no results on fault-panconnectivity were reported in the literature.

Many interconnection networks can be expanded into higher dimensional networks by connecting two lower dimensional networks. As a graph modeling of the expansion, we consider the graph obtained by connecting two graphs G_0 and G_1 with n vertices. We denote by V_i and E_i the vertex set and edge set of G_i, $i = 0, 1$, respectively. We let $V_0 = \{v_1, v_2, \ldots, v_n\}$ and $V_1 = \{w_1, w_2, \ldots, w_n\}$. With respect to a permutation $M = (i_1, i_2, \ldots, i_n)$ of $\{1, 2, \ldots, n\}$, we can "merge" the two graphs into a graph $G_0 \oplus_M G_1$ with $2n$ vertices in such a way that the vertex set $V = V_0 \cup V_1$ and the edge set $E = E_0 \cup E_1 \cup E_2$, where $E_2 = \{(v_j, w_{i_j}) | 1 \le j \le n\}$. We denote by $G_0 \oplus G_1$ a graph obtained by merging G_0 and G_1 w.r.t. an arbitrary permutation M. Here, G_0 and G_1 are called *components* of $G_0 \oplus G_1$.

Vaidya *et al.*[16] introduced a class of hypercube-like interconnection networks, called *HL-graphs*, which can be defined by applying the $\oplus$ operation repeatedly as follows: $HL_0 = \{K_1\}$; for $m \ge 1$, $HL_m = \{G_0 \oplus G_1 | G_0, G_1 \in HL_{m-1}\}$. Then, $HL_1 = \{K_2\}$; $HL_2 = \{C_4\}$; $HL_3 = \{Q_3,\ G(8, 4)\}$. Here, C_4 is a cycle graph with 4 vertices, Q_3 is a 3-dimensional hypercube, and $G(8, 4)$ is a recursive circulant which is isomorphic to twisted cube TQ_3 and Möbius ladder with 4 spokes as shown in Figure 1. It was shown by Park and Chwa in [11] that every nonbipartite HL-graph is hamiltonian-connected.



(a) $G(8, 4)$ (b) TQ_3 (c) Möbius ladder

Fig. 1. Isomorphic graphs.

In [13], a subclass of nonbipartite HL-graphs, called *restricted HL-graphs* was introduced which is defined recursively as follows: $RHL_m = HL_m$ for $0 \leq m \leq 2$; $RHL_3 = HL_3 \backslash Q_3 = \{G(8, 4)\}$; $RHL_m = \{G_0 \oplus G_1 | G_0, G_1 \in RHL_{m-1}\}$ for $m \geq 4$. A graph which belongs to RHL_m is called an *m-dimensional restricted HL-graph.* Many of the nonbipartite hypercube-like interconnection networks such as crossed cube, Möbius cube, twisted cube, multiply twisted cube, Mcube, generalized twisted cube, locally twisted cube, etc. proposed in the literature are restricted HL-graphs. It was shown in [13] that every m-dimensional restricted HL-graph, $m \geq 3$, is $m-3$-fault hamiltonian-connected and $m-2$-fault hamiltonian. The result was utilized in [14] to find disjoint paths which cover all the vertices between source-sink pairs in restricted HL-graphs.

We first investigate panconnectivity and pancyclicity of $G_0 \oplus G_1$ with faulty elements. It will be shown that if each G_i is f-fault q-panconnected and $f+1$-fault hamiltonian (with additional conditions $n \geq f+2q+1$ and $q \geq 2f+3$), then $G_0 \oplus G_1$ is $f+1$-fault $q+2$-panconnected for any $f \geq 2$. To study pancyclicity of $G_0 \oplus G_1$, the notion of *hypohamiltonian-connectivity* is introduced. A graph G is called *f-fault hypohamiltonian-connected* if each pair of vertices can be joined by a path of length $|V(G \backslash F)| - 2$, that is one less than the longest possible length, in $G \backslash F$ for any fault set F with $|F| \leq f$. We will show that if each G_i is f-fault hamiltonian-connected, f-fault hypohamiltonian-connected, and $f+1$-fault almost pancyclic, then $G_0 \oplus G_1$ is $f+2$-fault almost pancyclic for any $f \geq 1$.

Our main results are applied to restricted HL-graphs. We will show that every m-dimensional restricted HL-graph with $m \geq 3$ is $m-3$-fault $2m-3$-panconnected and $m-2$-fault almost pancyclic. Both bounds $m-3$ and $m-2$ on the number of acceptable faulty elements are the maximum possible. Notice that f-fault q-panconnected graph is f-fault hamiltonian-connected, and that f-fault almost pancyclic graph is f-fault hamiltonian. Our results are not only the extension of some works of [8, 17, 18] on fault-pancyclicity of restricted HL-graphs, but also a new investigation on fault-panconnectivity of restricted HL-graphs.

2 Panconnectivity and Pancyclicity of $G_0 \oplus G_1$

For a vertex v in $G_0 \oplus G_1$, we denote by $\bar{v}$ the vertex adjacent to v which is in a component different from the component in which v is contained. We denote by F the set of faulty elements. When we are to construct a path from s to t, s and t are called a *source* and a *sink*, respectively, and both of them are called *terminals*. Throughout this paper, a path in a graph is represented as a sequence of vertices.

Definition 4. *A vertex v in $G_0 \oplus G_1$ is called* free *if v is fault-free and not a terminal, that is, $v \notin F$ and v is neither a source nor a sink. An edge (v, w) is called* free *if v and w are free and $(v, w) \notin F$.*

We denote by V_i and E_i the sets of vertices and edges in G_i, $i = 0, 1$, and by E_2 the set of edges joining vertices in G_0 and vertices in G_1. We let $n = |V_0| = |V_1|$. F_0 and F_1 denote the sets of faulty elements in G_0 and G_1, respectively, and F_2 denotes the set of faulty edges in E_2, so that $F = F_0 \cup F_1 \cup F_2$. Let $f_0 = |F_0|$, $f_1 = |F_1|$, and $f_2 = |F_2|$.

When we find a path/cycle, sometimes we regard some fault-free vertices and/or edges as faulty elements. They are called *virtual* faults. If G_i is f-fault hamiltonian-connected and $f + 1$-fault hamiltonian, $i = 0, 1$, then

$$f \leq \delta(G_i) - 3, \text{ and thus } f + 4 \leq n,$$

where $\delta(G_i)$ is the minimum degree of G_i.

2.1 Panconnectivity of $G_0 \oplus G_1$

Hamiltonian-connectivity of $G_0 \oplus G_1$ with faulty elements was considered in [13]. In this subsection, we study panconnectivity of $G_0 \oplus G_1$ in the presence of faulty elements. We denote by f_v^0 and f_v^1 the numbers of faulty vertices in G_0 and G_1, respectively, and by f_v the number of faulty vertices in $G_0 \oplus G_1$, so that $f_v = f_v^0 + f_v^1$. Note that the length of a hamiltonian path in $G_0 \oplus G_1 \backslash F$ is $2n - f_v - 1$.

Theorem 1. *Let G_0 and G_1 be graphs with n vertices each. Let f and q be nonnegative integers satisfying $n \geq f + 2q + 1$ and $q \geq 2f + 3$. If each G_i is f-fault q-panconnected and $f + 1$-fault hamiltonian, then*
(a) for any $f \geq 2$, $G_0 \oplus G_1$ is $f + 1$-fault $q + 2$-panconnected,
(b) for $f = 1$, $G_0 \oplus G_1$ with $2(= f + 1)$ faulty elements has a path of every length $q + 2$ or more joining s and t unless s and t are contained in the same component and $\bar{s}$ and $\bar{t}$ are the faulty elements(vertices), and
(c) for $f = 0$, $G_0 \oplus G_1$ with $1(= f + 1)$ faulty element has a path of every length $q + 2$ or more joining s and t unless s and t are contained in the same component and the faulty element is contained in the other component.

Proof. To prove (a), assuming the number of faulty elements $|F| \leq f+1$, we will construct a path of every length l, $q+2 \leq l \leq 2n-f_v-1$, in $G_0 \oplus G_1 \backslash F$ joining any pair of vertices s and t.

Case 1: $f_0, f_1 \leq f$.
When both s and t are contained in G_0, there exists a path P_0 of length l_0 in G_0 joining s and t for every $q \leq l_0 \leq n-f_v^0-1$. We are to construct a longer path P_1 that passes through vertices in G_1 as well as vertices in G_0. We first claim that there exists an edge (x,y) on P_0 such that all of $\bar{x}$, $(x,\bar{x})$, $\bar{y}$, and $(y,\bar{y})$ are fault-free. There are l_0 candidate edges on P_0 and at most $f+1$ faulty elements can "block" the candidates, at most two candidates per one faulty element. By assumption $l_0 \geq q \geq 2f+3$, and the claim is proved. The path P_1 can be obtained by merging P_0 and a path P' in G_1 between $\bar{x}$ and $\bar{y}$ with the edges $(x,\bar{x})$ and $(y,\bar{y})$. Here, of course the edge (x,y) is discarded. Letting l' be the length of P', the length l_1 of P_1 can be anything in the range $2q+1 \leq l_1 = l_0+l'+1 \leq 2n-f_v-1$. Since $n \geq f+2q+1$, we have $2q+1 \leq n-f_v^0$ and we are done.

When s is in G_0 and t is in G_1, we first find a free edge $(x,\bar{x})$ in E_2 such that $(\bar{x},t)$ is an edge and fault-free. The existence of such a free edge $(x,\bar{x})$ is due to the fact that there are $\delta(G_1)$ candidates and that at most $f+1$ faulty elements and the source s can block the candidates. Remember $f \leq \delta(G_1)-3$. Assuming $x \in V_0$, a path joining s and x in G_0 and an edge $(\bar{x},t)$ are merged with $(x,\bar{x})$ into a path P_0. The length l_0 of P_0 is any integer in the range $q+2 \leq l_0 \leq n-f_v^0+1$. A longer path P_1 is obtained by replacing the edge $(\bar{x},t)$ with a path in G_1 between $\bar{x}$ and t of length l'', $q \leq l'' \leq n-f_v^1-1$. The length l_1 of P_1 is in the range $2q+1 \leq l_1 \leq 2n-f_v-1$. We are done since $2q+1 \leq n-f_v^0$ as shown in the previous subcase.

Case 2: $f_0 = f+1$ (or symmetrically, $f_1 = f+1$).
We have $f_1 = f_2 = 0$. First, we consider the subcase $s,t \in V_0$. Letting P' be a path in G_1 joining $\bar{s}$ and $\bar{t}$, we have a path $P_0 = (s, P', t)$ between s and t. The length l_0 of P_0 is any integer in the range $q+2 \leq l_0 \leq n+1$. To construct a longer path P_1, we select an arbitrary faulty element α in G_0. Regarding α as *a virtual fault-free element*, find a path P'' in G_0 between s and t. If α is a faulty vertex on P'', let x and y be the two vertices on P'' next to α; else if P'' passes through the faulty edge α, let x and y be the endvertices of α; else let (x,y) be an arbitrary edge on P''. The path P_1 is obtained by merging $P''\backslash\alpha$ and a path in G_1 joining $\bar{x}$ and $\bar{y}$ with edges $(x,\bar{x})$ and $(y,\bar{y})$. If α is faulty vertex on P'', the length l_1 of P_1 is in the range $2q \leq l_1 \leq 2n-f_v-1$; otherwise, we have $2q+1 \leq l_1 \leq 2n-f_v-1$. In any cases, we are done since $2q+1 \leq n+2$.

Secondly, we consider the subcase $s \in V_0$ and $t \in V_1$. We first find a hamiltonian cycle C in $G_0 \backslash F_0$ and let $C = (s = z_0, z_1, z_2, ..., z_k)$, where $k = n-f_v^0-1$. Assuming $\bar{z}_l \neq t$ without loss of generality, we can construct a path P_0 by merging $(z_0, z_1, ..., z_l)$ and a path in G_1 between $\bar{z}_l$ and t with the edge $(z_l, \bar{z}_l)$. The length l_0 of P_0 is any integer in the range $q+l+1 \leq l_0 \leq n-f_v^1+l$. Since l itself is any integer in the range $1 \leq l \leq n-f_v^0-1$, we have $q+2 \leq l_0 \leq 2n-f_v-1$.

Finally, we consider the subcase $s, t \in V_1$. We have a path P_0 in G_1 joining s and t, and the length l_0 of P_0 is in the range $q \leq l_0 \leq n-1$. To construct a longer path P_1, we let $C = (z_0, z_1, z_2, ..., z_k)$ be a hamiltonian cycle in $G_0 \backslash F_0$, where $k = n - f_v^0 - 1$. If $\bar{s} \notin F$, we assume w.l.o.g. $\bar{s} = z_0$. Then, letting w.l.o.g. $\bar{z_l} \neq t$, P_1 is a concatenation of $(s, z_0, z_1, \ldots, z_l)$ and a path in $G_1 \backslash s$ between $\bar{z_l}$ and t. The length l_1 of P_1 is in the range $q+3 \leq l_1 \leq 2n - f_v - 1$. If $\bar{s} \in F$, we let $(x, \bar{x})$ be a free edge such that $\bar{x}$ is adjacent to s. Then, letting w.l.o.g. $x = z_0$ and $\bar{z_l} \neq t$, P_1 is a concatenation of $(s, \bar{x}, z_0, z_1, \ldots, z_l)$ and a path in $G_1 \backslash \{s, \bar{x}\}$ between $\bar{z_l}$ and t. Here, the length l_1 of P_1 is in the range $q+4 \leq l_1 \leq 2n - f_v - 1$. By the condition of $n \geq f + 2q + 1$ and $q \geq 2f + 3$, we can observe $q + 4 \leq n$. Therefore, we are done. This completes the proof of (a).

It immediately follows from Case 1 and the first and second subcases of Case 2, where the assumption $f \geq 2$ is never used, that for $f = 0, 1$, $G_0 \oplus G_1$ with $f+1$ faulty elements has a path of every length $q+2$ or more joining s and t unless s and t are contained in the same component and all the faulty elements are contained in the other component. Thus, the proof of (c) is done. To prove (b), assuming w.l.o.g. $\bar{s} \notin F$, it suffices to employ the construction of the last subcase of Case 2. Note that in the construction, G_1 is 1-fault q-panconnected. This completes the proof. □

Corollary 1. *Let G_0 and G_1 be graphs with n vertices each. Let f and q be nonnegative integers satisfying $n \geq f + 2q + 1$ and $q \geq 2f + 3$. If each G_i is f-fault q-panconnected and $f+1$-fault hamiltonian, then $G_0 \oplus G_1$ is f-fault $q+2$-panconnected.*

2.2 Pancyclicity of $G_0 \oplus G_1$

In the presence of faulty elements, the existence of hamiltonian cycle in $G_0 \oplus G_1$ was considered in [13] as in Theorem 2. In this subsection, we investigate almost pancyclicity of $G_0 \oplus G_1$ with faulty elements. We denote by $H[v, w|G, F]$ a hamiltonian path in $G \backslash F$ joining a pair of fault-free vertices v and w in a graph G with a set F of faulty elements. $HH[v, w|G, F]$ is a hypohamiltonian path in $G \backslash F$ between v and w.

Theorem 2. *[13] Let a graph G_i be f-fault hamiltonian-connected and $f+1$-fault hamiltonian, $i = 0, 1$. Then,*
(a) for any $f \geq 1$, $G_0 \oplus G_1$ is $f+2$-fault hamiltonian, and
(b) for $f = 0$, $G_0 \oplus G_1$ with $2(= f+2)$ faulty elements has a hamiltonian cycle unless one faulty element is contained in G_0 and the other faulty element is contained in G_1.

Before presenting our theorem on pancyclicity, we will give two lemmas. The proofs are omitted. They imply that to show an f-fault hamiltonian graph is f-fault almost pancyclic, it is sufficient to consider only vertex faults and further the maximum number of vertex faults. We call a graph G to be *f-vertex-fault almost pancyclic*, if $G \backslash F_v$ contains a cycle of every length from 4 to $|V(G \backslash F_v)|$ for any set of faulty vertices F_v with $|F_v| \leq f$.

Lemma 1. *Let a graph G be f-fault hamiltonian and f-vertex-fault almost pancyclic. Then, G is f-fault almost pancyclic.*

Lemma 2. *Let a graph G be f-fault hamiltonian and almost pancyclic when the number of faulty vertices $f_v = f$. Then, G is f-vertex-fault almost pancyclic.*

Theorem 3. *Let G_i be f-fault hamiltonian-connected, f-fault hypohamiltonian-connected, and $f+1$-fault almost pancyclic, $i = 0, 1$. Then,*
(a) for any $f \geq 1$, $G_0 \oplus G_1$ is $f+2$-fault almost pancyclic, and
(b) for $f = 0$, $G_0 \oplus G_1$ with $2(= f+2)$ faulty elements is almost pancyclic unless one faulty element is contained in G_0 and the other faulty element is contained in G_1.

Proof. To prove (a), we let $|F| = f + 2$, and assume F has only vertex faults by virtue of the above two lemmas. Note that, by Theorem 2(a), $G_0 \oplus G_1$ is $f+2$-fault hamiltonian. Assuming $f_0 \geq f_1$ without loss of generality, we will construct cycles in $G_0 \oplus G_1 \backslash F$. By the condition in the theorem, there exist cycles of length from 4 to $n - f_1$ in $G_1 \backslash F_1$. Also, the cycle of length $2n - f_0 - f_1$ exists. So, the construction of remaining cycles of length from $n - f_1 + 1$ to $2n - f_0 - f_1 - 1$ will be given.

Case 1: $f_0 \leq f$.

Subcase 1.1: $n > f_0 + 2f_1$.

There exists a hamiltonian cycle C_0 of length $n - f_0$ in $G_0 \backslash F_0$. On C_0, we have $n - f_0$ different paths P_k's of length k for every $1 \leq k \leq n - f_0 - 1$. Among them, there exists a P_k joining x_k and y_k such that both $\bar{x_k}$ and $\bar{y_k}$ are fault-free, since we have $n - f_0$ candidates and each of f_1 faulty vertices in G_1 can block at most two candidates. Then, $C = (P_k, HH[\bar{y_k}, \bar{x_k}|G_1, F_1])$ is a cycle of length $n - f_1 + k$, $1 \leq k \leq n - f_0 - 1$.

Subcase 1.2: $n \leq f_0 + 2f_1$.

We find two free edges $(x, \bar{x})$ and $(y, \bar{y})$ in E_2. Such free edges exist since there are $n(\geq f+4)$ candidates and $f+2$ blocking elements. Note that there are no terminals. We will construct a cycle by merging $H[x, y|G_0, F']$ or $HH[x, y|G_0, F']$ with $H[\bar{x}, \bar{y}|G_1, F'']$ or $HH[\bar{x}, \bar{y}|G_1, F'']$. Here, F' (resp. F'') is a set of faulty elements in G_0 (resp. G_1) regarding some fault-free vertices as virtual faults. By taking account of $f - f_0$ vertices in $G_0 \backslash F_0$ excluding $\{x, y\}$ as virtual faults one by one, we can construct paths of length from $n - f - 2$ to $n - f_0 - 1$ between x and y. Also, by taking account of $f - f_1$ vertices in $G_1 \backslash F_1$ excluding $\{\bar{x}, \bar{y}\}$ as virtual faults one by one, we can construct paths of length from $n - f - 2$ to $n - f_1 - 1$ between $\bar{x}$ and $\bar{y}$. By merging two paths in G_0 and G_1, we can obtain cycles of length from $2n - 2f - 2$ to $2n - f_0 - f_1$. If $2n - 2f - 2 \leq n - f_1 + 1$, we will have all cycles of desired lengths. First, we have $2n - 2f - 2 \leq n - f_1 + 2$ since $(2n - 2f - 2) - (n - f_1 + 2) = n - 2f + f_1 - 4 \leq (f_0 + 2f_1) - 2f + f_1 - 4 = f_0 + 3f_1 - 2f - 4 = 2f_1 - f - 2 \leq 0$. Furthermore, careful observation on the above equation leads to $2n - 2f - 2 \leq n - f_1 + 1$ unless $n = f_0 + 2f_1$ and $f_0 = f_1$.

For the remaining case that $n = f_0 + 2f_1$ and $f_0 = f_1$, it is sufficient to construct a cycle of length $n - f_1 + 1$. To do this, we claim that there exists an edge (x, y) in G_0 such that both $\bar{x}$ and $\bar{y}$ are fault-free. Let $W = \{w | w \in V_0 \backslash F_0,$

$\bar{w} \notin F\}$, and let $B = V_0 \backslash (F_0 \cup W)$. It holds true that $|W| \geq |B|$ since $|W| \geq n - f_0 - f_1 = f_1$ and $|B| \leq f_1$. Let C_0 be a hamiltonian cycle in $G_0 \backslash F_0$. If there is an edge (a, b) on C_0 such that $a, b \in W$, we are done. Suppose otherwise, we have $|W| = |B|$ and the vertices on C_0 should alternate in W and B. Since $G_0 \backslash F_0$ is hamiltonian-connected, we always have such an edge (x, y) joining vertices in W. Note that $|W|, |B| \geq 2$, and that if there are no edges between vertices in W, there can not exist a hamiltonian path joining vertices in B. Then, we have a desired cycle $(x, y, HH[\bar{y}, \bar{x}|G_1, F_1])$ of length $n - f_1 + 1$.

Case 2: $f_0 = f + 1$.
We find a hamiltonian cycle C_0 in $G_0 \backslash F_0$, and let x_k and y_k be two vertices in C_0 such that both $\bar{x_k}$ and $\bar{y_k}$ are fault-free and there is a path of length k between x_k and y_k on C_0, $1 \leq k \leq n - f_0 - 1$. The existence of such x_k and y_k is due to the fact that the length of C_0 is at least three and $f_1 = 1$. Let P_k be the path of length k on C_0 whose endvertices are x_k and y_k. We construct cycles $(P_k, HH[\bar{y_k}, \bar{x_k}|G_1, F_1])$, $1 \leq k \leq n - f_0 - 1$, of length from $n - f_1 + 1$ to $2n - f_0 - f_1 - 1$. The hypohamiltonian path in G_1 between $\bar{y_k}$ and $\bar{x_k}$ exists since $f_1 = 1 \leq f$.

Case 3: $f_0 = f + 2$.
We select an arbitrary faulty vertex v_f in G_0, regarding it as *a virtual fault-free vertex*, find a hamiltonian cycle C_0 in $G_0 \backslash F'$, where $F' = F_0 \backslash v_f$. The existence of C_0 is due to $|F'| = f + 1$. Let P_k be an arbitrary path of length k on $C_0 \backslash v_f$ whose endvertices are x_k and y_k, $1 \leq k \leq n - f_0 - 1$. Then, we have a cycle $(P_k, HH[\bar{y_k}, \bar{x_k}|G_1, \emptyset])$ of length $n - f_1 + k$ for every $1 \leq k \leq n - f_0 - 1$.

The proof of (b) follows immediately from the proof of (a), where the assumption $f \geq 1$ is used only when $f_1 = 1$ in Case 2. □

3 Restricted HL-graphs

In this section, we will show that every m-dimensional restricted HL-graph is $m - 3$-fault $2m - 3$-panconnected and $m - 2$-fault almost pancyclic. Fault-hamiltonicity of restricted HL-graphs was studied in [13] as follows.

Theorem 4. *[13] Every m-dimensional restricted HL-graph, $m \geq 3$, is $m - 3$-fault hamiltonian-connected and $m - 2$-fault hamiltonian.*

3.1 Panconnectivity of restricted HL-graphs

By induction on m, we will prove that every m-dimensional restricted HL-graph, $m \geq 3$, is $m - 3$-fault $2m - 3$-panconnected. The proofs of lemmas are omitted.

Lemma 3. *The 3-dimensional restricted HL-graph is 0-fault 3-panconnected.*

To prove Lemmas 5 and 6, we employ a property on disjoint paths in $G(8, 4) \oplus G(8, 4)$ shown in Lemma 4. Two paths joining $\{s_1, s_2\}$ and $\{t_1, t_2\}$ such that $\{s_1, s_2\} \cap \{t_1, t_2\} = \emptyset$ are defined to be either s_1-t_1 and s_2-t_2 paths or s_1-t_2 and s_2-t_1 paths. Two paths P_1 and P_2 in a graph G are called *disjoint covering paths* if $V(P_1) \cap V(P_2) = \emptyset$ and $V(P_1) \cup V(P_2) = V(G)$, where $V(P_i)$ is the set of vertices in P_i.

Lemma 4. *For any four distinct vertices s_1, s_2, t_1, and t_2 in $G(8,4) \oplus G(8,4)$, there exists a vertex $z \notin \{s_1, s_2, t_1, t_2\}$ such that $G(8,4) \oplus G(8,4) \backslash z$ has two disjoint covering paths joining $\{s_1, s_2\}$ and $\{t_1, t_2\}$.*

Similar to Lemma 4, we can show that $G(8,4) \oplus G(8,4)$ has two disjoint covering paths joining every $\{s_1, s_2\}$ and $\{t_1, t_2\}$ with $\{s_1, s_2\} \cap \{t_1, t_2\} = \emptyset$.

Lemma 5. *Every 4-dimensional restricted HL-graph is 1-fault 5-panconnected.*

Lemma 6. *Every 5-dimensional restricted HL-graph is 2-fault 7-panconnected.*

By an inductive argument utilizing Theorem 1(a) and Lemmas 3, 5, and 6, we have Theorem 5.

Theorem 5. *Every m-dimensional restricted HL-graph, $m \geq 3$, is $m-3$-fault $2m-3$-panconnected.*

Corollary 2. *Every m-dimensional restricted HL-graph, $m \geq 3$, is $m-3$-fault hypohamiltonian-connected.*

A graph G is called *f-fault q-edge-pancyclic* if for any faulty set F with $|F| \leq f$, there exists a cycle of every length from q to $|V(G\backslash F)|$ that passes through an arbitrary fault-free edge. Of course, an f-fault q-panconnected graph is always f-fault $q+1$-edge-pancyclic. From Theorem 5, we have the following.

Theorem 6. *Every m-dimensional restricted HL-graph, $m \geq 3$, is $m-3$-fault $2m-2$-edge-pancyclic.*

3.2 Pancyclicity of restricted HL-graphs

To show that every m-dimensional restricted HL-graph is $m-2$-fault almost pancyclic, due to Lemmas 1 and 2, we assume that the faulty set F contains $m-2$ faulty vertices. The proofs of lemmas are omitted.

Lemma 7. *The 3-dimensional restricted HL-graph is 1-fault almost pancyclic.*

Lemma 8. *Every 4-dimensional restricted HL-graph is 2-fault almost pancyclic.*

From Theorem 3(a) and Lemmas 7 and 8, we have Theorem 7.

Theorem 7. *Every m-dimensional restricted HL-graph, $m \geq 3$, is $m-2$-fault almost pancyclic.*

Corollary 3. *(a) Twisted cube TQ_m, $m \geq 3$, is $m-2$-fault almost pancyclic[18].*
(b) Crossed cube CQ_m, $m \geq 3$, is $m-2$-fault almost pancyclic[17].
(c) Multiply twisted cube MQ_m, $m \geq 3$, is $m-2$-fault almost pancyclic.
(d) Both 0-Möbius cube and 1-Möbius cube of dimension m, $m \geq 3$, are $m-2$-fault almost pancyclic[8].
(e) The m-Mcube, $m \geq 3$, is $m-2$-fault almost pancyclic.
(f) Generalized twisted cube GQ_m, $m \geq 3$, is $m-2$-fault almost pancyclic.
(g) Locally twisted cube LTQ_m, $m \geq 3$, is $m-2$-fault almost pancyclic.
(h) $G(2^m, 4)$, m odd and $m \geq 3$, is $m-2$-fault almost pancyclic[12].

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