Fault Hamiltonicity of Two-Dimensional Torus Networks *

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Abstract. In this paper, we consider the hamiltonian properties of two-dimensional torus networks with the set F of faulty elements(vertices and/or edges). We will show that (1) if $m \geq 3$, $n \geq 3$, and n is odd, $m \times n$ torus $- F$ is hamiltonian-connected for any F with $|F| \leq 1$, (2) if both m and n with $m, n \geq 4$, are even, $m \times n$ torus $- F$ is bihamiltonian-connected for any F with $|F| \leq 1$, and (3) if $m \geq 3$, $n \geq 3$, and n is odd, $m \times n$ torus $- F$ has a hamiltonian cycle for any F with $|F| \leq 2$.

1 Introduction

Torus networks have recently received significant attention as interconnection networks for massively parallel computing. In this paper, we address some issues related to ring or linear array embedding in two-dimensional torus networks with faulty nodes(vertices) and/or links(edges). Ring embedding in the interconnection networks is closely related to a hamiltonian problem which is one of the well known problems in the graph theory. If a interconnection network has hamiltonian cycle or hamiltonian path, ring or linear array can be implemented in this network. Ring embedding in the interconnection networks with faulty nodes and links has been widely studied in [2, 5–7].

A graph G is called *k-vertex-fault hamiltonian-connected* if for each pair of vertices, s and t, there is a path from s to t which contains all the nonfaulty vertices when there are k or less faulty vertices, and a graph G is called *k-edge-fault hamiltonian-connected* if for each pair of vertices, s and t, there is a path from s to t which contains all vertices and contains only nonfaulty edges when there are k or less faulty edges. A graph G is called *k-fault hamiltonian-connected* if for each pair of vertices, s and t, there is a path from s to t which contains all nonfaulty vertices and contains only nonfaulty edges when there are k or less faulty vertices and/or edges.

A graph G is called *k-vertex-fault hamiltonian* if there is a cycle which contains all the nonfaulty vertices when there are k or less faulty vertices, and a graph G is called *k-edge-fault hamiltonian* if there is a cycle which contains all the vertices and contains only nonfaulty edges when there are k or less faulty edges. A graph G is called *k-fault hamiltonian* if there is a cycle which contains all the nonfaulty vertices and contains only nonfaulty edges when there

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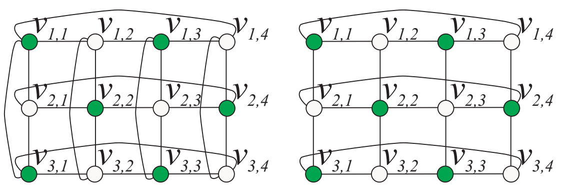
are k or less faulty vertices and/or edges. Note that if a graph G is k-fault hamiltonian, then $k \leq \delta(G) - 2$ where $\delta(G)$ is the minimum degree of vertices of G. If a graph G is k-fault hamiltonian-connected, then $k \leq \delta(G) - 3$. The bipartite graph G cannot be hamiltonian-connected because for bipartite sets X and Y of G, there is no hamiltonian path from s to t for $s, t \in Y$ such that $|X| \geq |Y|$. A bipartite graph G with bipartite sets X and Y ($|X| \geq |Y|$), is called *bihamiltonian-connected* if one of the following conditions are satisfied:
(1) if $|X| = |Y|$, there is a hamiltonian path from s to t for all $s \in X$ and $t \in Y$.
(2) if $|X| = |Y| + 1$, there is a hamiltonian path from s to t for all $s, t \in X$.
A bipartite graph G is *k-fault bihamiltonian-connected* if for any faulty set F of vertices and/or edges such that $|F| \leq k$, $G - F$ is bihamiltonian-connected.

In this paper, we will investigate the above hamiltonian properties in two-dimensional $m \times n$ torus with faults of vertices and/or edges. Note that two-dimensional $m \times n$ torus with $m \geq 3$ and $n \geq 3$ is 4-regular. If both m and n are even, then the torus is bipartite. Therefore it is not k-fault hamiltonian-connected for any k. In this case, we consider k-fault bihamiltonian-connectedness. In section 3 and 4, it will be shown that $m \times n$ torus with $m \geq 3$ and odd $n \geq 3$ is 1-fault hamiltonian-connected, and that $m \times n$ torus with even m and n, $m, n \geq 4$, is 1-fault bihamiltonian-connected. In section 5, $m \times n$ torus with $m \geq 3$ and odd $n \geq 3$ is 2-fault hamiltonian. Finally, concluding remarks are given in Section 6.

2 Notations and Terminologies

For a positive integer n, let $[n]= \{i|1 \leq i \leq n\}$. Let $+_n$ denote cyclic addition and $-_n$ denote cyclic subtraction. That is, for $i, j \in [n]$, $i +_n j = k$ where $k = i + j$ if $i + j \leq n$; otherwise, $k = (i + j) - n$. $i -_n j = k$ where $k = i - j$ if $i - j > 0$; otherwise, $k = (i - j) + n$. Two-dimensional $m \times n$ torus denoted by $Torus(m, n)$ is a network consisting of mn vertices, each vertex is identified by $v_{i,j}$ where $1 \leq i \leq m$ and $1 \leq j \leq n$. Two vertices, $v_{i,j}$ and $v_{i',j'}$, are adjacent if either $i' = i$ and $j' = j \pm_n 1$ or $j' = j$ and $i' = i \pm_m 1$. For i with $1 \leq i \leq m$, an edge $(v_{i,1}, v_{i,n})$ is called *row-wraparound edge*, for j with $1 \leq j \leq n$, an edge $(v_{1,j}, v_{m,j})$ is called *column-wraparound edge*. Figure 1 (a) shows $Torus(3, 4)$. If $m = 2$ or $n = 2$, there are multiple wraparound edges, therefore it is assumed that in $Torus(m, n)$, $m \geq 3$ and $n \geq 3$. Note that $Torus(m, n)$ is a graph of degree 4.



(a) $Torus(3, 4)$ (b) *Row-*$Torus(3, 4)$

Figure 1: $Torus(3, 4)$ and *Row-*$Torus(3, 4)$

The resulting graph when all wraparound edges are deleted in $Torus(m, n)$ is called grid, which is denoted by $Grid(m, n)$. The resulting graph when all column-wraparound edges in $Torus(m, n)$ are deleted is denoted by *Row-*$Torus(m, n)$. Note that $Torus(m, n)$ is a product of a cycle of length m and a cycle of length n, and $Grid(m, n)$ is a product of a path of length m and a path of length n, and *Row-*$Torus(m, n)$ is a product of a path of length m and a cycle of length n.

An edge $(v_{i,j}, v_{i',j'})$ is called *row-edge* if $i = i'$. An edge $(v_{i,j}, v_{i',j'})$ is called *column-edge* if $j = j'$. $Row(i : j)$ is a subgraph induced by $\{v_{k,l} | i \leq k \leq j, 1 \leq l \leq n\}$. $Col(i : j)$ is a subgraph induced by $\{v_{k,l} | 1 \leq k \leq m, i \leq l \leq j\}$. $Row(i : i)$ and $Col(j : j)$ are denoted by $Row(i)$ and $Col(j)$, respectively. $Grid(i_1 : i_2, j_1 : j_2)$ is the resulting graph such that all wraparound edges are deleted in the subgraph induced by $\{v_{k,l} | i_1 \leq k \leq i_2, j_1 \leq l \leq j_2\}$.

Note that $Torus(m, n)$ is vertex-symmetric but not edge-symmetric. In Figure 1 (a), there is a cycle of length 3 which contains the edge $(v_{1,2}, v_{2,2})$, but there is no cycle of length 3 which contains the edge $(v_{1,2}, v_{1,3})$. However, any two row-edges(or column-edges) are similar, i.e. for any two row-edges(or column-edges), (u, v) and (u', v'), there is an automorphism g such that $g(u) = u'$ and $g(v) = v'$.

If one of m and n is odd, then $Torus(m, n)$ is not bipartite, and if both m and n are even, $Torus(m, n)$ is bipartite. Let $v_{i,j}$ be a vertex of $Torus(m, n)$ or $Row\text{-}Torus(m, n)$ or $Grid(m, n)$. A vertex $v_{i,j}$ is called *black* if $i + j$ is even; otherwise, $v_{i,j}$ is called *white*. If n is odd, every row-wraparound edge joins two vertices with the same color. Graph theoretic terms not defined here can be found in [1].

The following lemmas are useful in showing the hamiltonian properties of Torus with faults.

Lemma 1 *[5] $Grid(m, n)$ with $m, n \geq 4$ is bihamiltonian-connected.*

Lemma 2 *[3] In $Grid(m, n)$, $m \geq 2$ and $n \geq 2$, there is a hamiltonian path from a corner vertex s, a vertex of degree 2, to every other vertex t which has a different color from s if mn is even.*

3 1-Vertex-Fault Hamiltonian-Connectedness of Torus

In this section, we will investigate vertex-fault hamiltonian properties of two-dimensional torus with 1 or less faulty vertex. To begin with, we give hamiltonian-connectedness of $Row\text{-}Torus(m, n)$ which is crucial to prove the hamiltonian properties of torus with faults.

Lemma 3 *$Row\text{-}Torus(2, 2n + 1)$ with $n \geq 1$ is hamiltonian-connected. Futhermore, for every pair of vertices s, t, there exists a hamiltonian path from s to t which contains the edge $(v_{1,2n+1}, v_{1,2n})$ or the edge $(v_{1,2n+1}, v_{1,1})$.*

. *Proof* We will show that for any two vertices s, t, there is a hamiltonian path from s to t. Without loss of generality, s is assumed to be $v_{1,2n+1}$. The following cases will be shown based on Lemma 2.

Case 1 $t = v_{2,2n+1}$.
There is a hamiltonian path P' from $v_{1,2n}$ to $v_{2,2n}$ in $Col(1 : 2n)$. Then the path $s = v_{1,2n+1}$, $v_{1,2n}$, P', $v_{2,2n}$, $t = v_{2,2n+1}$ becomes a hamiltonian path.

Case 2 $t = v_{1,2n}$.
There is a hamiltonian path P' from $v_{1,1}$ to $v_{2,1}$ in $Col(1 : 2n - 1)$. Let P be the path $s = v_{1,2n+1}$, $v_{1,1}$, P', $v_{2,1}$, $v_{2,2n+1}$, $v_{2,2n}$, $t = v_{1,2n}$. Then P becomes a hamiltonian path.

Case 3 $t = v_{2,2n}$.
There is a hamiltonian path P' from $v_{1,2n-1}$ to $v_{2,1}$ in $Col(1 : 2n - 1)$. Let P be the path $s = v_{1,2n+1}$, $v_{1,2n}$, $v_{1,2n-1}$, P', $v_{2,1}$, $v_{2,2n+1}$, $t = v_{2,2n}$. Then P becomes a hamiltonian path.

Case 4 t is in $Col(1 : 2n - 1)$. Note that s is black.

Case 4-1 t is black.
Since $v_{2,1}$ is white, there is a hamiltonian path P' from $v_{2,1}$ to t in $Col(1 : 2n - 1)$. Then the path $s = v_{1,2n+1}$, $v_{1,2n}$, $v_{2,2n}$, $v_{2,2n+1}$, $v_{2,1}$, P', t becomes a hamiltonian path.

Case 4-2 t is white and $t \neq v_{2,1}$.
There is a hamiltonian path P' from $v_{2,2n}$ to t in $Col(2:2n)$ since $v_{2,2n}$ is black. The path $s = v_{1,2n+1}$, $v_{1,1}$, $v_{2,1}$, $v_{2,2n+1}$, $v_{2,2n}$, P', t becomes a hamiltonian path.

Case 4-3 $t = v_{2,1}$ (t is white).
There is a hamiltonian path P' from $v_{1,2}$ to $v_{2,2n}$ in $Col(2:2n)$ since $v_{1,2}$ is white and $v_{2,2n}$ is black. The path $s = v_{1,2n+1}$, $v_{1,1}$, $v_{1,2}$, P', $v_{2,2n}$, $v_{2,2n+1}$, $t = v_{2,1}$ becomes a hamiltonian path.

Every hamiltonian path from s to t constructed in the above contains $(v_{1,2n+1}, v_{1,2n})$ or $(v_{1,2n+1}, v_{1,1})$. If neither s nor t is equal to $v_{1,2n+1}$, the hamiltonian path from s to t must contain $(v_{1,2n+1}, v_{1,2n})$ or $(v_{1,2n+1}, v_{1,1})$ because the degree of $v_{1,2n+1}$ is 3. □

Next we will extend the above lemma to general Row-Torus.

Theorem 1 *Row-Torus*$(m, 2n+1)$ *with* $m \geq 2$ *and* $n \geq 1$ *is hamiltonian-connected. And for two vertices* s *and* t*, there exists a hamiltonian path from* s *to* t *which contains* $(v_{1,2n+1}, v_{1,2n})$ *or* $(v_{1,2n+1}, v_{1,1})$.

Proof We will prove the theorem by induction on m. When $m = 2$, the theorem is true by Lemma 3. Assume that the theorem is true for *Row-Torus*$(m-1, 2n+1)$, $m > 1$. We will show that the theorem holds for *Row-Torus*$(m, 2n+1)$. Let X be the set of vertices of $Row(1:m-1)$ and Y be the set of vertices of $Row(m)$. Now it will be shown that for any two vertices, s and t, there exists a hamiltonian path from s to t in $Torus(m, 2n+1)$.

Case 1 Both s and t are in X.
Since $Row(1:m-1)$ is hamiltonian connected by induction hypothesis, there is a hamiltonian path P from s to t in $Row(1:m-1)$ such that it contains $(v_{1,2n+1}, v_{1,2n})$ or $(v_{1,2n+1}, v_{1,1})$. The path P must contain some row-edge $e_1 = (u, v)$ of $Row(m-1)$. Let C be a hamiltonian cycle in $Row(m)$. Note that C consists of all row-edges of $Row(m)$. In C, there is an edge $e_2 = (u', v')$ such u, u', v', v, u forms a cycle of length 4. We merge P and C as follows: $P - e_1 + (u, u') + C - e_2 + (v', v)$. Then it becomes a hamiltonian path from s to t in $Torus(m, 2n+1)$.

Case 2 Both s and t are in Y.
Let C be a hamiltonian cycle in $Row(m)$. From C, we can construct two vertex-disjoint paths P_1, P_2 such that P_1 and P_2 begin at s and t, respectively, and $Y = V(P_1) \cup V(P_2)$ where $V(P_1)$ and $V(P_2)$ are the set of the vertices of P_1 and P_2, respectively. Let s' and t' be the ending vertices of P_1 and P2, respectively, and let s'' and t'' be the vertices of X which are adjacent to s' and t', respectively. Then there exists a hamiltonian path P_3 in $Row(1:m-1)$ from s'' to t'' which contains $(v_{1,2n+1}, v_{1,2n})$ or $(v_{1,2n+1}, v_{1,1})$. Then $s, P_1, s', s'', P_3, t'', t', P_2, t$ forms a hamiltonian path which satisfies the condition of the theorem.

Case 3 s is in Y and t is in X.
There are two hamiltonian paths beginning at s in $Row(m)$. Between these paths, let P_1 be the path whose ending vertex is not adjacent to t. Let s' be the ending vertex of P_1, and t' be the vertex of X adjacent to s', and P_2 be a hamiltonian path from t' to t in $Row(1:m-1)$. Then s, P_1, s', t', P_2, t forms a hamiltonian path which satisfies the condition of the theorem. □

Now we are ready to show the vertex-fault hamiltonian-connectedness of Torus.

Lemma 4 $Torus(m, 2n+1)$ *with* $m \geq 4$ *and* $n \geq 1$ *is* 1*-vertex-fault hamiltonian-connected.*

Proof If there is no faulty vertex, $Torus(m, 2n+1)$ is hamiltonian-connected by Theorem 1. Now it is assumed that the faulty vertex is $v_{1,2n+1}$ without loss of generality since the torus is vertex-symmetric. Let X be the set of vertices of $Row(1:2) - v_{1,2n+1}$, and Y be the set of vertices of $Row(3:m)$. Let $C = v_{2,1}$, $v_{1,1}$, $v_{1,2}$, $v_{2,2}$, $v_{2,3}$, $v_{1,3}$, $v_{1,4}$, $v_{2,4}$, $\cdots$, $v_{1,2n}$, $v_{2,2n}$, $v_{2,2n+1}$, $v_{2,1}$ (see Figure 2). C is a cycle which contains all vertices of X.

Case 1 Both s and t are in X.
From C, we can construct two vertex-disjoint paths P_1, P_2 such that P_1 and P_2 begin at s and

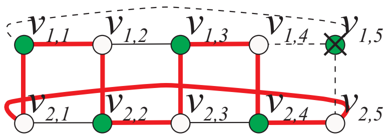


Figure 2: The cycle C which contains X

t, respectively and $X = V(P_1) \cup V(P_2)$ where $V(P_1)$ and $V(P_2)$ are the set of the vertices of P_1 and P_2, respectively. Let s' and t' be the ending vertices of P_1 and P_2, respectively. Let s'' and t'' be the vertices of Y adjacent to s' and t', respectively. By Theorem 1, there is a hamiltonian path P_3 from s'' to t'' in $Row(3:m)$. As a consequence, $s, P_1, s', s'', P_3, t'', t', P_2, t$ forms a hamiltonian path from s to t in $Torus(m, 2n+1) - v_{1,2n+1}$.

Case 2 Both s and t are in Y.

By Theorem 1, there is a hamiltonian path P from s to t in $Row(3:m)$ which contains $e_1 = (v_{3,2n+1}, v_{3,2n})$ or $e_2 = (v_{3,2n+1}, v_{3,1})$. Note that C contains both $e'_1 = (v_{2,2n+1}, v_{2,2n})$ and the edge $e'_2 = (v_{2,2n+1}, v_{2,1})$. If P contains e_1, then $P - e_1 + (v_{3,2n+1}, v_{2,2n+1}) + C - e'_1 + (v_{2,2n}, v_{3,2n})$ forms a hamiltonian path from s to t in $Torus(m, 2n+1) - v_{1,2n+1}$. In case that P contains e_2, a hamiltonian path from s to t in $Torus(m, 2n+1) - v_{1,2n+1}$ can be similarly constructed.

Case 3 s is in X and t is in Y.

From C, there are two hamiltonian paths from s in $Row(1:2) - v_{1,2n+1}$. Between these paths, let P_1 the path whose ending vertex is not adjacent to t. Let s' be the ending vertex of P_1, and t' be the vertex of Y adjacent to s'. By Theorem 1, there exists a hamiltonian path P_2 from t' to t in $Row(3:m)$. Consequently, s, P_1, s', t', P_2, t forms a hamiltonian path from s to t in $Torus(m, 2n+1) - v_{1,2n+1}$. □

Theorem 2 *$Torus(m, 2n+1)$ with $m \geq 3$ and $n \geq 1$ is 1-vertex-fault hamiltonian-connected.*

Proof If $m > 3$ and $n \geq 1$, the theorem holds by Lemma 4. If $m = 3$ and $n > 1$, the theorem also holds by Lemma 4 when we interchange rows and columns. If $m = 3$ and $n = 1$, it can be easily seen that the theorem holds. □

If the number of rows and that of columns are both even, torus is bipartite. Therefore, $Torus(m, n)$ with even m and even n is not 1-vertex-fault hamiltonian-connected. However, it is 1-vertex-fault bihamiltonian connected, which will be shown in the following. We begin with the following Lemma.

Lemma 5 *Row-$Torus(2, 2n)$, $n \geq 1$ is 1-vertex-fault bihamiltonian-connected.*

Proof If there is no faulty vertex, Row-$Torus(2, 2n)$ is bihamiltonian-connected by Lemma 2. Without loss of generality, it is assumed that the faulty vertex is $v_{1,2n}$ which is white. Since the number of nonfaulty black vertices is greater than the number of nonfaulty white vertices by 1, we will show that for any two black vertices s and t, there is a hamiltonian path from s to t in $Row - Torus(2, 2n) - v_{1,2n}$. Without loss of generality, s is on the left of t. If t is $v_{2,2n}$, a hamiltonian path from s to t in Row-$Torus(2, 2n) - v_{1,2n}$ can be constructed as follow: s, P, $v_{2,2n-1}$, t, where P is hamiltonian path from s to $v_{2,2n-1}$ in $Col(1 : 2n-1)$. Note that $v_{2,2n-1}$ is white. If t is not $v_{2,2n}$, let k be the column-position of s. A hamiltonian path from s to t in Row-$Torus(2, 2n) - v_{1,2n}$ can be constructed as follows: s, P_1, $v_{2,1}$, $v_{2,2n}$, $v_{2,2n-1}$, P_2, t, where P_1 is a hamiltonian path from s to $v_{2,1}$ in $Col(1 : k)$ and P_2 is a hamiltonian path from $v_{2,2n-1}$ to t in $Col(k+1, 2n-1)$. Note that $v_{2,1}$ is white. □

Theorem 3 *$Torus(2m, 2n)$ with $m, n \geq 2$ is 1-vertex-fault bihamiltonian-connected.*

Proof If there is no faulty vertex, $Torus(2m, 2n)$ is bihamiltonian-connected by Lemma 1. Without loss of generality, it is assumed that the faulty vertex is $v_{1,2n}$ which is white. Let X be the set of nonfaulty vertices of $Row(1:2)$, and Y be the set of vertices in $Row(3:2m)$. Let s and t be any two black vertices.

Case 1 Both s and t is in X.
By Lemma 5, there is a hamiltonian path P_1 from s to t in $Row(1:2) - v_{1,2n}$. P_1 must contain one edge in $\{(v_{2,1}, v_{2,2}), (v_{2,1}, v_{2,2n})\}$. Let $e = (s', t')$ be such an edge. Note that the color of s' and that of t' are different. Let s'' and t'' be the vertices of Y adjacent to s' and t', respectively. By Lemma 2 there is a hamiltonian path P_2 from s'' to t'' in $Row(3:2m)$ since s'' or t'' is a corner vertex of $Row(3:2m)$. Replace the edge e in P_1 as the path s', s'', P_2, t'', t', then the resulting path becomes a hamiltonian path in $Row(2m:2n) - v_{1,2n}$.

Case 2 s is in X and t is in Y.
Let $s'(\neq s)$ be a vertex in $\{v_{1,1}, v_{2,2n}\}$. Note that s' is black. By Lemma 5, there is a hamiltonian path P_1 from s to s' in $Row(1:2) - v_{1,2n}$. Let t' be the vertex in Y adjacent to s'. Note that t' is white and corner vertex of $Row(3:2m)$. There is a hamiltonian path P_2 from t' to t in $Row(3:2m)$. Then the path s, P_1, s', t', P_2, t forms a hamiltonian path from s to t in $Row(2m:2n) - v_{1,2n}$.

Case 3 Both s and t are in Y.
If s and t are not on the same row, the row-position of s is assumed to be less than that of t. Let r_s and r_t be the row-position of s and t, respectively. $Row(3:2m)$ is divided into two parts, $Row(3:r_s)$ and $Row(r_s+1:2m)$. If $r_s = 3$, let s' be the vertex in $Row(3)$ adjacent to s; otherwise, let s' be $v_{3,2n}$. And if $r_t = 2m$, let t' the vertex in $Row(2m)$ adjacent to t; otherwise, let t' be $v_{2m,1}$. Note that both s' and t' are white, and $s \neq s'$, and $t \neq t'$. Then there is a hamiltonian path P_1 from s to s' in $Row(3:r_s)$ and there is a hamiltonian path P_2 from t to t' in $Row(r_s+1:2m)$. Let s'' and t'' be the vertices of X adjacent to s' and t', respectively. By Lemma 5, there is a hamiltonian path P_3 from s'' to t'' in $Row(1:2) - v_{1,2n}$. Consequently, s, P_1, s', s'', P_3, t'', t', P_2, t forms a hamiltonian path from s to t in $Row(2m:2n) - v_{1,2n}$.

If s and t are on the same row, s is assumed to be on the left of t. Let c_s and c_t be the column-position of s and t, respectively. $Row(3:2m)$ can be divided into two grids, $Grid(3:2m, 1:k)$, $Grid(3:2m, k+1:2n)$ such that if r_s is 1, then $k = 2$; otherwise $k = c_s$. Note that the number of columns of $Grid(3:2m, 1:k)$ and that of $Grid(3:2m, k+1:2n)$ are both greater than 1. Let s' and t' be the white corner verticse of $Grid(3:2m, 1:k)$ and $Grid(3:2m, k+1:2n)$, respectively. By Lemma 2, there is a hamiltonian path P_1 from s to s' in $Grid(3:2m, 1:k)$ and there is a hamiltonian path P_2 from t to t' in $Grid(3:2m, k+1:2n)$. Let s'' and t'' be the vertices in X which are adjacent to s' and t', respectively. Since both s'' and t'' are black, there is a hamiltonian path P_3 from s'' to t'' in $Row(1:2) - v_{1,2n}$ by Lemma 5. Consequently, s, P_1, s', s'', P_3, t'', t', P_2, t forms a hamiltonian path from s to t in $Row(2m:2n) - v_{1,2n}$. $\square$

4 1-Edge-Fault Hamiltonian-Connectedness of Torus

In this section, we will consider the hamiltonian-connectedness when there is 1 or less faulty edge in Torus.

Lemma 6 *$Torus(m, 2n+1)$ with $m \geq 4$ and $n \geq 1$ is 1-edge-fault hamiltonian-connected.*

Proof If there is no faulty edge, $Torus(m, 2n+1)$ is 1-edge-fault hamiltonian-connected by Theorem 1. Assume there is one faulty edge. The faulty edge is row-edge or column-edge. To prove the lemma, we consider both the case of faulty row-edge and the case of faulty column-edge together. Let e_f and e'_f be the faulty row-edge and the faulty column-edge, respectively.

Since all row-edges are similar and all column-edges are similar in torus, let $e_f = (v_{1,1}, v_{1,2})$, and let $e'_f = (v_{1,2n+1}, v_{2,2n+1})$. Let X be the set of vertices of $Row(1:2)$ and Y be the set of vertices of $Row(3:m)$. Let $C = v_{1,1}, v_{1,2n+1}, v_{1,2n}, v_{1,2n-1}, \cdots, v_{1,2}, v_{2,2}, v_{2,3}, \cdots, v_{2,2n}, v_{2,2n+1}, v_{2,1}, v_{1,1}$, which is a hamiltonian cycle in $Row(1:2) - e_f - e'_f$ (See Figure 3).

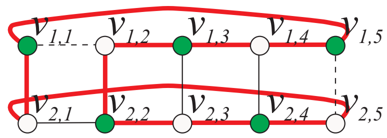


Figure 3: The cycle C which contains X

Case 1 Both s and t are in X.
From C, we can obtain two vertex-disjoint paths P_1 and P_2 such that P_1 and P_2 begins at s and t, respectively and $X = V(P_1) \cup V(P_2)$ where $V(P_1)$ and $V(P_2)$ are the set of the vertices of P_1 and P_2, respectively. Let s' and t' be the ending vertices of P_1 and P_2, respectively. Let s'' and t'' be the vertices in Y adjacent to s' and t', respectively. Let P_3 be a hamiltonian path from s'' to t'' in $Row(3:m)$ which is hamiltonian connected. Then s, P_1, s', s'', P_3, t'', t', P_2, t becomes a hamiltonian path from s to t in $Torus(m, 2n+1) - e_f - e'_f$.

Case 2 Both s and t are in Y.
By Theorem 1, there is a hamiltonian path P from s to t in $Row(3:m)$ which contains the row-edge $(v_{3,2n+1}, v_{3,2n})$ or the row-edge $(v_{3,2n+1}, v_{3,1})$. Since C contains $(v_{2,2n+1}, v_{2,2n})$ and $(v_{2,2n+1}, v_{2,1})$, we can merge C and P and obtain a hamiltonian path from s to t in $Torus(m, 2n+1) - e_f - e'_f$.

Case 3 s is in X and t is in Y.
From C, we can obtain two hamiltonian paths beginning at s in $Row(1:2) - e_f - e'_f$. Between these paths, let P_1 be the path whose ending vertex is not adjacent to t. Let s' be the ending vertex of P_1, and let t' be the vertex in Y adjacent to s', and let P_2 be a hamiltonian path from t' to t in $Row(3:m)$. Then s, P_1, s', t', P_2, t forms a hamiltonian path in $Torus(m, 2n+1) - e_f - e'_f$. □

Theorem 4 *$Torus(m, 2n+1)$ with $m \geq 3$ and $n \geq 1$ is 1-edge-fault hamiltonian-connected.*

Proof In case of $m = 3$ and $n = 3$, it can be easily shown that the theorem holds. In all the other cases, it follows from Lemma 6. □

$Torus(m, n)$ with even m and even n is bipartite. Therefore it is not 1-edge-fault hamiltonian-connected. However, it is 1-edge-fault bihamiltonian-connected, which will be shown in the following.

Theorem 5 *$Torus(2m, 2n)$ with $m, n \geq 2$ is 1-egde-fault bihamiltonian-connected.*

Proof $Torus(2m, 2n)$ with 1 or less faulty edge has a spanning subgraph isomorphic to $Grid(2m, 2n)$, which is bihamiltonian-connected by Lemma 1. □

5 Hamiltonian Cycle in Torus with 2 or less Faulty Elements

In this section, we investigate whether there is a hamiltonian cycle in Torus when there are 2 or less faulty vertices and/or edges. Note that $Torus(2m, 2n)$ with $m, n \geq 2$ is not 2-fault hamiltonian because it is bipartite. Therefore we consider $Torus(m, 2n+1)$ with $m \geq 3$ and $n \geq 1$. What $Torus(m, 2n+1)$ with $m \geq 3$ and $n \geq 1$ is 2-fault hamiltonian are shown by dividing into three cases: vertex faults, edge faults, and one vertex fault and one edge fault. The details of the proof are omitted.

Lemma 7 *$Torus(m, 2n+1)$ with $m \geq 4$ and $n \geq 1$ is 2-vertex fault hamiltonian.*

Lemma 8 *$Torus(m, 2n+1)$ with $m \geq 4$ and $n \geq 1$ is 2-edge fault hamiltonian.*

Lemma 9 *$Torus(m, 2n+1)$ with $m \geq 4$ and $n \geq 1$ has a hamiltonian cycle when there are one faulty vertex and one faulty edge.*

The following theorem follows from the above three lemmas.

Theorem 6 *$Torus(m, 2n+1)$ with $m \geq 3$ and $n \geq 1$ is 2-fault hamiltonian.*

Proof If $m = 3$ and $n = 1$, it can be easily shown that the theorem holds. In all the other cases, it follows from Lemma 7, Lemma 8, and Lemma 9 that the theorem holds. $\square$

6 Concluding Remarks

In this paper, we suggested hamiltonian properties of two-dimensional torus with faulty vertices and/or edges. We showed that $m \times n$ torus with even $m \geq 3$ and odd $n \geq 3$, is 1-fault hamiltonian-connected. We also showed that $m \times n$ torus with even $m \geq 4$ and even $n \geq 4$ which is bipartite, is 1-fault bihamiltonian-connected. Finally it was shown that $m \times n$ torus with even $m \geq 3$ and odd $n \geq 3$, has a cycle which contains all the fault-free vertices and consists of only fault-free edges when there are 2 or less faulty vertices and/or edges. Fault-hamiltonian properties of k-dimensional torus should be further studied.

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