An Approach to Conditional Diagnosability Analysis under the PMC Model and its Application to Torus Networks

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Abstract

A general technique is proposed for determining the conditional diagnosability of interconnection networks under the PMC model. Several graph invariants are involved in the approach, such as the length of the shortest cycle, the minimum number of neighbors, γ_p (resp. γ'_p), over all p-vertex subsets (resp. cycles), and a variant of connectivity, called the *r-super-connectivity*. An n-dimensional torus network is defined as a Cartesian product of n cycles, $C_{k_1} \times \cdots \times C_{k_n}$, where C_{k_j} is a cycle of length k_j for $1 \le j \le n$. The proposed technique is applied to the two or higher-dimensional torus networks, and their conditional diagnosabilities are established completely: the conditional diagnosability of every torus network G is equal to $\gamma'_4(G) + 1$, excluding the three small ones $C_3 \times C_3$, $C_3 \times C_4$, and $C_4 \times C_4$. In addition, $\gamma_p(G)$ as well as $\gamma'_4(G)$ is derived for $2 \le p \le 4$ and the r-super-connectivity is also derived for $1 \le r \le 3$.

Keywords: Fault diagnosis, PMC model, conditional diagnosability, torus network, minimum neighborhood, r-super-connectivity.

1. Introduction

In a multiprocessor system, the probability that failure occurs increases as the number of processors increases. A system consisting of a large number of processors is required to continue operating even if failure occurs in the processors. Fault tolerance is an essential feature of such systems due to the catastrophic consequences of not tolerating faults. One of the major issues in fault tolerance of a multiprocessor system is fault diagnosis, which is to identify the faulty processors in the system. Several models for self-diagnosis of a system have been proposed [7, 17, 18].

Preparata *et al.* [18] introduced a model, the so-called *PMC model*, for system-level diagnosis in multiprocessor systems. In the PMC model, a system consists of processors, and only processors with a direct link are allowed to test each other. When processor u tests processor v, u evaluates v as fault-free or faulty. The test result is reliable only if the testing processor is fault-free. A system is *t-diagnosable* if from the test results, all the faulty processors can always be identified provided the number of faulty processors does not exceed t [18]. The *diagnosability* of a system is the maximum value of t such that the system is t-diagnosable. The t-diagnosable system was characterized by Hakimi and Amin [9], and a polynomial-time algorithm for finding the diagnosability of a system was designed by Sullivan [20].

In the event of a random processor failure, it is very unlikely that all of the processors adjacent to a single processor fail simultaneously. Motivated by this, Lai *et al.* [15] introduced a new diagnosability measure, called conditional diagnosability. A system is *conditionally t-diagnosable* if from the test results, all faulty processors can always be identified provided the number of faulty processors does not exceed t and also, all adjacent processors to each processor are not faulty at the same time. The *conditional diagnosability* of a system is the maximum value of t such that

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the system is conditionally t-diagnosable. The conditional diagnosability under the PMC model has been studied for several interconnection networks, such as hypercubes [15], k-ary n-cubes [5], matching composition networks [22], BC networks [27], augmented cubes [4], balanced hypercubes [23], folded hypercubes [28], and alternating group graphs [10].

An alternative model for fault diagnosis in multiprocessor systems, the so-called *MM model*, was introduced by Maeng and Malek [17]. This model is a comparison-based model in which a processor sends the same task to each pair of its neighbors. Upon the receipt of the two responses, the processor compares them and proclaims that the two neighbors are both fault-free or at least one of them is faulty. Conditional diagnosability for interconnection networks under the MM model has been investigated in various studies [6, 11, 12, 13, 14, 16, 19, 25, 26].

An interconnection network of a multiprocessor system is represented as a graph, where vertices correspond to processors and edges correspond to communication links. In this paper, we suggest an approach for determining the conditional diagnosability of an interconnection network under the PMC model. The approach involves a few graph invariants, including the length of a shortest cycle, called the *girth*, the minimum number of neighbors, denoted by γ_p (resp. γ'_p), over all p-vertex subsets (resp. cycles) for some p, and a variant of connectivity, called the *r-super-connectivity*, for some integer r. The proposed technique is applied to two or higher-dimensional torus networks to determine their conditional diagnosabilities.

A torus network is one of the most popular interconnection networks. An n-dimensional torus, denoted by $T(k_1, k_2, \ldots, k_n)$, is defined as a Cartesian product of n cycles, $C_{k_1} \times \cdots \times C_{k_n}$, where C_{k_j} is a cycle of length k_j for every $1 \leq j \leq n$. The k-ary n-cube [3, 5, 8] is a special type of an n-dimensional torus where $k_j = k$ for every $1 \leq j \leq n$. For more discussion on torus networks, refer to [1, 21, 24]. For all n-dimensional torus networks where $n \geq 2$, their conditional diagnosabilities are established completely, and the aforementioned graph invariants are determined: γ_p for $2 \leq p \leq 4$, γ'_4, and the r-super-connectivity for $1 \leq r \leq 3$. The conditional diagnosability of $T(k_1, \ldots, k_n)$, where $n \geq 2$ and $3 \leq k_1 \leq k_2 \leq \cdots \leq k_n$, is

$$\begin{cases} 4 & \text{if } (n, k_1, k_2) = (2, 3, 3), \\ 5 & \text{if } (n, k_1, k_2) = (2, 3, 4), \\ 7 & \text{if } (n, k_1, k_2) = (2, 4, 4), \\ 8n - 7 & \text{if } k_1 \geq 4, (n, k_1, k_2) \neq (2, 4, 4), \\ 8n - 9 & \text{if } k_1 = 3 \ \&\ k_2 \geq 4, (n, k_1, k_2) \neq (2, 3, 4), \\ 8n - 11 & \text{if } k_1 = k_2 = 3, (n, k_1, k_2) \neq (2, 3, 3). \end{cases}$$

The organization of this paper is as follows. In the next section, definitions and notation are given. In Section 3, a general approach is addressed for determining the conditional diagnosability under the PMC model. In Section 4, the graph invariants of torus networks are investigated and then their conditional diagnosabilities are established. Finally, the paper is concluded in Section 5.

2. Preliminaries

Let G be a graph, where $V(G)$ and $E(G)$ represent the vertex set and the edge set of G, respectively. If $(u, v) \in E(G)$, u is adjacent to v or u is a neighbor of v. The *degree* of a vertex is the number of vertices adjacent to it. A *path* between v_1 and v_k is a sequence of vertices, $(v_1, v_2, \ldots, v_k)$, such that $(v_j, v_{j+1}) \in E(G)$ for every $1 \leq j < k$. The length of this path is $k - 1$. A *cycle* is a closed path $(v_1, v_2, \ldots, v_k)$ such that $k \geq 3$ and $(v_k, v_1) \in E(G)$. The length of this cycle is k. The *connected component* of G is a maximal connected subgraph of G. The size of a connected component is the number of vertices in it. The *connectivity* of G, $\kappa(G)$, is the minimum number of vertices whose removal results in a trivial graph or a disconnected graph.

For a vertex subset $S \subseteq V(G)$, the *subgraph of G induced by S*, denoted by $G\langle S \rangle$, is a graph whose vertex set is S and for every pair of vertices $u, v \in S$, (u, v) is an edge of the graph $G\langle S \rangle$ if and only if $(u, v) \in E(G)$. For a vertex subset $S \subseteq V(G)$, we denote by $G \setminus S$ the resultant subgraph obtained from G by deleting all the vertices of S (including the edges incident to them). Note that $G \setminus S$ is the subgraph of G induced by $V(G) \setminus S$. The *neighborhood* of a vertex v, denoted by $N_G(v)$, is $\{u \in V(G) : (u, v) \in E(G)\}$. The neighborhood of a vertex subset S, denoted by $N_G(S)$, is $\bigcup_{v \in S} N_G(v) \setminus S$. A vertex subset S is a *conditional set* if $N_G(v) \not\subseteq S$ for every $v \in V(G)$. Graph theoretic terms not defined here can be found in [2].

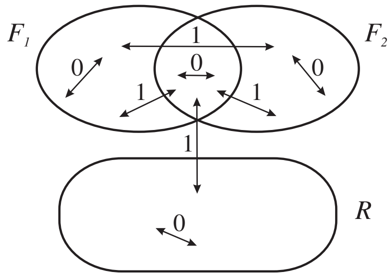


Figure 1: F_1 and F_2 are consistent with this syndrome.

Definition 1 (Minimum-neighborhood set). *A p-vertex subset, T, of G is said to be a* minimum-neighborhood set of order p *if* $|N_G(T)| = \gamma_p(G)$, *where* $\gamma_p(G)$ *is the minimum cardinality of neighborhoods over all p-vertex subsets of G, i.e.,* $\gamma_p(G) = \min\{|N_G(S)| : S \subseteq V(G), |S| = p\}$.

Definition 2 (Minimum-neighborhood cycle). *A cycle of length p, C, of G is said to be a* minimum-neighborhood cycle of order p *if* $|N_G(V(C))| = \gamma'_p(G)$, *where* $V(C)$ *denotes the vertex set of C and* $\gamma'_p(G) = \min\{|N_G(S)| : S \subseteq V(G), |S| = p$, *and* $G\langle S\rangle$ *contains a cycle of length* $p\}$.

$\gamma'_p(G)$ is left undefined if G contains no cycle of length p. Obviously, $\gamma'_p(G) \geq \gamma_p(G)$ if both $\gamma'_p(G)$ and $\gamma_p(G)$ are well-defined.

Definition 3 (r-Super-connectivity). *For a nonnegative integer r, the* r-super-connectivity *of a graph G, denoted by* $\kappa_s^r(G)$, *is defined as the minimum number of vertices whose removal results in a trivial graph or a disconnected graph composed of one large connected component and the remaining connected components with more than r vertices in total.*

Clearly, $\kappa_s^r(G) \leq |V(G)| - 1$. The r-super-connectivity is a generalization of the ordinary connectivity in that $\kappa_s^0(G)$ is nothing but $\kappa(G)$.

In the PMC model, each processor has a capability of testing adjacent processors. Let a graph G represent the interconnection network of a multiprocessor system. It is assumed that for every edge $(u, v) \in E(G)$, u tests v and v tests u. Each processor can be either fault-free or faulty. The test outcome is 0 (resp. 1) if the testing processor evaluates the tested processor as fault-free (resp. faulty). The test outcome is reliable only if the testing processor is fault-free. The collection of all test outcomes is called the *syndrome* of the system. The test performed by processor u on processor v is represented as a test (u, v). For a syndrome σ, $\sigma(u, v)$ represents the outcome of the test (u, v). For a given syndrome σ, a subset F of $V(G)$ is called a *consistent fault set* if $\sigma(u, v) = 1$ for every test (u, v) such that $u \in V(G) \setminus F$ and $v \in F$, and $\sigma(u, v) = 0$ for every test (u, v) such that $u, v \in V(G) \setminus F$.

The same syndrome can come from different fault sets, that is, there might be more than one fault set consistent with the syndrome. We say that F_1 and F_2 are *indistinguishable* if there is a syndrome for which they are consistent fault sets; otherwise, F_1 and F_2 are *distinguishable*. Figure 1 shows a syndrome for which both F_1 and F_2 are consistent fault sets; thus, the two fault sets are indistinguishable. A system G is t-diagnosable if and only if for each pair of distinct sets $F_1, F_2 \subset V(G)$ with $|F_1|, |F_2| \leq t$, F_1 and F_2 are distinguishable [15]. A fault set $F \subset V(G)$ is a *conditional fault set* if it is a conditional set, i.e., $N_G(v) \nsubseteq F$ for every $v \in V(G)$. A system G is conditionally t-diagnosable if and only if F_1 and F_2 are distinguishable for each pair of distinct conditional fault sets $F_1, F_2 \subset V(G)$ with $|F_1|, |F_2| \leq t$ [15]. The conditional diagnosability of G is denoted by $t_c(G)$.

3. Approach for Determining Conditional Diagnosability

In this section, a general technique is developed for determining the conditional diagnosability of connected graphs, especially graphs proposed as interconnection networks. Those graphs G are usually regular or almost regular, and the minimum degree, $\delta(G)$, is much smaller than the number of vertices, $|V(G)|$, say, logarithmic or sublogarithmic in $|V(G)|$; $\gamma_{p+1}(G)$ is greater than $\gamma_p(G)$ for p sufficiently smaller than $|V(G)|$.

We begin with some lemmas on fundamental properties of a graph that has two conditional fault sets which are indistinguishable. We denote by $F_1 \triangle F_2$ the symmetric difference of F_1 and F_2, i.e. $F_1 \triangle F_2 \equiv (F_1 \setminus F_2) \cup (F_2 \setminus F_1)$.

Lemma 1. *Let F_1, F_2 be distinct conditional fault sets of a graph G that are indistinguishable, and let $R = V(G) \setminus (F_1 \cup F_2)$.*
(a) There exists no edge joining a pair of vertices $u \in F_1 \triangle F_2$ and $v \in R$.
(b) $N_G(F_1 \triangle F_2) \subseteq F_1 \cap F_2$.
(c) $F_1 \nsubseteq F_2$ and $F_2 \nsubseteq F_1$.

Proof. To prove (a), let F_1 and F_2 be consistent with some syndrome σ. Suppose such an edge (u, v) exists. If u is in $F_1 \setminus F_2$, then the outcome of the test from v to u should be 1 since $v \notin F_1$ and $u \in F_1$, while its outcome should be 0 since $u, v \notin F_2$, which is a contradiction. Similarly, if u is in $F_2 \setminus F_1$, a contradiction also arises. Therefore, no such edge (u, v) exists. The statement (b) is a direct consequence of (a). To prove (c), suppose to the contrary that $F_1 \subseteq F_2$. From (b), $N_G(F_2 \setminus F_1) \subseteq F_1 \cap F_2$. This implies $N_G(v) \subseteq F_2$ for every $v \in F_2 \setminus F_1$, which is a contradiction to the fact that F_2 is a conditional set. Symmetrically, $F_2 \nsubseteq F_1$ can also be derived. Thus, the proof is completed. □

Lemma 2. *Let F_1, F_2 be distinct conditional fault sets of a graph G that are indistinguishable.*
(a) Every vertex of $F_i \setminus F_j$ has at least two neighbors: one in $F_i \setminus F_j$ and the other in $F_j \setminus F_i$, where $\{i, j\} = \{1, 2\}$.
(b) $|F_1 \setminus F_2|, |F_2 \setminus F_1| \geq 2$ and thus $|F_1 \triangle F_2| \geq 4$.
(c) If $|F_1 \triangle F_2| = 4$, then the subgraph induced by $F_1 \triangle F_2$ contains a cycle of length four, C_4, as a spanning subgraph.

Proof. Let v be a vertex in $F_1 \setminus F_2$. Then, $N_G(v) \subseteq F_1 \cup F_2$ by Lemma 1(a). Since both F_1 and F_2 are conditional, $N_G(v) \nsubseteq F_1$ and $N_G(v) \nsubseteq F_2$. Thus, there exists a neighbor $u \in F_1 \setminus F_2$ of v and there exists a neighbor $x \in F_2 \setminus F_1$ of v. Similarly, a vertex of $F_2 \setminus F_1$ also has two neighbors: one in $F_1 \setminus F_2$ and the other in $F_2 \setminus F_1$, proving (a). The statement (b) is direct from Lemmas 1(c) and 2(a). To prove (c), let $y \in F_2 \setminus F_1$ be a neighbor of x. From $|F_1 \triangle F_2| = 4$, $F_1 \triangle F_2 = \{u, v, x, y\}$, which forms a path of length three, (u, v, x, y). If (y, u) is an edge of G, the path becomes a cycle of length four and we are done. If $(y, u) \notin E(G)$, both (y, v) and (u, x) should be edges of G by (a), forming a cycle of length four (u, v, y, x). This completes the proof. □

Remark 1. If $|F_1 \triangle F_2| = 4$, the subgraph induced by $F_1 \triangle F_2$ is isomorphic to either C_4 or C_4 with additional chord edges. In many interconnection networks such as meshes, hypercube-like graphs, recursive circulants, etc., two adjacent vertices have at most one common neighbor. This is the case in the torus networks considered in this paper, which will be shown later in Lemma 7. In such networks, the induced subgraph is isomorphic to C_4.

Lemma 3. *Let F_1, F_2 be distinct conditional fault sets of a graph G that are indistinguishable, and let $R = V(G) \setminus (F_1 \cup F_2)$. Then, $R = \emptyset$ or $F_1 \cap F_2$ is a conditional vertex cut of G separating $F_1 \triangle F_2$ and R.*

Proof. Suppose $R \neq \emptyset$. Due to Lemma 1(a), $F_1 \cap F_2$ is a vertex cut of G separating $F_1 \triangle F_2$ and R. Since every subset of a conditional set is also conditional, $F_1 \cap F_2$ is a conditional cut. □

To establish the conditional diagnosability of a graph G, denoted by $t_c(G)$, we need to find two conditional fault sets F_1 and F_2 such that they are indistinguishable and $\max\{|F_1|, |F_2|\}$ is as small as possible. Then, $t_c(G)$ will be $\max\{|F_1|, |F_2|\} - 1$. In this case, we claim that $N_G(F_1 \triangle F_2) = F_1 \cap F_2$. By Lemma 1(b), $N_G(F_1 \triangle F_2) \subseteq F_1 \cap F_2$. It will be shown that $N_G(F_1 \triangle F_2)$ is not a proper subset of $F_1 \cap F_2$. Suppose to the contrary that $N_G(F_1 \triangle F_2)$ is a proper subset of $F_1 \cap F_2$. Let $F_1' = (F_1 \setminus F_2) \cup N_G(F_1 \triangle F_2)$ and $F_2' = (F_2 \setminus F_1) \cup N_G(F_1 \triangle F_2)$. F_1' and F_2' become conditional fault sets since $F_i' \subseteq F_i$ for $1 \leq i \leq 2$. They are also indistinguishable. This is a contradiction to the fact that $\max\{|F_1|, |F_2|\}$ is the minimum possible, proving the claim.

For the purpose of finding F_1 and F_2 that suggest the conditional diagnosability of G, we concentrate on $F_1 \triangle F_2$, in fact, on the subgraph of G induced by $F_1 \triangle F_2$ since it can be assumed that $F_1 \cap F_2 = N_G(F_1 \triangle F_2)$. Let H be the subgraph of G induced by $F_1 \triangle F_2$. Every vertex of H has two neighbors, one in $F_1 \setminus F_2$ and the other in $F_2 \setminus F_1$, due to Lemma 2(a). This property can be rephrased as the coloring of vertices, which differs from ordinary vertex coloring, as follows: the vertices of H can be colored in two colors, blue and orange, such that every vertex has a blue-colored neighbor and an orange-colored neighbor. This bicoloring is said to be *embraceable*. A nonempty graph that admits an embraceable bicoloring has at least four vertices, as Lemma 2(b) suggests. Not every graph with a minimum degree of at least two has an embraceable bicoloring, which is apparent in a cycle of length six. A graph may have multiple embraceable bicolorings.

For a bicoloring $\phi : V(H) \to \{\mathsf{blue}, \mathsf{orange}\}$ of a nonempty induced subgraph H of G, $\nu(\phi, H)$ denotes the maximum of $|V_b|$ and $|V_o|$ if ϕ is embraceable, where $V_b = \{v \in V(H) : \phi(v) = \mathsf{blue}\}$ and $V_o = \{v \in V(H) : \phi(v) = \mathsf{orange}\}$; $\nu(\phi, H) = \infty$ if ϕ is not embraceable. Then, the conditional diagnosability $t_c(G)$ of G can be stated in terms of its induced subgraphs and their embraceable bicolorings, as shown in the following.

Lemma 4. *$t_c(G)$ is the minimum of $\nu(\phi, H) + |N_G(V(H))| - 1$ over all pairs of a nonempty induced subgraph H of G and its (embraceable) bicoloring ϕ such that $V_b \cup N_G(V(H))$ and $V_o \cup N_G(V(H))$ are conditional sets.*

Proof. The proof is straightforward from the discussion above. □

One might expect that the first step of our approach will be to identify the smallest induced subgraph H that admits an embraceable bicoloring. If G has a cycle of length four, then H will be any cycle of length four possibly with additional chords, which clearly has an embraceable bicoloring. Actually, many interconnection networks, such as torus networks, hypercubes, and recursive circulants, contain a cycle of length four as a subgraph. The following theorem serves as a starting point for this discussion. Let $g(G)$ denote the *girth* of a graph G, which is defined to be the length of the shortest cycle contained in the graph.

Theorem 1. *Let G be a graph of girth $4q - 4 < g(G) \leq 4q$ for some integer $q \geq 1$. If a nonempty induced subgraph H of G admits an embraceable bicoloring, then H contains a path of $4q$ vertices, and whenever $|V(H)| = 4q$, H has a (hamiltonian) cycle of length $4q$.*

Proof. Suppose that every vertex of H is colored by some embraceable bicoloring. By the definition of an embraceable bicoloring, H has a path of four vertices, and whenever $|V(H)| = 4$, H has a cycle of length four, as suggested by Lemma 2. It suffices to consider $q \geq 2$. Let a blue-colored vertex $u_1 \in V(H)$ have a blue-colored neighbor u_2 and have an orange-colored neighbor v_1. The vertex v_1 also has an orange-colored neighbor v_2. Also, for each even i such that $i \leq 2q - 2$, u_i has an orange-colored neighbor v_{i+1} and v_i has a blue-colored neighbor u_{i+1}; for each odd i such that $i \leq 2q - 2$, u_i has a blue-colored neighbor u_{i+1} and v_i has an orange-colored neighbor v_{i+1}. Then, since $g(G) > 4q - 4$, the subgraph of H induced by $B \cup O$, where $B = \{u_1, \ldots, u_{2q-2}\}$ and $O = \{v_1, \ldots, v_{2q-2}\}$, forms an induced path of $4q - 4$ vertices. Moreover, the $4q - 2$ vertices of $B' \cup O'$ are all distinct and form a path joining u_{2q-1} and v_{2q-1}, where $B' = B \cup \{u_{2q-1}\}$ and $O' = O \cup \{v_{2q-1}\}$.

Suppose for the first case that u_{2q-1} and v_{2q-1} have a blue-colored neighbor $u_{2q} \notin B$ and an orange-colored neighbor $v_{2q} \notin O$, respectively. Then, there are $2q$ blue vertices and $2q$ orange vertices, forming a path P of $4q$ vertices between u_{2q} and v_{2q}, where $P = (u_{2q}, u_{2q-1}, v_{2q-2}, v_{2q-3}, \ldots, u_{2q-3}, u_{2q-2}, v_{2q-1}, v_{2q})$. It suffices to show $(u_{2q}, v_{2q}) \in E(H)$ if $|V(H)| = 4q$. u_{2q} has no neighbor in O; suppose otherwise, i.e., $(u_{2q}, v_i) \in E(H)$ for some $v_i \in O$, then the edge (u_{2q}, v_i) and the subpath P' starting at u_{2q} of the path P such that the length of P' is $4q - 5$ would create a cycle of length less than or equal to $4q - 4$, which contradicts the fact that $g(G) > 4q - 4$. (Note that $u_{2q-3}, u_{2q-2}, v_{2q-1}, v_{2q} \notin O$.) Similarly, v_{2q} has no neighbor in B. Thus, $(u_{2q}, v_{2q}) \in E(H)$; suppose otherwise, both (u_{2q}, v_{2q-1}) and (v_{2q}, u_{2q-1}) should be edges of H, which is impossible since the two edges would create a cycle of length four $(v_{2q-1}, v_{2q}, u_{2q-1}, u_{2q})$.

For the remaining case, it is assumed w.l.o.g. (without loss of generality) that every blue neighbor of u_{2q-1} is contained in B. Then, u_{2q-2} is a unique blue neighbor of u_{2q-1}; suppose otherwise, then H would have a cycle of length at most $4q - 4$, which is a contradiction to $g(G) > 4q - 4$. Similarly, v_{2q-1} has no neighbor contained in $O \setminus \{v_{2q-2}\}$. In addition, v_{2q-2} cannot be a neighbor of v_{2q-1} since suppose otherwise, there would be a cycle of length four $(v_{2q-2}, v_{2q-1}, u_{2q-2}, u_{2q-1})$, which is a contradiction. So, v_{2q-1} has an orange neighbor $v_{2q} \notin O$. v_{2q} should have a blue neighbor u. Then, u cannot be a vertex of B': suppose $u \in \{u_{2q-1}, u_{2q-2}, u_{2q-3}\}$, then there would exist a cycle of length three or four, which is a contradiction; supposing $u \in B \setminus \{u_{2q-2}, u_{2q-3}\}$ would lead to a cycle of length at most $4q - 4$, which is also a contradiction. Thus, there exists a path of $4q$ vertices joining u and u_{2q-1}. To show that H has a cycle of length $4q$ whenever $|V(H)| = 4q$, let u_j be a blue neighbor of u. We claim $j = 2q - 3$. By our assumption, $j \neq 2q - 1$. If $j = 2q - 2$, there would be a cycle of length four, which is a contradiction. If $j \leq 2q - 4$, then $q \geq 3$ and there would be a path joining u_j and u_{2q-2} of length at most $2q-2$ lying on a cycle induced by $B' \cup O$, where the path and $(u_{2q-2}, v_{2q-1}, v_{2q}, u, u_j)$ would create a cycle of length at most $(2q-2)+4 \leq 4q-4$, which is a contradiction. Thus, the claim is proved. For $u_j = u_{2q-3}$, merging of the path induced by $(B \cup O) \setminus \{u_{2q-2}\}$ and path $(v_{2q-2}, u_{2q-1}, u_{2q-2}, v_{2q-1}, v_{2q}, u, u_{2q-3})$ results in a hamiltonian cycle of length $4q$. This completes the entire proof. □

This theorem leads to an upper bound on the conditional diagnosability $t_c(G)$ of a graph G.

Theorem 2. *Let G be a graph of girth $4q-4 < g(G) \leq 4q$ for some q and let G have a cycle of length $4q$, $C_{4q} = (u_1, v_1, x_1, y_1, u_2, v_2, x_2, y_2, \ldots, u_q, v_q, x_q, y_q)$. If both $F_1 \equiv N_G(V(C_{4q})) \cup \bigcup_{i=1}^{q}\{u_i, v_i\}$ and $F_2 \equiv N_G(V(C_{4q})) \cup \bigcup_{i=1}^{q}\{x_i, y_i\}$ are conditional sets, then $t_c(G) \leq |N_G(V(C_{4q}))| + 2q - 1$.*

Proof. Let $R = V(G) \setminus (N_G(V(C_{4q})) \cup V(C_{4q}))$. Consider a syndrome σ such that the test outcome $\sigma(u, v) = 1$ if and only if $u \in F_i$ & $v \notin F_i$ or $u \notin F_i$ & $v \in F_i$ for some i such that $1 \leq i \leq 2$, as shown in Figure 1. Then, both F_1 and F_2 are consistent fault sets with σ. Thus, $t_c(G) < \max\{|F_1|, |F_2|\} = |N_G(V(C_{4q}))| + 2q$ and the theorem follows. □

Remark 2. $t_c(G) \leq \gamma'_{4q}(G) + 2q - 1$ if there exists a cycle C_{4q} such that (i) $|N_G(V(C_{4q}))| = \gamma'_{4q}(G)$ and (ii) F_1 and F_2 of Theorem 2 are conditional. Recall that $\gamma'_{4q}(G)$ denotes the minimum $|N_G(V(C_{4q}))|$ over all cycles of length $4q$, C_{4q}, in G.

To develop a lower bound on the conditional diagnosability, every induced subgraph possessing an embraceable bicoloring needs to be touched. Accordingly, developing the lower bound would be harder than the upper bound for which it suffices to pick up a good induced subgraph. Let G be a graph of girth $4q-4 < g(G) \leq 4q$ for some q. To prove $t_c(G) \geq t$ for some t, we suppose to the contrary that $t_c(G) < t$. Then, there exist distinct conditional fault sets F_1 and F_2 that are indistinguishable such that $|F_1|, |F_2| \leq t$. Let H be the subgraph of G induced by $F_1 \triangle F_2$. Then,

$$\max\{|F_1|, |F_2|\} \geq |F_1 \cap F_2| + \lceil |V(H)|/2 \rceil.$$

Notice that since F_1 and F_2 are conditional fault sets, H admits an embraceable bicoloring with all vertices of $F_1 \setminus F_2$ being blue-colored and all vertices of $F_2 \setminus F_1$ being orange-colored. Furthermore, H contains a path P of $4q$ vertices by Theorem 1. To derive $\max\{|F_1|, |F_2|\} > t$, which is a contradiction to the assumption of $|F_1|, |F_2| \leq t$, we may utilize the $(4q-1)$-super-connectivity of G, $\kappa_s^{4q-1}(G)$.

1. If $|F_1 \cap F_2| < \kappa_s^{4q-1}(G)$, then $G \setminus (F_1 \cap F_2)$ either is connected or has one large connected component and the remaining connected components with at most $4q-1$ vertices in total. Therefore, the $4q$ vertices of path P should be included in the large connected component. Let $R = V(G) \setminus (F_1 \cup F_2)$. There is no edge between R and $V(H)$, and thus the large connected component is contained in H which contains P. Since $|R| \leq 4q-1$, H may be large enough.
2. If $|F_1 \cap F_2| \geq \kappa_s^{4q-1}(G)$, then $F_1 \cap F_2$ may be large enough when, hopefully, $\kappa_s^{4q-1}(G)$ is big.

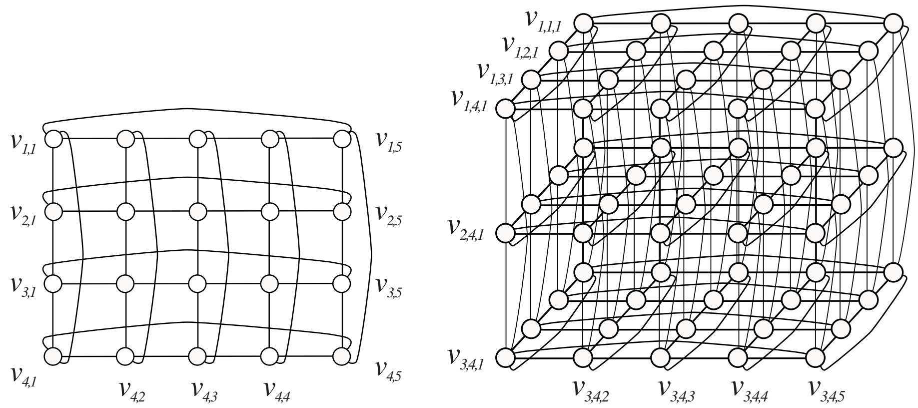
Based on these observations, a lower bound on $t_c(G)$ can be derived for a graph having a cycle of length $4 \cdot \lceil g(G)/4 \rceil$.

Theorem 3. *Let G be a graph of girth $4q-4 < g(G) \leq 4q$ for some q and let G have a cycle of length $4q$. Then, $t_c(G) \geq t$ for an integer $t \leq \gamma'_{4q}(G) + 2q - 1$ if (i) $\lceil |V(G)|/2 \rceil \geq t+1$, (ii) $\gamma_p(G) \geq p$ for every $1 \leq p \leq 4q-1$, and (iii) $\kappa_s^{4q-1}(G) \geq t - 2q$.*

Proof. Suppose to the contrary that $t_c(G) < t$. Then, there exist distinct conditional fault sets F_1 and F_2 with $|F_1|, |F_2| \leq t$, which are indistinguishable. Let H be the aforementioned subgraph induced by $F_1 \triangle F_2$. Then H contains a path of $4q$ vertices by Theorem 1. Therefore, $|V(H)| \geq 4q$.

Case 1: $|F_1 \cap F_2| < \kappa_s^{4q-1}(G)$.
Let $R = V(G) \setminus (F_1 \cup F_2)$. If $R = \emptyset$, then $F_1 \cup F_2 = V(G)$. Thus,

$$\max\{|F_1|, |F_2|\} \geq \lceil |V(G)|/2 \rceil \geq t+1,$$



(a) $T(4, 5)$ (b) $T(3, 4, 5)$

Figure 2: Examples of torus networks.

which is a contradiction. Now, $R \neq \emptyset$. From the fact that $|F_1 \cap F_2| < \kappa_s^{4q-1}(G)$ and H has a path of length $4q$, it follows that $|R| \leq 4q - 1$. Furthermore, $N_G(R) \subseteq F_1 \cap F_2$ by Lemma 1(a), and $|N_G(R)| \geq \gamma_{|R|}(G) \geq |R|$ by the condition (ii). Thus,

$$\begin{aligned}
\max\{|F_1|, |F_2|\} &\geq |F_1 \cap F_2| + \lceil |V(H)|/2 \rceil \\
&= |F_1 \cap F_2| + \lceil (|V(G)| - |F_1 \cap F_2| - |R|)/2 \rceil \\
&= \lceil (|V(G)| + |F_1 \cap F_2| - |R|)/2 \rceil \\
&\geq \lceil (|V(G)| + |F_1 \cap F_2| - |N_G(R)|)/2 \rceil \\
&\geq \lceil |V(G)|/2 \rceil \\
&\geq t + 1,
\end{aligned}$$

which is also a contradiction.

Case 2: $|F_1 \cap F_2| \geq \kappa_s^{4q-1}(G)$.
If $|V(H)| \geq 4q + 1$, then

$$\max\{|F_1|, |F_2|\} \geq |F_1 \cap F_2| + \lceil |V(H)|/2 \rceil \geq \kappa_s^{4q-1}(G) + (2q + 1) \geq t + 1,$$

which is a contradiction. If $|V(H)| = 4q$, then H contains a cycle of length $4q$ due to Theorem 1. Moreover, $N_G(V(H)) \subseteq F_1 \cap F_2$ by Lemma 1(b). Thus,

$$\max\{|F_1|, |F_2|\} \geq |F_1 \cap F_2| + \lceil |V(H)|/2 \rceil \geq |N_G(V(H))| + 2q \geq \gamma'_{4q}(G) + 2q \geq t + 1,$$

which is a contradiction. This completes the entire proof. □

4. Torus Networks

An n-dimensional torus network, denoted by $T(k_1, k_2, \ldots, k_n)$ where $n \geq 1$ and $k_j \geq 3$ for $1 \leq j \leq n$, is a graph consisting of $k_1 k_2 \cdots k_n$ vertices, each of which is identified by $v_{i_1, i_2, \ldots, i_n}$ where $1 \leq i_j \leq k_j$ for $1 \leq j \leq n$. Two vertices $v_{i_1, i_2, \ldots, i_n}$ and $v_{i'_1, i'_2, \ldots, i'_n}$ of the torus network are adjacent if $i'_p = (i_p \mod k_p) + 1$ for some p such that $1 \leq p \leq n$, and $i'_j = i_j$ for every j other than p. $T(k_1, k_2, \ldots, k_n)$ can also be defined as a Cartesian product of cycles, $C_{k_1} \times C_{k_2} \times \cdots \times C_{k_n}$. See Figure 2 for examples of torus networks. If $(k'_1, k'_2, \ldots, k'_n)$ is a permutation of $(k_1, k_2, \ldots, k_n)$, then $T(k'_1, k'_2, \ldots, k'_n)$ is isomorphic to $T(k_1, k_2, \ldots, k_n)$. So, it is assumed w.l.o.g. that $3 \leq k_1 \leq k_2 \leq \cdots \leq k_n$ throughout this paper.

Lemma 5. *Let G be an n-dimensional torus $T(k_1, k_2, \ldots, k_n)$ where $n \geq 1$.*
(a) G is a $2n$-regular graph of connectivity $\kappa(G) = 2n$.
(b) If $n \geq 2$, G has a cycle of length four; moreover, $g(G) = 3$ if $k_1 = 3$; $g(G) = 4$ if $k_1 \geq 4$, where $g(G)$ denotes the girth of G.

Proof. The proofs by induction on n are straightforward. □

An n-dimensional torus $T(k_1, k_2, \ldots, k_n)$ where $n \geq 2$, can be defined recursively as $C_{k_1} \times T(k_2, k_3, \ldots, k_n)$. This recursive structure of the torus networks will be fully exploited in this section. Let G represent an n-dimensional torus $T(k_1, k_2, \ldots, k_n)$. For $1 \leq j \leq k_1$, we denote by V_j a subset of $V(G)$ such that $V_j \equiv \{v_{i_1,i_2,\ldots,i_n} : i_1 = j, 1 \leq i_p \leq k_p$ for $2 \leq p \leq n\}$, and by G_j the subgraph of G induced by V_j. If we define $\mathbb{V} \equiv \{V_1, V_2, \ldots, V_{k_1}\}$, then $\mathbb{V}$ becomes a partition of $V(G)$. Let $\mathbb{E}$ represent an adjacency relation on $\mathbb{V}$ so that $(V_j, V_{j'}) \in \mathbb{E}$ if there exists an edge $(x, y) \in E(G)$ such that $x \in V_j$ and $y \in V_{j'}$. The graph $\mathbb{G}$ whose vertex set and edge set are respectively $\mathbb{V}$ and $\mathbb{E}$ is referred to as the *skeleton* of G.

Lemma 6. *Let $\mathbb{G}$ be the skeleton of G, an n-dimensional torus $T(k_1, k_2, \ldots, k_n)$ where $n \geq 2$.*
(a) $\mathbb{G}$ is isomorphic to a cycle of length k_1, C_{k_1}.
(b) For $1 \leq j \leq k_1$, G_j is isomorphic to an $(n-1)$-dimensional torus $T(k_2, \ldots, k_n)$.
(c) If $(V_j, V_{j'}) \in \mathbb{E}$, there are $|V_j|$ edges of G joining V_j and $V_{j'}$. These edges form a perfect matching of the induced subgraph of G, $G\langle V_j \cup V_{j'}\rangle$.

Proof. The proofs are trivial. □

Here, a *matching* of a graph is a set of pairwise nonadjacent edges, and a matching that covers all vertices of the graph is called *perfect*.

For analysis of the conditional diagnosability of a two or higher-dimensional torus network, Theorems 2 and 3 will be employed. To apply these theorems, we need to find out several structural properties, including the minimum-neighborhood set of order p for $2 \leq p \leq 4$, the minimum-neighborhood cycle of order four, and the 3-super-connectivity. The minimum-neighborhood set/cycle and the 3-super-connectivity will be studied in Sections 4.1 and 4.2 respectively, and then the conditional diagnosability will be determined in Section 4.3.

4.1. The Minimum-Neighborhood Set/Cycle

Let G be an n-dimensional torus $T(k_1, k_2, \ldots, k_n)$ where $n \geq 1$. In this subsection, $\gamma_p(G)$ will be determined for $1 \leq p \leq 4$ ($\gamma_4(G)$ will be used later in Section 4.2 for the 3-super-connectivity). Recall that $\gamma_p(G) = |N_G(T)|$, where T is a minimum-neighborhood set of order p of G. Also, $\gamma'_4(G)$ will be derived, where $\gamma'_4(G) = |N_G(V(C))|$ for a minimum-neighborhood cycle, C, of order four of G. It is obvious that $\gamma_1(G) = \delta(G) = 2n$. As a basic property for the minimum neighborhood, the number of common neighbors of two distinct vertices x, y of G is counted first. Keep in mind that every G_j, the subgraph induced by V_j, $1 \leq j \leq k_1$, is isomorphic to $T(k_2, \ldots, k_n)$ when $n \geq 2$.

Lemma 7. *Let G be an n-dimensional torus $T(k_1, k_2, \ldots, k_n)$ where $n \geq 1$.*
(a) For a pair of nonadjacent vertices x and y, $|N_G(x) \cap N_G(y)| \leq 2$.
(b) For a pair of adjacent vertices x and y, $|N_G(x) \cap N_G(y)| = 0$ if $k_1 \geq 4$; $|N_G(x) \cap N_G(y)| \leq 1$ if $k_1 = 3$.

Proof. The proof is by induction on n. It is straightforward to verify the base case of $n = 1$, where G is isomorphic to C_{k_1}. For the inductive step of $n \geq 2$, it may be assumed that $x \in V_1$. The case first considered is when $y \in V_1$. In $V(G) \setminus V_1$, there is no common neighbor of x and y. It follows that $N_G(x) \cap N_G(y) = N_{G_1}(x) \cap N_{G_1}(y)$, where G_1 is isomorphic to $T(k_2, \ldots, k_n)$. If $(x, y) \notin E(G_1)$, then by the induction hypothesis, $|N_{G_1}(x) \cap N_{G_1}(y)| \leq 2$, proving (a). Suppose $(x, y) \in E(G_1)$. Then, also by the induction hypothesis, $|N_{G_1}(x) \cap N_{G_1}(y)| = 0$ if $k_2 \geq 4$; $|N_{G_1}(x) \cap N_{G_1}(y)| \leq 1$ if $k_2 = 3$. Therefore, if $k_1 \geq 4$, then $k_2 \geq 4$ and thus $|N_G(x) \cap N_G(y)| = 0$; if $k_1 = 3$, then $|N_G(x) \cap N_G(y)| \leq 1$ whether $k_2 \geq 4$ or $k_2 = 3$, proving (b).

Next consider the case when $y \notin V_1$. Assume w.l.o.g. $y \in V_p$ for some p such that $2 \leq p < k_1$. Then, $N_G(y) = N_{G_p}(y) \cup Y$ where $Y = \{y', y''\}$ for some $y' \in V_{p+1}$ and $y'' \in V_{p-1}$. Similarly, $N_G(x) = N_{G_1}(x) \cup X$ where $X = \{x', x''\}$ for some $x' \in V_{k_1}$ and $x'' \in V_2$. It is clear that $N_{G_1}(x) \cap N_{G_p}(y) = \emptyset$, $|N_{G_1}(x) \cap Y| \leq 1$, $|N_{G_p}(y) \cap X| \leq 1$, and

$|X \cap Y| \leq 2$. If $(x, y) \notin E(G)$, then $X \cap Y = \emptyset$ for $p = 2$ and $N_{G_1}(x) \cap Y = N_{G_p}(y) \cap X = \emptyset$ for $p \neq 2$. Thus, $|N_G(x) \cap N_G(y)| = |N_{G_1}(x) \cap Y| + |X \cap N_{G_p}(y)| + |X \cap Y| \leq \max\{1 + 1 + 0, 0 + 0 + 2\} = 2$, proving (a). Suppose $(x, y) \in E(G)$. Then, $p = 2$, i.e., $y \in V_2$. In addition, $y'' = x$, $x'' = y$, $y' \in V_3$, and $x' \in V_{k_1}$, where $k_1 \geq 3$. Therefore, $N_G(x) \cap N_G(y) = (N_{G_1}(x) \cap \{y'\}) \cup (\{x'\} \cap N_{G_2}(y)) \cup (\{x'\} \cap \{y'\}) = \{x'\} \cap \{y'\}$. Furthermore, $|\{x'\} \cap \{y'\}| = 1$ if and only if $k_1 = 3$, proving (b). □

Let $T_2 = \{x, y\}$ be a vertex subset of G where

$$x = v_{1,1,\ldots,1} \text{ and } y = v_{2,1,\ldots,1}.$$

Then, $x \in V_1$, $y \in V_2$, and $(x, y) \in E(G)$. It will be shown in the following lemma that T_2 is a minimum-neighborhood set of order two of G.

Lemma 8. *Let G be an n-dimensional torus $T(k_1, \ldots, k_n)$ where $n \geq 1$. Then,*

$$\gamma_2(G) = \begin{cases} 4n - 2 & \text{if } k_1 \geq 4, \\ 4n - 3 & \text{if } k_1 = 3. \end{cases}$$

Proof. It will be proved that T_2 is a minimum-neighborhood set of order two of G. The proof for $n = 1$ is trivial, so let $n \geq 2$. $|N_G(T_2)|$ is first calculated. $|N_G(T_2)| = 2(n - 1) + 2(n - 1) + 1 + 1 = 4n - 2$ if $k_1 \geq 4$; $|N_G(T_2)| = 2(n - 1) + 2(n - 1) + 1 = 4n - 3$ if $k_1 = 3$. It remains to show that $|N_G(S)| \geq |N_G(T_2)|$ for every two-vertex subset $S = \{u, v\}$ of G. Suppose $k_1 \geq 4$ for the first case. Then, by Lemma 7, $|N_G(S)| \geq 2n + 2n - 2 = 4n - 2$ if $(u, v) \notin E(G)$; $|N_G(S)| = (2n - 1) + (2n - 1) = 4n - 2$ if $(u, v) \in E(G)$. Thus, $|N_G(S)| \geq |N_G(T_2)|$ for $k_1 \geq 4$. Now, suppose $k_1 = 3$ for the second case. Then, also by Lemma 7, $|N_G(S)| \geq 4n - 2$ if $(u, v) \notin E(G)$; $|N_G(S)| \geq (2n - 1) + (2n - 1) - 1 = 4n - 3$ if $(u, v) \in E(G)$. Thus, $|N_G(S)| \geq |N_G(T_2)|$ for $k_1 = 3$. This completes the proof. □

$\gamma_3(G)$ for $n \geq 2$ will be obtained in the following lemma. When $n = 1$, it is straightforward that $\gamma_3(G) = 2$ if $k_1 \geq 5$; $\gamma_3(G) = 1$ if $k_1 = 4$; $\gamma_3(G) = 0$ if $k_1 = 3$. For $n \geq 2$, let $T_3 = \{x, y, z\}$ be a subset of $V(G)$, where

$$x = v_{1,1,1,\ldots,1}, y = v_{1,2,1,\ldots,1}, \text{ and } z = v_{2,1,1,\ldots,1}.$$

Then, the subgraph induced by T_3 is a path of three vertices (y, x, z), where $x, y \in V_1$ and $z \in V_2$. Notice that $\{x, y\}$ is a minimum-neighborhood set of order two of G_1, which follows from the construction of T_2.

Lemma 9. *Let G be an n-dimensional torus $T(k_1, \ldots, k_n)$ where $n \geq 2$. Then,*

$$\gamma_3(G) = \begin{cases} 6n - 5 & \text{if } k_1 \geq 4, \\ 6n - 6 & \text{if } k_1 = 3 \,\&\, k_2 \geq 4, \\ 6n - 7 & \text{if } k_1 = k_2 = 3. \end{cases}$$

Proof. In order to prove that T_3 is a minimum-neighborhood set of order three of G, $|N_G(T_3)|$ will be calculated first as before. Since $\{x, y\}$ is a minimum-neighborhood set of G_1, $|N_{G_1}(\{x, y\})| = \gamma_2(G_1)$, i.e., $|N_{G_1}(\{x, y\})| = 4(n - 1) - 2$ if $k_2 \geq 4$; $|N_{G_1}(\{x, y\})| = 4(n-1)-3$ if $k_2 = 3$. (Recall that G_1 is isomorphic to an $(n-1)$-dimensional torus $T(k_2, \ldots, k_n)$.) For $k_1 \geq 4$,

$$|N_G(T_3)| = |N_{G_1}(\{x, y\})| + |N_{G_2}(z)| + |N_G(\{x, y\}) \cap V_{k_1}| + |N_G(z) \cap V_3|.$$

Thus, $|N_G(T_3)| = (4(n - 1) - 2) + 2(n - 1) + 2 + 1 = 6n - 5$. For $k_1 = 3$,

$$|N_G(T_3)| = |N_{G_1}(\{x, y\})| + |N_{G_2}(z)| + |N_G(\{x, y\}) \cap V_3|.$$

Thus, if $k_1 = 3$ & $k_2 \geq 4$, then $|N_G(T_3)| = (4(n - 1) - 2) + 2(n - 1) + 2 = 6n - 6$; if $k_1 = k_2 = 3$, then $|N_G(T_3)| = (4(n - 1) - 3) + 2(n - 1) + 2 = 6n - 7$.

In the remaining part of this proof, we will show, by induction on n, that $|N_G(S)| \geq |N_G(T_3)|$ for every three-vertex subset S of G. Let $S_j = S \cap V_j$ for $1 \leq j \leq k_1$, and assume w.l.o.g. $|S_1| \geq |S_j|$ for every j. There are three cases depending on the size of S_1.

Case 1: $|S_1| = 3$.
In this case, $S_1 = S$ and

$$|N_G(S)| = |N_{G_1}(S_1)| + |N_G(S_1) \cap V_2| + |N_G(S_1) \cap V_{k_1}| = |N_{G_1}(S_1)| + 6.$$

Suppose $n = 2$ for the first subcase. Note that G_1 is isomorphic to C_{k_2}, a cycle of length k_2. Thus, $|N_{G_1}(S_1)| \geq 1$ if $k_2 \geq 4$; $|N_{G_1}(S_1)| = 0$ if $k_2 = 3$. Thus, if $k_1 \geq 4$ or $k_1 = 3$ & $k_2 \geq 4$, then $|N_G(S)| \geq 1 + 6 = 7 \geq |N_G(T_3)|$; if $k_1 = k_2 = 3$, then $|N_G(S)| \geq 0 + 6 > |N_G(T_3)|$. Suppose $n \geq 3$ for the second subcase. Then, $|N_{G_1}(S_1)| \geq \gamma_3(G_1)$, where by the induction hypothesis, $\gamma_3(G_1)$ is $6(n-1) - 5$ if $k_2 \geq 4$; $6(n-1) - 6$ if $k_2 = 3$ & $k_3 \geq 4$; $6(n-1) - 7$ if $k_2 = k_3 = 3$. Therefore, if $k_1 \geq 4$ or $k_1 = 3$ & $k_2 \geq 4$, then $k_2 \geq 4$ and thus $|N_G(S)| \geq (6(n-1) - 5) + 6 = 6n - 5 \geq |N_G(T_3)|$; if $k_1 = k_2 = 3$, then $|N_G(S)| \geq (6(n-1) - 7) + 6 = 6n - 7 = |N_G(T_3)|$.

Case 2: $|S_1| = 2$.
Assume w.l.o.g. $|S_p| = 1$ for some p such that $2 < p \leq k_1$. (The case of $p = 2$ is symmetric to the case of $p = k_1$.) Consider the first subcase where $k_1 \geq 4$. Let $q = p + 1$ if $p = 3$; let $q = p - 1$ otherwise, so that $(V_p, V_q) \in \mathbb{E}$ and $q \notin \{1, 2\}$ where $\mathbb{E}$ is the edge set of the skeleton of G. Then,

$$|N_G(S)| \geq |N_{G_1}(S_1)| + |N_{G_p}(S_p)| + |N_G(S_1) \cap V_2| + |N_G(S_p) \cap V_q|,$$

where $|N_{G_1}(S_1)| \geq \gamma_2(G_1) = 4(n-1)-2$ by Lemma 8. Thus, $|N_G(S)| \geq (4(n-1)-2)+2(n-1)+2+1 = 6n-5 = |N_G(T_3)|$. Now, consider the second subcase where $k_1 = 3$. Then, $p = 3$, i.e., $|S_3| = 1$, and

$$|N_G(S)| \geq |N_{G_1}(S_1)| + |N_{G_3}(S_3)| + |N_G(S_1) \cap V_2|.$$

If $k_1 = 3$ & $k_2 \geq 4$, then $|N_G(S)| \geq (4(n-1) - 2) + 2(n-1) + 2 = 6n - 6 = |N_G(T_3)|$; if $k_1 = k_2 = 3$, then $|N_G(S)| \geq (4(n-1) - 3) + 2(n-1) + 2 = 6n - 7 = |N_G(T_3)|$.

Case 3: $|S_1| = 1$.
There exist two integers p and q, $2 \leq p < q \leq k_1$, such that $|S_p| = |S_q| = 1$. Consider the first subcase where $k_1 \geq 4$. Let $r \notin \{1, p, q\}$ be an index such that $(V_r, V_j) \in \mathbb{E}$ for some $j \in \{1, p, q\}$. Then,

$$|N_G(S)| \geq |N_{G_1}(S_1)| + |N_{G_p}(S_p)| + |N_{G_q}(S_q)| + |N_G(S) \cap V_r|.$$

Thus, $|N_G(S)| \geq 2(n-1) + 2(n-1) + 2(n-1) + 1 = 6n - 5 = |N_G(T_3)|$. For the remaining subcase where $k_1 = 3$,

$$|N_G(S)| \geq |N_{G_1}(S_1)| + |N_{G_p}(S_p)| + |N_{G_q}(S_q)|.$$

Thus, $|N_G(S)| \geq 6n - 6 \geq |N_G(T_3)|$, whether $k_1 = 3$ & $k_2 \geq 4$ or $k_1 = k_2 = 3$. This completes the proof. □

Theorem 4 determines $\gamma_4(G)$ for $n \geq 2$. When $n = 1$, it is obvious that $\gamma_4(G) = 2$ if $k_1 \geq 6$; $\gamma_4(G) = 1$ if $k_1 = 5$; $\gamma_4(G) = 0$ if $k_1 = 4$. $\gamma_4(G)$ is left undefined if $n = 1$ and $k_1 = 3$. For $n \geq 2$, let $T_4 = \{x, y, z, w\}$ be a vertex subset of G, where

$$(x, y, z, w) = \begin{cases} (v_{1,1}, v_{2,1}, v_{3,1}, v_{2,2}) & \text{if } k_1 \geq 4 \ \& \ n = 2, \\ (v_{1,1,1,1,\ldots,1}, v_{1,1,2,1,\ldots,1}, v_{1,2,1,1,\ldots,1}, v_{2,1,1,1,\ldots,1}) & \text{if } k_1 \geq 4 \ \& \ n \geq 3, \\ (v_{1,1,1,\ldots,1}, v_{1,2,1,\ldots,1}, v_{2,2,1,\ldots,1}, v_{2,1,1,\ldots,1}) & \text{if } k_1 = 3. \end{cases}$$

The subgraph induced by T_4 is isomorphic to a complete bipartite graph, $K_{1,3}$, if $k_1 \geq 4$ (whether $n = 2$ or $n \geq 3$); the induced subgraph is isomorphic to a cycle of length four, C_4, if $k_1 = 3$. It will be shown below that T_4 is a minimum-neighborhood set of order four of G.

Theorem 4. *Let G be an n-dimensional torus $T(k_1, \ldots, k_n)$ where $n \geq 2$. Then,*

$$\gamma_4(G) = \begin{cases} 8 \ (= 8n - 8) & \text{if } k_1 \geq 5 \ \& \ n = 2, \\ 8n - 9 & \text{if } k_1 \geq 4 \ \& \ n \geq 3 \text{ or } k_1 = 4 \ \& \ n = 2, \\ 8n - 10 & \text{if } k_1 = 3 \ \& \ k_2 \geq 4, \\ 8n - 12 & \text{if } k_1 = k_2 = 3. \end{cases}$$

Proof. Let us calculate $|N_G(T_4)|$ first as before. Firstly, suppose $k_1 \geq 4$ & $n = 2$. Then, $|N_G(T_4)| = |N_{G_1}(x)| + |N_{G_2}(\{y, w\})| + |N_{G_3}(z)| + |(N_G(x) \cap V_{k_1}) \cup (N_G(z) \cap V_4)|$, where the last term is equal to two if $k_1 \geq 5$; it is equal to one if $k_1 = 4$. Notice that $|N_{G_1}(x)| = |N_{G_2}(\{y, w\})| = |N_{G_3}(z)| = 2$. Thus, $|N_G(T_4)| = 2 + 2 + 2 + 2 = 8(= 8n - 8)$ if $k_1 \geq 5$; $|N_G(T_4)| = 2+2+2+1 = 7(= 8n-9)$ if $k_1 = 4$. Secondly, suppose $k_1 \geq 4$ & $n \geq 3$. Observe that $\{x, y, z\}$ is a minimum-neighborhood set of order three of G_1 from the construction of T_3. Thus, $|N_{G_1}(\{x, y, z\})| = \gamma_3(G_1) = 6(n - 1) - 5$ by Lemma 9. Furthermore, the neighborhood in V_2 of $\{y, z\}$ is a subset of $N_{G_2}(w)$. Then, $|N_G(T_4)| = |N_{G_1}(\{x, y, z\})| + |N_{G_2}(w)| + |N_G(\{x, y, z\}) \cap V_{k_1}| + |N_G(w) \cap V_3| = (6(n - 1) - 5) + 2(n - 1) + 3 + 1 = 8n - 9$. Finally, suppose $k_1 = 3$. Then, $|N_G(T_4)| = |N_{G_1}(\{x, y\})| + |N_{G_2}(\{w, z\})| + |N_G(\{x, y\}) \cap V_3|$. Note that $\{x, y\}$ and $\{w, z\}$ are minimum-neighborhood sets of order two of G_1 and G_2, respectively. Hence by Lemma 8, $|N_{G_1}(\{x, y\})| = |N_{G_2}(\{w, z\})| = 4(n - 1) - 2$ if $k_2 \geq 4$; $|N_{G_1}(\{x, y\})| = |N_{G_2}(\{w, z\})| = 4(n - 1) - 3$ if $k_2 = 3$. Thus, $|N_G(T_4)| = (4(n - 1) - 2) + (4(n - 1) - 2) + 2 = 8n - 10$ if $k_1 = 3$ & $k_2 \geq 4$; $|N_G(T_4)| = (4(n - 1) - 3) + (4(n - 1) - 3) + 2 = 8n - 12$ if $k_1 = k_2 = 3$.

Now, it suffices to show that $|N_G(S)| \geq |N_G(T_4)|$ for every four-vertex subset S of G. The proof is by induction on n. Let $S_j = S \cap V_j$ for $1 \leq j \leq k_1$, and assume w.l.o.g. $|S_1| \geq |S_j|$ for every j.

Case 1: $|S_1| = 4$.
In this case, $S_1 = S$. Then,

$$|N_G(S)| = |N_{G_1}(S_1)| + |N_G(S_1) \cap V_2| + |N_G(S_1) \cap V_{k_1}| = |N_{G_1}(S_1)| + 8.$$

If $n = 2$, then it is clear that $|N_G(S)| \geq 8 \geq |N_G(T_4)|$. Suppose $n \geq 3$. If $k_1 \geq 4$ or $k_1 = 3$ & $k_2 \geq 4$, then $k_2 \geq 4$ and thus $|N_{G_1}(S_1)| \geq 8(n - 1) - 9$ by the induction hypothesis. Thus $|N_G(S)| \geq 8n - 9 \geq |N_G(T_4)|$. If $k_1 = k_2 = 3$, then $|N_G(S)| \geq 8n - 12 = |N_G(T_4)|$ since $|N_{G_1}(S_1)| \geq 8(n - 1) - 12$ by the induction hypothesis.

Case 2: $|S_1| = 3$.
Assume w.l.o.g. $|S_p| = 1$ for some p such that $2 < p \leq k_1$. Suppose $k_1 \geq 4$ for the first subcase. Let $q = p + 1$ if $p = 3$; let $q = p - 1$ otherwise, so that $(V_p, V_q) \in \mathbb{E}$ and $q \notin \{1, 2\}$. Then,

$$|N_G(S)| \geq |N_{G_1}(S_1)| + |N_{G_p}(S_p)| + |N_G(S_1) \cap V_2| + |N_G(S_p) \cap V_q|.$$

For $n = 2$, it is clear that $|N_{G_1}(S_1)| \geq 2$ if $k_2 \geq 5$; $|N_{G_1}(S_1)| = 1$ if $k_2 = 4$. Thus, $|N_G(S)| \geq 2 + 2 + 3 + 1 = 8 = |N_G(T_4)|$ if $k_1 \geq 5$; $|N_G(S)| \geq 1 + 2 + 3 + 1 = 7 = |N_G(T_4)|$ if $k_1 = 4$. Let $n \geq 3$. Then, $|N_{G_1}(S_1)| \geq \gamma_3(G_1) = 6(n - 1) - 5$ by Lemma 9. Thus, $|N_G(S)| \geq (6(n - 1) - 5) + 2(n - 1) + 3 + 1 = 8n - 9 = |N_G(T_4)|$.

Suppose $k_1 = 3$ for the second subcase. Then, $p = 3$ and

$$|N_G(S)| \geq |N_{G_1}(S_1)| + |N_{G_3}(S_3)| + |N_G(S_1) \cap V_2|.$$

For $n = 2$, $|N_G(S)| \geq 1 + 2 + 3 = 6 = |N_G(T_4)|$ if $k_1 = 3$ & $k_2 \geq 4$; $|N_G(S)| \geq 0 + 2 + 3 = 5 > |N_G(T_4)|$ if $k_1 = k_2 = 3$. Let $n \geq 3$. If $k_1 = 3$ & $k_2 \geq 4$, then $|N_G(S)| \geq (6(n - 1) - 5) + 2(n - 1) + 3 = 8n - 10 = |N_G(T_4)|$. If $k_1 = k_2 = 3$, then $|N_G(S)| \geq (6(n - 1) - 7) + 2(n - 1) + 3 = 8n - 12 = |N_G(T_4)|$.

Case 3: $|S_1| = 2$.

Case 3.1: $|S_p| = 2$ *for some* p *such that* $2 \leq p \leq k_1$.
Assume w.l.o.g. $p \neq 2$. Suppose $k_1 \geq 4$ for the first subcase. As in Case 2, let $q = p + 1$ if $p = 3$; let $q = p - 1$ otherwise. Then,

$$|N_G(S)| \geq |N_{G_1}(S_1)| + |N_{G_p}(S_p)| + |N_G(S_1) \cap V_2| + |N_G(S_p) \cap V_q|.$$

Here, $|N_{G_1}(S_1)|, |N_{G_p}(S_p)| \geq \gamma_2(G_1) = 4(n-1)-2$ by Lemma 8. Thus, $|N_G(S)| \geq (4(n-1)-2)+(4(n-1)-2)+2+2 = 8n - 8 \geq |N_G(T_4)|$, which holds true whether $n = 2$ or $n \geq 3$. Suppose $k_1 = 3$ for the second subcase. Then $p = 3$ and

$$|N_G(S)| \geq |N_{G_1}(S_1)| + |N_{G_3}(S_3)| + |N_G(S_1) \cap V_2|.$$

If $k_1 = 3$ and $k_2 \geq 4$, then $|N_G(S)| \geq (4(n - 1) - 2) + (4(n - 1) - 2) + 2 = 8n - 10 = |N_G(T_4)|$. If $k_1 = k_2 = 3$, then $|N_G(S)| \geq (4(n - 1) - 3) + (4(n - 1) - 3) + 2 = 8n - 12 = |N_G(T_4)|$.

Case 3.2: $|S_p| = 1$ *and* $|S_q| = 1$ *for some* p, q *such that* $2 \leq p < q \leq k_1$.
Suppose $k_1 \geq 4$ for the first subcase. There exists an index $r \notin \{1, p, q\}$ such that $(V_r, V_j) \in \mathbb{E}$ for some $j \in \{1, p, q\}$. Then,

$$|N_G(S)| \geq |N_{G_1}(S_1)| + |N_{G_p}(S_p)| + |N_{G_q}(S_q)| + |N_G(S) \cap V_r|.$$

Thus, $|N_G(S)| \geq (4(n-1)-2)+2(n-1)+2(n-1)+1 = 8n-9 = |N_G(T_4)|$ except for only the case when $k_1 \geq 5$ & $n = 2$. For the exceptional case, we can pick up another index $r' \notin \{1, p, q, r\}$ such that $(V_{r'}, V_j) \in \mathbb{E}$ for some $j \in \{1, p, q\}$ since $k_1 \geq 5$. Then, $|N_G(S) \cap V_{r'}| \geq 1$ and thus $|N_G(S)| \geq |N_{G_1}(S_1)| + |N_{G_p}(S_p)| + |N_{G_q}(S_q)| + |N_G(S) \cap V_r| + |N_G(S) \cap V_{r'}| \geq 2+2+2+1+1 = 8 = |N_G(T_4)|$.

Suppose $k_1 = 3$ for the second subcase. Then, $p = 2$, $q = 3$, and

$$|N_G(S)| \geq |N_{G_1}(S_1)| + |N_{G_2}(S_2)| + |N_{G_3}(S_3)|.$$

If $k_1 = 3$ & $k_2 \geq 4$, then $|N_G(S)| \geq (4(n-1)-2) + 2(n-1) + 2(n-1) = 8n-10 = |N_G(T_4)|$. If $k_1 = k_2 = 3$, then $|N_G(S)| \geq (4(n-1)-3) + 2(n-1) + 2(n-1) = 8n-11 > |N_G(T_4)|$.

Case 4: $|S_1| = 1$.

In this case, $k_1 \geq 4$ and $|S_p| = |S_q| = |S_r| = 1$ for some $2 \leq p < q < r \leq k_1$. Thus $|N_G(S)| \geq |N_{G_1}(S_1)| + |N_{G_p}(S_p)| + |N_{G_q}(S_q)| + |N_{G_r}(S_r)| = 2(n-1)+2(n-1)+2(n-1)+2(n-1) = 8n-8 \geq |N_G(T_4)|$. This completes the entire proof. □

Finally, $\gamma'_4(G)$ will be determined, where $\gamma'_4(G) = |N_G(V(C))|$ for a minimum-neighborhood cycle C of order four of G. Recall that for $k_1 = 3$, the subgraph of G induced by T_4 which is a minimum-neighborhood set of order four contains a cycle of length four. Thanks to Theorem 4, it suffices to consider the case when $k_1 \geq 4$. Notice $\gamma'_4(G) \geq \gamma_4(G)$ by the definition. Let C be the cycle of length four (x, y, z, w), such that

$$(x, y, z, w) = (v_{1,1,1,\ldots,1}, v_{1,2,1,\ldots,1}, v_{2,2,1,\ldots,1}, v_{2,1,1,\ldots,1}).$$

Notice that $V(C)$ is equal to T_4 for $k_1 = 3$, which is a minimum neighborhood set for $k_1 = 3$. It will be shown below that C is a minimum-neighborhood cycle of order four of G, whether $k_1 = 3$ or $k_1 \geq 4$.

Theorem 5. *Let G be an n-dimensional torus $T(k_1, \ldots, k_n)$ where $n \geq 2$. Then,*

$$\gamma'_4(G) = \begin{cases} 8n-8 & \text{if } k_1 \geq 4, \\ \gamma_4(G) & \text{if } k_1 = 3. \end{cases}$$

Proof. Suppose $k_1 \geq 4$ due to Theorem 4. Then, $|N_G(V(C))| = |N_{G_1}(\{x,y\})| + |N_{G_2}(\{z,w\})| + |N_G(\{x,y\}) \cap V_{k_1}| + |N_G(\{z,w\}) \cap V_3| = (4(n-1)-2) + (4(n-1)-2) + 2 + 2 = 8n-8$. It remains to show that $|N_G(S)| \geq |N_G(V(C))|$ for every four-vertex subset S of G by which the induced subgraph, $G\langle S\rangle$, contains a cycle of length four. The proof is by induction on n. Let $S_j = S \cap V_j$ for $1 \leq j \leq k_1$, and assume w.l.o.g. $|S_1| \geq |S_j|$ for every j. Then, $|S_1| \neq 3$ by the structure of a torus network.

Case 1: $|S_1| = 4$.

Obviously, $|N_G(S)| = |N_{G_1}(S_1)| + |N_G(S_1) \cap V_2| + |N_G(S_1) \cap V_{k_1}| = |N_{G_1}(S_1)| + 8$. It follows that if $n = 2$, then $|N_G(S)| \geq 8 = |N_G(V(C))|$; if $n \geq 3$, where $|N_{G_1}(S_1)| \geq 8(n-1) - 8$ by the induction hypothesis, then $|N_G(S)| \geq 8n-8 = |N_G(V(C))|$.

Case 2: $|S_1| = 2$.

Assume w.l.o.g. $|S_2| = 2$. Then, $|N_G(S)| = |N_{G_1}(S_1)| + |N_{G_2}(S_2)| + |N_G(S_1) \cap V_{k_1}| + |N_G(S_2) \cap V_3| \geq (4(n-1)-2) + (4(n-1)-2) + 2 + 2 = 8n-8 = |N_G(V(C))|$.

Case 3: $|S_1| = 1$.

In this case, $k_1 = 4$ and $|S_2| = |S_3| = |S_4| = 1$. Thus, $|N_G(S)| = |N_{G_1}(S_1)| + |N_{G_2}(S_2)| + |N_{G_3}(S_3)| + |N_{G_4}(S_4)| = 4 \cdot 2(n-1) = 8n-8 = |N_G(V(C))|$. The proof is completed. □

4.2. The Super-Connectivity

Let G be an n-dimensional torus network, $T(k_1, k_2, \ldots, k_n)$, where $n \geq 2$. In this subsection, the 3-super-connectivity, $\kappa_s^3(G)$, of a torus network will be determined: $\kappa_s^3(G) = \gamma_4(G)$ for every torus network with only two exceptions, $T(3,3)$ and $T(4,4)$. As by-products of this result, the 2- and 1-super-connectivities of torus networks are also obtained. We begin with the two exceptional tori.

Lemma 10. *(a) Let G be $T(3,3)$. Then, $\kappa_s^3(G) = \kappa_s^2(G) = 8$ and $\kappa_s^1(G) = 5$.*
(b) Let G be $T(4,4)$. Then, $\kappa_s^3(G) = \kappa_s^2(G) = \kappa_s^1(G) = 6$.

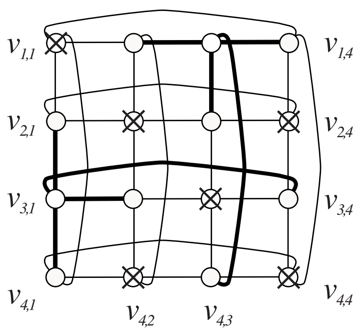


Figure 3: $T(4,4)$ with a vertex subset F, where the six vertices of F are marked as "×".

Proof. For a vertex subset F of a graph G, let G' denote the largest connected component of $G \setminus F$ and let $R = V(G) \setminus (F \cup V(G'))$. To prove (a), let G be $T(3,3)$. If $|V(G')| = 1$, then each component of $G \setminus F$ is a trivial graph and thus F contains at least two vertices in G_j for $1 \leq j \leq 3$. Recall that G_j is the subgraph of G induced by V_j. Thus $|F| \geq 6$ and $|R| \leq 2$. If $|V(G')| \geq 2$, let S be a subset of $V(G')$ such that $|S| = 2$. Since $N_G(S) \subset V(G') \cup F$, it follows that $|V(G') \cup F| \geq |S| + |N_G(S)| \geq 2 + 5 = 7$, which means $|R| \leq 2$. Hence, there is no F such that the total number of vertices contained in the connected components of $G \setminus F$ other than G' is greater than two. From the definition of $\kappa_s^r(G)$, $\kappa_s^3(G) = \kappa_s^2(G) = |V(G)| - 1 = 8$. To show $\kappa_s^1(G) = 5$, we claim $\kappa_s^1(G) \geq 5$ first. Suppose to the contrary $\kappa_s^1(G) < 5$, i.e., there exists a vertex subset F such that $|F| \leq 4$ and $|R| \geq 2$. Clearly $|F| = 4$ since the connectivity of G is four. Also, $|V(G')| \geq 2$; suppose otherwise, then there are two isolated vertices u, v in $G \setminus F$ and $N_G(\{u,v\}) \subseteq F$, but $|N_G(\{u,v\})| \geq 6$ by Lemma 7(a), which is a contradiction. Furthermore, $|V(G')|, |R| \geq 4$ since $N_G(V(G')), N_G(R) \subseteq F$ and $\gamma_2(G) = \gamma_3(G) = 5 > |F|$ by Lemmas 8 and 9. This leads to a contradiction that $|V(G)| = |F| + |V(G')| + |R| \geq 4 + 4 + 4 > |V(G)|$, proving the claim. For the set, S, of two end-vertices of an edge of G, let $F = N_G(S)$. Then $|F| = 5$, $|V(G')| = 2$, and $|R| = 2$. Thus $\kappa_s^1(G) \leq |F| = 5$.

To prove (b), let G be $T(4,4)$. Figure 3 shows a vertex subset F of G, where $|F| = 6$, such that $G \setminus F$ has two connected components, each of size five. This implies $\kappa_s^r(G) \leq 6$ for $1 \leq r \leq 3$ (in fact, for $1 \leq r \leq 4$). It remains to show $\kappa_s^r(G) \geq 6$ for $1 \leq r \leq 3$. Let $F \subset V(G)$ be of size at most five. Suppose $G \setminus F$ is disconnected. Then, $|F| = 4$ or $|F| = 5$. We denote by H_1 and H_2 respectively the subgraphs of G induced by $V_1 \cup V_2$ and $V_3 \cup V_4$. Observe that every vertex of H_1 has a unique neighbor contained in H_2, and vice versa (G is isomorphic to the 4-dimensional hypercube, and each of H_1 and H_2 is isomorphic to the 3-dimensional one). Assume w.l.o.g. that $|V(H_1) \cap F| \leq 2$. Then, $H_1 \setminus F$ is connected, and thus, the large connected component, containing $V(H_1) \setminus F$, of $G \setminus F$ also contains every vertex of $H_2 \setminus F$ whose unique neighbor in H_1 is not a member of F. Since $|V(H_1) \cap F| \leq 2$, there may be at most two vertices of $H_2 \setminus F$ not contained in the large component. We claim the number of such vertices is one, which leads to completing the proof of (b). Suppose to the contrary that the number is two. Let x, y be the two vertices of $H_2 \setminus F$ not contained in the large component. Then, $|V(H_1) \cap F| = 2$ and $|V(H_2) \cap F| \leq 3$. It is obvious that $|N_{H_2}(\{x,y\})| \geq 4$. Thus, at least one of $N_{H_2}(\{x,y\})$ is contained in the large component since $|V(H_2) \cap F| \leq 3$. This implies that at least one of x, y is also contained in the large component, which is a contradiction. □

Hereafter, we consider torus networks where either $n = 2$ & $(k_1, k_2) \notin \{(3,3),(4,4)\}$ or $n \geq 3$. An upper bound on $\kappa_s^3(G)$ can be derived without difficulty as follows. Note that if $n = 2$, then $k_1 = 3$ & $k_2 \geq 4$, or $k_1 = 4$ & $k_2 \geq 5$, or $k_1 \geq 5$.

Lemma 11. *Let G be an n-dimensional torus $T(k_1, k_2, \ldots, k_n)$ such that $n \geq 2$ and $(n, k_1, k_2) \notin \{(2,3,3),(2,4,4)\}$. Then, $\kappa_s^3(G) \leq \gamma_4(G)$.*

Proof. Consider the case where $(n, k_1, k_2) = (2,3,4)$ first. For a vertex subset $F = \{v_{1,1}, v_{1,3}, v_{2,1}, v_{2,3}, v_{3,2}, v_{3,4}\}$, $G \setminus F$ is a disconnected graph which has four connected components of size 2, 2, 1, and 1. This means that $\kappa_s^3(G) \leq |F| = 6 = \gamma_4(G)$. For the remaining cases, let S be a minimum-neighborhood set of order four of G and let $F = N_G(S)$. Then

$|F| = \gamma_4(G)$. We claim $|R| \geq 4$ where $R = V(G) \setminus (S \cup F)$. If $n = 2$, then

$$|R| = |V(G)| - (4 + \gamma_4(G)) \geq \begin{cases} k_1k_2 - (4+8) \geq 4 & \text{if } k_1 \geq 5, \\ k_1k_2 - (4+7) \geq 4 & \text{if } k_1 = 4 \ \& \ k_2 \geq 5, \\ k_1k_2 - (4+6) \geq 4 & \text{if } k_1 = 3 \ \& \ k_2 \geq 5. \end{cases}$$

If $n \geq 3$, then $|R| \geq 3^n - (4 + (8n-9)) = (3^n - 8n) + 5 \geq 4$. Thus, the claim is proved. This implies that for every connected component of $G \setminus F$, its vertex set is completely contained either in S or in R. The total number of vertices contained in the connected components of $G \setminus F$ other than the largest one is at least four. Thus $\kappa_s^3(G) \leq \gamma_4(G)$. □

To prove $\kappa_s^3(G) \geq \gamma_4(G)$, let F be an arbitrary subset of $V(G)$ such that $|F| < \gamma_4(G)$. We will show that $G \setminus F$ has a *large* connected component with size of at least $|V(G)| - |F| - 3$. The two-dimensional torus problem is first considered and the higher-dimensional torus problem will be discussed later. Let $F_j = V_j \cap F$ for $1 \leq j \leq k_1$. Assume w.l.o.g. $|F_1| \leq |F_j|$ for every j. Let G be a two-dimensional torus $T(k_1, k_2)$. Then $|F_1| \leq 1$; suppose otherwise, $|F| \geq 2k_1 \geq \gamma_4(G)$ by Theorem 4, which contradicts the condition for $|F|$. Recall that for vertex set $\mathbb{V}$ of the skeleton $\mathbb{G}$ of G, $\mathbb{V} = \{V_1, V_2, \ldots, V_{k_1}\}$. Let $\mathbb{Z}$ be the subset of $\mathbb{V}$ such that

$$\mathbb{Z} \equiv \{V_j \in \mathbb{V} : |F_j| \geq 2\} \text{ and let } \bar{\mathbb{Z}} \equiv \mathbb{V} \setminus \mathbb{Z}.$$

Then, for every $V_j \in \bar{\mathbb{Z}}$, $G_j \setminus F$ is connected. This is because $\kappa(G_j) = 2$ where $\kappa(G_j)$ is the connectivity of G_j. Notice that G_j denotes $G\langle V_j \rangle$, the subgraph induced by V_j, and remember $V_1 \in \bar{\mathbb{Z}}$.

Lemma 12. *Let G be a two-dimensional torus $T(k_1, k_2)$ where $(k_1, k_2) \notin \{(3,3), (4,4)\}$. Let F be an arbitrary subset of $V(G)$ such that $|F| < \gamma_4(G)$.*
(a) If $|\mathbb{Z}| \leq 2$, there is a path in $G \setminus F$ between any pair of vertices of $\bigcup_{V_j \in \bar{\mathbb{Z}}} V_j \setminus F$.
(b) If $|\mathbb{Z}| \geq 3$, then $F_1 = \emptyset$.

Proof. For the proof of (a), it will be shown that for every $V_p \in \bar{\mathbb{Z}}$ where $p \neq 1$, there exists a path in $G \setminus F$ joining a vertex of $V_p \setminus F_p$ and a vertex of $V_1 \setminus F_1$. If $k_1 = 3$ & $k_2 \geq 4$, then $p \in \{2, 3\}$ and thus there is an edge (x, y) for some $x \in V_p \setminus F_p$ and $y \in V_1 \setminus F_1$. Suppose $k_1 \geq 4$ & $k_2 \geq 5$ or $k_1 \geq 5$. There exists a four-vertex subset W of $V_p \setminus F_p$ since $k_2 \geq 5$ and $|F_p| \leq 1$. Let $W = \{v_{p,i_1}, v_{p,i_2}, v_{p,i_3}, v_{p,i_4}\}$. From each vertex v_{p,i_j} of W, we define two paths in G to $v_{1,i_j} \in V_1$: an *upward path* $P_j^u = (v_{p,i_j}, v_{p-1,i_j}, \ldots, v_{1,i_j})$ and a *downward path* $P_j^d = (v_{p,i_j}, v_{p+1,i_j}, \ldots, v_{k_1,i_j}, v_{1,i_j})$. These eight paths in total are internally vertex-disjoint. If $|F_1| = 0$, at least one of the eight paths is a path of $G \setminus F$ between $V_p \setminus F_p$ and $V_1 \setminus F_1$ since $|F| < \gamma_4(G) \leq 8$. Consider the case when $|F_1| = 1$. In this case, $|F_p| = 1$. Among the eight paths above, there are at least six paths such that the last vertex of each path is not in F_1. At least one of these six paths is also a path of $G \setminus F$ between $V_p \setminus F_p$ and $V_1 \setminus F_1$ since $|F| - |F_1| - |F_p| = |F| - 2 < \gamma_4(G) - 2 \leq 6$, completing the proof of (a).

To prove (b), suppose to the contrary that $F_1 \neq \emptyset$. Then, $|F| \geq 2|\mathbb{Z}| + (k_1 - |\mathbb{Z}|) \geq k_1 + 3 \geq \gamma_4(G)$ by Theorem 4, which contradicts the condition for $|F|$. □

For a subset F of $V(G)$, let $B = \bigcup_{V_j \in \bar{\mathbb{Z}}} V_j \setminus F$ if $|\mathbb{Z}| \leq 2$, and let $B = V_1$ if $|\mathbb{Z}| \geq 3$. Note that in $G \setminus F$, there is a path joining any pair of vertices of B. The connected component of $G \setminus F$ that contains B will be referred to as a *big* component. Observe that the big component is of size at least $|V(G)| - |F| - 3$ if and only if $R_W \cap B \neq \emptyset$ for every four-vertex subset W of $G \setminus F$, where R_W is the set of vertices reachable from some vertex $w \in W$ along a path of $G \setminus F$. By definition, $W \subseteq R_W$. The condition is equivalent to saying that at least one vertex of W is contained in the big component. The subgraph of G induced by R_W is not necessarily connected. It will be proved in the following lemma that the big component is indeed the large connected component of size at least $|V(G)| - |F| - 3$.

Lemma 13. *Let G be a two-dimensional torus $T(k_1, k_2)$ where $(k_1, k_2) \notin \{(3,3), (4,4)\}$. Let F be an arbitrary subset of $V(G)$ such that $|F| < \gamma_4(G)$. For every four-vertex subset W of $G \setminus F$, $R_W \cap B \neq \emptyset$.*

Proof. First, consider the case when $|\mathbb{Z}| \leq 2$. In this case, $B = \bigcup_{V_j \in \bar{\mathbb{Z}}} V_j \setminus F$. Suppose $R_W \cap B = \emptyset$ for some four-vertex subset W. Then, $R_W \subset \bigcup_{V_p \in \mathbb{Z}} V_p$. For each $V_p \in \mathbb{Z}$, there exists some $V_q \in \bar{\mathbb{Z}}$ such that $(V_p, V_q) \in \mathbb{E}$. Since $|F_q| \leq 1$ and there exists a perfect matching in $G\langle V_p \cup V_q \rangle$ joining V_p and V_q, at most one vertex of V_p is contained in R_W. This implies $|R_W| \leq 2$, which is a contradiction to $|R_W| \geq |W| = 4$.

Next, consider the case when $|\mathbb{Z}| \geq 3$. Note that $F_1 = \emptyset$ from Lemma 12(b), and $B = V_1 \in \bar{\mathbb{Z}}$. Also, we have $k_1 = |\mathbb{Z}| + |\bar{\mathbb{Z}}| \geq 3 + 1 = 4$, and moreover $k_2 \geq 5$. Suppose to the contrary that $R_W \cap V_1 = \emptyset$ for some four-vertex subset W. Let $R_W^j = R_W \cap V_j$ for $V_j \in \mathbb{V}$. Let $V_p \in \mathbb{V}$ such that $|R_W \cap V_p| \geq |R_W \cap V_j|$ for every $V_j \in \mathbb{V}$. Then, $2 \leq p \leq k_1$. There are three cases depending on $|R_W^p|$.

Case 1: $|R_W^p| \geq 4$.
From each vertex of R_W^p, there exist two paths in G to some vertex of V_1, one upward path and one downward path defined in the proof of Lemma 12. The $2|R_W^p|$ paths in total are internally vertex-disjoint and $R_W \cap V_1 = \emptyset$, which implies $|F| \geq 2|R_W^p| \geq 8$. Moreover, $8 \geq \gamma_4(G)$ from Theorem 4. This contradicts the condition for $|F|$.

Case 2: $|R_W^p| = 3$.
In this case, $|F_p| \geq 2$ since $|V_p| = k_2 \geq 5$. Similar to Case 1, there are six internally vertex-joint paths between R_W^p and V_1. Furthermore, $R_W \cap V_1 = \emptyset$. Thus, $|F| = |F_p| + 6 \geq 2 + 6 \geq \gamma_4(G)$, which is a contradiction.

Case 3: $|R_W^p| \leq 2$.
Let $J = \{j : R_W^j \neq \emptyset\}$. For each $j \in J$, $|F_j| \geq 2$; suppose otherwise, $|R_W^j| \geq k_2 - 1 \geq 4 > |R_W^p|$, which is a contradiction. Let $q = \min J$, $r = \max J$. Then, either $|J| \geq 3$ or $|J| = 2$ & $|R_W^q| = |R_W^r| = 2$. Recall $|R_W| \geq |W| = 4$. There exist $|R_W^q|$ upward paths (between R_W^q and V_1) and $|R_W^r|$ downward paths (between R_W^r and V_1). These paths are internally vertex-disjoint. Thus, $|F| \geq \sum_{j \in J} |F_j| + (|R_W^q| + |R_W^r|) \geq \min\{2 \cdot 3 + (1 + 1), 2 \cdot 2 + (2 + 2)\} = 8 \geq \gamma_4(G)$, which is a contradiction. This completes the entire proof. □

Theorem 6. *Let G be a two-dimensional torus $T(k_1, k_2)$. Then,*

$$\kappa_s^3(G) = \begin{cases} 8 & \text{if } k_1 = k_2 = 3, \\ 6 & \text{if } k_1 = k_2 = 4, \\ \gamma_4(G) & \text{otherwise.} \end{cases}$$

Proof. The proof is a direct consequence of Lemmas 10, 11, and 13. □

Once the $(r + 1)$-super-connectivity of a graph is determined for some r, the r-super-connectivity of the graph can be obtained simply as suggested by the following lemma.

Lemma 14. *Let G be a graph and r be a nonnegative integer. If $\kappa_s^{r+1}(G) \geq \gamma_{r+1}(G)$, then $\kappa_s^r(G) \geq \gamma_{r+1}(G)$.*

Proof. Consider a vertex subset F of G such that $|F| < \gamma_{r+1}(G)$. Let H be the largest connected component of $G \setminus F$ and let $R = V(G) \setminus (F \cup V(H))$. Since $\kappa_s^{r+1}(G) \geq \gamma_{r+1}(G)$, it follows that $|R| \leq r + 1$. Furthermore, $|R| \neq r + 1$. Suppose $|R| = r + 1$, then $\gamma_{r+1}(G) \leq |N_G(R)| \leq |F| < \gamma_{r+1}(G)$ from the fact that $N_G(R) \subseteq F$, which is a contradiction. Therefore, it is concluded that $|R| \leq r$, proving the lemma. □

Theorem 7. *Let G be a two-dimensional torus $T(k_1, k_2)$.*

$$(a)\ \kappa_s^2(G) = \begin{cases} 8 & \text{if } k_1 = k_2 = 3, \\ 6 & \text{if } k_1 = k_2 = 4, \\ \gamma_3(G) & \text{otherwise.} \end{cases}$$

$$(b)\ \kappa_s^1(G) = \begin{cases} 5 & \text{if } k_1 = k_2 = 3, \\ 6 & \text{if } k_1 = k_2 = 4, \\ \gamma_2(G) & \text{otherwise.} \end{cases}$$

Proof. Suppose $(k_1, k_2) \notin \{(3, 3), (4, 4)\}$. From Lemmas 8 and 9 and Theorem 4 of Section 4.1, we can see that $\gamma_4(G) \geq \gamma_3(G) \geq \gamma_2(G)$. Thus, by Theorem 6 and Lemma 14, it follows that $\kappa_s^2(G) \geq \gamma_3(G)$ and $\kappa_s^1(G) \geq \gamma_2(G)$. It remains to show $\kappa_s^2(G) \leq \gamma_3(G)$ and $\kappa_s^1(G) \leq \gamma_2(G)$. For a minimum-neighborhood set of order three, S, of G where $|N_G(S)| = \gamma_3(G)$, we have $|V(G)| - (|S| + |N_G(S)|) = k_1k_2 - (3 + \gamma_3(G)) \geq \min\{4 \cdot 4 - (3 + 7),\ 3 \cdot 4 - (3 + 6)\} \geq |S| = 3$. Thus, every connected component of $G \setminus N_G(S)$ is of a size at most $|V(G)| - \gamma_3(G) - 3$, proving $\kappa_s^2(G) \leq \gamma_3(G)$. Finally, for a minimum-neighborhood set of order two, S, of G where $|N_G(S)| = \gamma_2(G)$, we have $|V(G)| - (|S| + |N_G(S)|) = k_1k_2 - (2 + \gamma_2(G)) \geq \min\{4 \cdot 4 - (2 + 6),\ 3 \cdot 4 - (2 + 5)\} \geq |S| = 2$. Thus, the size of every connected component of $G \setminus N_G(S)$ is at most $|V(G)| - \gamma_2(G) - 2$, proving $\kappa_s^1(G) \leq \gamma_2(G)$. □

The remaining part of this subsection is devoted to the higher-dimensional torus networks. Let G be an n-dimensional torus $T(k_1, k_2, \ldots, k_n)$ where $n \geq 3$. To conclude $\kappa_s^3(G) \geq \gamma_4(G)$, it will be shown as before that for an arbitrary subset F of $V(G)$ such that $|F| < \gamma_4(G)$, $G \setminus F$ has a *large* connected component with size of at least $|V(G)| - |F| - 3$. A *two*-dimensional skeleton of G will be defined and exploited, instead of the one-dimensional skeleton used for the two-dimensional torus problem. A similar approach, adjusted to the two-dimensional skeleton, will be taken for the higher-dimensional torus problem.

$T(k_1, k_2, \ldots, k_n)$ where $n \geq 3$ can be viewed as $T(k_1, k_2) \times T(k_3, \ldots, k_n)$. This recursive structure allows for the definition of a two-dimensional skeleton isomorphic to $T(k_1, k_2)$. We denote by $V_{p,q}$ a subset of $V(G)$ such that $V_{p,q} \equiv \{v_{i_1,i_2,\ldots,i_n} : i_1 = p, i_2 = q, 1 \leq i_j \leq k_j \text{ for } 3 \leq j \leq n\}$, where $1 \leq p \leq k_1$ and $1 \leq q \leq k_2$. We redefine

$$\mathbb{V} \equiv \{V_{p,q} : 1 \leq p \leq k_1, 1 \leq q \leq k_2\}.$$

$\mathbb{V}$ is still a partition of $V(G)$. Let $\mathbb{E}$ represent an adjacency relation on $\mathbb{V}$ so that $(V_{p,q}, V_{p',q'}) \in \mathbb{E}$ if there exists an edge $(x, y) \in E(G)$ for some $x \in V_{p,q}$ and $y \in V_{p',q'}$. The graph $\mathbb{G}$ whose vertex set and edge set respectively are $\mathbb{V}$ and $\mathbb{E}$ is said to be a *two-dimensional skeleton* of G. A vertex of $\mathbb{G}$ is referred to as a *supernode*.

Lemma 15. *Let $\mathbb{G}$ be the two-dimensional skeleton of G, where G is an n-dimensional torus $T(k_1, k_2, \ldots, k_n)$, $n \geq 3$.*
(a) $\mathbb{G}$ is isomorphic to a two-dimensional torus $T(k_1, k_2)$.
(b) Every induced subgraph of G by a supernode $V_{p,q}$ of $\mathbb{G}$ is isomorphic to an $(n-2)$-dimensional torus $T(k_3, \ldots, k_n)$.
(c) If $(V_{p,q}, V_{p',q'}) \in \mathbb{E}$, there are $|V_{p,q}|$ edges of G joining $V_{p,q}$ and $V_{p',q'}$, which form a perfect matching of the induced subgraph of G, $G\langle V_{p,q} \cup V_{p',q'}\rangle$.
(d) For every subset $\mathbb{X}$ of $\mathbb{V}$ such that $|\mathbb{X}|, |\bar{\mathbb{X}}| \geq 4$, there exists a matching of size four in $\mathbb{G}$ joining $\mathbb{X}$ and $\bar{\mathbb{X}}$, where $\bar{\mathbb{X}} = \mathbb{V} \setminus \mathbb{X}$.

Proof. The proofs of (a), (b), and (c) are trivial. To prove (d), suppose that the size of a maximum matching between $\mathbb{X}$ and $\bar{\mathbb{X}}$, denoted by m^*, is less than four. Let $\mathbb{G}'$ be the spanning subgraph of $\mathbb{G}$ obtained by removing all the edges of $\mathbb{G}$ joining two supernodes both of which are contained in $\mathbb{X}$ or in $\bar{\mathbb{X}}$. Then, $\mathbb{G}'$ is bipartite since every edge of $\mathbb{G}'$ joins $\mathbb{X}$ and $\bar{\mathbb{X}}$. Furthermore, the size of its maximum matching is m^*. This implies that there exists a vertex cover $\mathbb{C}$ of size m^* in $\mathbb{G}'$, where a *vertex cover* of a graph is defined to be a set of vertices such that each edge of the graph is incident to at least one vertex of the set. This is because, by the König-Egerváry Theorem [2], the size of a maximum matching in a bipartite graph is equal to the size of its minimum vertex cover. Since $\mathbb{G}' \setminus \mathbb{C}$ contains no edge at all, $\mathbb{G} \setminus \mathbb{C}$ contains no edge between $\mathbb{X} \setminus \mathbb{C}$ and $\bar{\mathbb{X}} \setminus \mathbb{C}$. This means that $\mathbb{C}$ is a vertex cut of size $m^* < 4$, separating $\mathbb{X} \setminus \mathbb{C}$ and $\bar{\mathbb{X}} \setminus \mathbb{C}$, which contradicts the fact that the connectivity of $\mathbb{G}$ is four. Note that $\mathbb{X} \setminus \mathbb{C}$ and $\bar{\mathbb{X}} \setminus \mathbb{C}$ are nonempty. Therefore, $\mathbb{G}$ has a matching of size four between $\mathbb{X}$ and $\bar{\mathbb{X}}$. □

Let $\mathbb{Z}$ denote the subset of $\mathbb{V}$ such that

$$\mathbb{Z} \equiv \{V_{p,q} \in \mathbb{V} : |F_{p,q}| \geq \kappa(G_{p,q})\} \text{ and let } \bar{\mathbb{Z}} \equiv \mathbb{V} \setminus \mathbb{Z},$$

where $F_{p,q} \equiv F \cap V_{p,q}$ and $G_{p,q}$ is the subgraph of G induced by $V_{p,q}$. Then, for every $V_{p,q} \in \bar{\mathbb{Z}}$, $G_{p,q} \setminus F$ is connected. Note that $\kappa(G_{p,q}) = 2n - 4$. Hereafter, when referring to a supernode of $\mathbb{V}$, its index will be dropped for notational simplicity.

Lemma 16. *Let G be an n-dimensional torus $T(k_1, k_2, \ldots, k_n)$ where $n \geq 3$. Let F be an arbitrary subset of $V(G)$ such that $|F| < \gamma_4(G)$.*
(a) If $|\mathbb{Z}| \leq 3$, the subgraph of G induced by $\bigcup_{X \in \bar{\mathbb{Z}}} X \setminus F$ is connected.
(b) If $|\mathbb{Z}| \geq 4$, there exists a supernode $Y \in \bar{\mathbb{Z}}$ such that $Y \cap F = \emptyset$.

Proof. To prove (a), we claim that $G\langle X \cup Y\rangle \setminus F$ is connected for each pair $X, Y \in \bar{\mathbb{Z}}$ such that $(X, Y) \in \mathbb{E}$ where $\mathbb{E}$ is the edge set of the two-dimensional skeleton of G. It suffices to show that there exists an edge $(x, y) \in E(G)$ for some $x \in X \setminus F$ and $y \in Y \setminus F$. This is because $G\langle X \setminus F\rangle$ and $G\langle Y \setminus F\rangle$ are both connected graphs. Furthermore, $|X \cap F| + |Y \cap F| \leq 2(2n - 5) = 4n - 10$. Suppose that no such edge (x, y) exists, then $|X \cap F| + |Y \cap F| \geq |X| \geq 3^{n-2}$. It is impossible that $3^{n-2} \leq 4n - 10$ for every $n \geq 3$, thus the claim is proved. In addition, $\mathbb{G} \setminus \mathbb{Z}$ is connected since the connectivity of $\mathbb{G}$ is four. Recall that $\mathbb{G}$ is isomorphic to $T(k_1, k_2)$. This implies that the subgraph induced by $\bigcup_{X\in\bar{\mathbb{Z}}} X \setminus F$ is connected, completing the proof of (a).

To prove (b), suppose to the contrary that $X \cap F \neq \emptyset$ for every $X \in \bar{\mathbb{Z}}$. Then, $|F| \geq (2n-4)|\mathbb{Z}| + (k_1k_2 - |\mathbb{Z}|) = (2n-5)|\mathbb{Z}| + k_1k_2 \geq 4(2n-5) + k_1k_2$. It follows that if $k_1 = k_2 = 3$, then $|F| \geq 4(2n-5) + 9 = 8n - 11 > \gamma_4(G)$; if $k_1 = 3$ & $k_2 \geq 4$, then $|F| \geq 4(2n-5) + 12 = 8n - 8 > \gamma_4(G)$; if $k_1 \geq 4$, then $|F| \geq 4(2n-5) + 16 = 8n - 4 > \gamma_4(G)$. This contradicts the condition for $|F|$. □

For a subset F of $V(G)$, let $B \equiv \bigcup_{X\in\bar{\mathbb{Z}}} X \setminus F$, if $|\mathbb{Z}| \leq 3$; let $B \equiv Y$ for an arbitrary supernode $Y \in \bar{\mathbb{Z}}$ such that $Y \cap F = \emptyset$, if $|\mathbb{Z}| \geq 4$. As before, it will be shown that the size of the connected component of $G \setminus F$ that contains B, called a *big* component, is at least $|V(G)| - |F| - 3$. For a four-vertex subset W of $G \setminus F$, R_W is defined as the set of vertices reachable, via a path of $G \setminus F$, from some vertex of W. It will be proved that $R_W \cap B \neq \emptyset$. To begin with, the problem of how many supernodes of $\mathbb{G}$ intersect with R_W is considered. Let $\mathbb{W}$ denote the subset of $\mathbb{V}$ such that

$$\mathbb{W} \equiv \{X \in \mathbb{V} : X \cap R_W \neq \emptyset\} \text{ and let } \bar{\mathbb{W}} \equiv \mathbb{V} \setminus \mathbb{W}.$$

Lemma 17. *Let G be an n-dimensional torus $T(k_1, k_2, \ldots, k_n)$ where $n \geq 3$, and let F be an arbitrary subset of $V(G)$ such that $|F| < \gamma_4(G)$. For every four-vertex subset W of $G \setminus F$, $|\mathbb{W}| \geq 4$.*

Proof. Suppose to the contrary that $|\mathbb{W}| \leq 3$. Let $\mathbb{W} = \{X_1, \ldots, X_q\}$ for $q = |\mathbb{W}|$. Assume w.l.o.g. $|X_j \cap R_W| \geq |X_{j+1} \cap R_W|$ for all $1 \leq j < q$. Let $\mathbb{X}$ be a set of arbitrary $4-q$ supernodes in $N_{\mathbb{G}}(X_1) \cap \bar{\mathbb{W}}$. Let $\{(X_j, Y_j) : X_j \in \mathbb{W}, Y_j \in \mathbb{V} \setminus (\mathbb{W} \cup \mathbb{X}), 1 \leq j \leq q\}$ be a matching of size q between $\mathbb{W}$ and $\mathbb{V} \setminus (\mathbb{W} \cup \mathbb{X})$. Such a matching can be obtained from a matching M of size four between $\mathbb{W} \cup \mathbb{X}$ and $\mathbb{V} \setminus (\mathbb{W} \cup \mathbb{X})$. The existence of a matching M follows from Lemma 15(d) since $|\mathbb{W} \cup \mathbb{X}| = 4$ and $|\mathbb{V} \setminus (\mathbb{W} \cup \mathbb{X})| = |\mathbb{V}| - 4 \geq 5$. Let $\mathbb{Y} = N_{\mathbb{G}}(\mathbb{W}) \setminus (\{Y_j : 1 \leq j \leq q\} \cup \mathbb{X})$. The vertices of $\bigcup_{1\leq j\leq q} (X_j \cup Y_j)$, $\bigcup_{X\in\mathbb{X}} X$, and $\bigcup_{Y\in\mathbb{Y}} Y$ which belong to F are to be counted.

We pick up a subset W_j of $X_j \cap R_W$ for $1 \leq j \leq q$ such that (i) $|W_j| \geq 1$, (ii) $\sum_{j=1}^{q} |W_j| = 4$, and (iii) $|W_j| \geq |W_{j+1}|$ for all $1 \leq j < q$. This is always possible since $q \leq 3$ and $|R_W| \geq |W| = 4$. For all j such that $1 \leq j \leq q$, we claim that (i) for every vertex $x \in W_j$, we have $y \in F$ where $y \in Y_j$ is the neighbor of x, and (ii) for every vertex $x \in \Gamma_j$ where $\Gamma_j \equiv N_{G\langle X_j\rangle}(W_j)$, we have $x \in F$ or $y \in F$, where $y \in Y_j$ is the neighbor of x. Suppose (i) or (ii) is violated, then Y_j would be a supernode of $\mathbb{W}$, which is a contradiction. Thus, $X_j \cup Y_j$ contains at least $|W_j| + |\Gamma_j|$ vertices of F. It follows that for $1 \leq j \leq q$,

$$|F \cap (X_j \cup Y_j)| \geq |W_j| + |\Gamma_j| \geq p_j + \gamma_{p_j}(G\langle X_j\rangle), \tag{1}$$

where $p_j = |W_j|$. For each $X \in \mathbb{X}$, we have $N_G(W_1) \cap X \subset F$; suppose otherwise, X would be a supernode of $\mathbb{W}$, which is a contradiction. So,

$$|F \cap \bigcup_{X\in\mathbb{X}} X| \geq |\mathbb{X}| \cdot |W_1| = (4-q)p_1. \tag{2}$$

For each $Y \in \mathbb{Y}$, there exists X_j such that $(X_j, Y) \in \mathbb{E}$, and moreover, we have $N_G(W_j) \cap Y \subset F$. Thus,

$$|F \cap \bigcup_{Y\in\mathbb{Y}} Y| \geq |\mathbb{Y}| \cdot |W_q| = (|N_{\mathbb{G}}(\mathbb{W})| - q - |\mathbb{X}|)p_q \geq (\gamma_q(\mathbb{G}) - 4)p_q. \tag{3}$$

From the three inequalities, we obtain

$$|F| \geq \textstyle\sum_{j=1}^{q} \{p_j + \gamma_{p_j}(G\langle X_j\rangle)\} + \{(4-q)p_1 + (\gamma_q(\mathbb{G}) - 4)p_q\}. \tag{4}$$

In the remaining part of this proof, it will be shown that $|F| \geq \gamma_4(G)$ from inequality (4), which contradicts the condition for $|F|$. In regards to the term $\gamma_{p_j}(G\langle X_j\rangle)$, recall that $G\langle X_j\rangle$ is isomorphic to an $(n-2)$-dimensional torus $T(k_3, \ldots, k_n)$. For simplicity, H will be used, instead of $G\langle X_j\rangle$, to denote the subgraph of G induced by a supernode of $\mathbb{G}$. From the results of Section 4.1, we can derive the following inequalities, which hold true for every $n \geq 3$.

$$\gamma_2(H) \geq \begin{cases} 4(n-2) - 2 = 4n - 10 & \text{if } k_3 \geq 4, \\ 4(n-2) - 3 = 4n - 11 & \text{if } k_3 = 3. \end{cases}$$

$$\gamma_3(H) \geq \begin{cases} 6(n-2) - 5 = 6n - 17 & \text{if } k_3 \geq 4, \\ 6(n-2) - 7 = 6n - 19 & \text{if } k_3 = 3. \end{cases}$$

$$\gamma_4(H) \geq \begin{cases} 8(n-2)-9 = 8n-25 & \text{if } k_3 \geq 4, \\ 8(n-2)-12 = 8n-28 & \text{if } k_3 = 3. \end{cases}$$

The right-hand sides of the inequalities are negative in some cases, but these cause no problem; $\gamma_4(H)$ is left undefined and never used when $|V(H)| = 3$, i.e., $n = 3$ & $k_3 = 3$. There are three cases depending on $|\mathbb{W}|$. Remember that $\mathbb{G}$ is isomorphic to $T(k_1, k_2)$.

Case 1: $|\mathbb{W}| = 1$ ($q = 1$).
In this case, $p_1 = 4$ and the right-hand side of inequality (4) is equal to

$$(4 + \gamma_4(H)) + (3 \cdot 4 + 0) = \gamma_4(H) + 16.$$

If $k_1 \geq 4$ or $k_1 = 3$ & $k_2 \geq 4$, then $k_3 \geq 4$, and thus, $|F| \geq (8n-25)+16 = 8n-9 \geq \gamma_4(G)$ by Theorem 4. If $k_1 = k_2 = 3$, then $|F| \geq (8n-28) + 16 = 8n - 12 = \gamma_4(G)$.

Case 2: $|\mathbb{W}| = 2$ ($q = 2$).
If $p_1 = 3$ & $p_2 = 1$, the right-hand side of inequality (4) is equal to

$$(3 + \gamma_3(H)) + (1 + \gamma_1(H)) + (2 \cdot 3 + (\gamma_2(\mathbb{G}) - 4) \cdot 1) = \gamma_3(H) + \gamma_1(H) + \gamma_2(\mathbb{G}) + 6;$$

if $p_1 = p_2 = 2$, it is equal to

$$(2 + \gamma_2(H)) + (2 + \gamma_2(H)) + (2 \cdot 2 + (\gamma_2(\mathbb{G}) - 4) \cdot 2) = 2\gamma_2(H) + 2\gamma_2(\mathbb{G}).$$

Here, $\gamma_2(\mathbb{G}) = 6$ if $k_1 \geq 4$; $\gamma_2(\mathbb{G}) = 5$ if $k_1 = 3$ by Lemma 8. Thus, if $k_1 \geq 4$, then

$$\begin{aligned} |F| &\geq \min\{\gamma_3(H) + \gamma_1(H) + \gamma_2(\mathbb{G}) + 6,\ 2\gamma_2(H) + 2\gamma_2(\mathbb{G})\} \\ &\geq \min\{(6n-17) + (2n-4) + 6 + 6,\ 2(4n-10) + 2 \cdot 6\} = 8n - 9 = \gamma_4(G). \end{aligned}$$

If $k_1 = 3$ & $k_2 \geq 4$, then

$$|F| \geq \min\{(6n-17) + (2n-4) + 5 + 6,\ 2(4n-10) + 2 \cdot 5\} = 8n - 10 = \gamma_4(G).$$

Finally, if $k_1 = k_2 = 3$, then

$$|F| \geq \min\{(6n-19) + (2n-4) + 5 + 6,\ 2(4n-11) + 2 \cdot 5\} = 8n - 12 = \gamma_4(G).$$

Case 3: $|\mathbb{W}| = 3$ ($q = 3$).
In this case, $p_1 = 2$ and $p_2 = p_3 = 1$. The the right-hand side of inequality (4) is equal to

$$(2 + \gamma_2(H)) + (1 + \gamma_1(H)) + (1 + \gamma_1(H)) + (1 \cdot 2 + (\gamma_3(\mathbb{G}) - 4) \cdot 1) = \gamma_2(H) + 2\gamma_1(H) + \gamma_3(\mathbb{G}) + 2,$$

where $\gamma_3(\mathbb{G})$ is, by Lemma 9, equal to 7, 6, and 5, respectively, if $k_1 \geq 4$, $k_1 = 3$ & $k_2 \geq 4$, and $k_1 = k_2 = 3$. Thus, if $k_1 \geq 4$, then $|F| \geq (4n-10)+2(2n-4)+7+2 = 8n-9 = \gamma_4(G)$; if $k_1 = 3$ & $k_2 \geq 4$, then $|F| \geq (4n-10)+2(2n-4)+6+2 = 8n - 10 = \gamma_4(G)$; if $k_1 = k_2 = 3$, then $|F| \geq (4n-11) + 2(2n-4) + 5 + 2 = 8n - 12 = \gamma_4(G)$. This completes the entire proof of this lemma. □

Lemma 17 achieves the target of proving $R_W \cap B \neq \emptyset$ for the case when $|\mathbb{Z}| \leq 3$. In this case, there exists at least one supernode X' in $\mathbb{W} \cap \bar{\mathbb{Z}}$. For every $x \in X' \setminus F$, we have $x \in R_W$ and $x \in B$, since B is defined to be $\bigcup_{X \in \bar{\mathbb{Z}}} X \setminus F$ and the subgraph of G induced by B is connected by Lemma 16(a). This implies $R_W \cap B \neq \emptyset$. Lemma 18 below deals with the remaining case when $|\mathbb{Z}| \geq 4$, where B is defined to be an arbitrary supernode $Y \in \mathbb{V}$ such that $Y \cap F = \emptyset$. (In fact, this lemma holds true if there exists a supernode $Y \in \mathbb{V}$ such that $Y \cap F = \emptyset$, whether or not $|\mathbb{Z}| \geq 4$.)

Lemma 18. *Let G be an n-dimensional torus $T(k_1, k_2, \ldots, k_n)$ where $n \geq 3$, and let F be an arbitrary subset of $V(G)$ such that $|F| < \gamma_4(G)$. Let $|\mathbb{Z}| \geq 4$.*
(a) For every supernode $X \in \mathbb{V} \setminus B$ such that $X \cap F = \emptyset$, there exists a path in $G \setminus F$ joining a vertex of X and a vertex of B.
(b) For every four-vertex subset W of $G \setminus F$, $R_W \cap B \neq \emptyset$.

Proof. We prove (a). Since the connectivity of $\mathbb{G}$ is four, there exist four pairwise internally vertex-disjoint paths $\mathbb{P}_j$, $1 \le j \le 4$, in $\mathbb{G}$ between X and B. From each $\mathbb{P}_j$, we can construct $|B|$ vertex-disjoint paths of G between X and B. Thus there are a total of $4|B|$ pairwise internally vertex-disjoint paths of G between X and B. Suppose that none of them is a path of $G \setminus F$. Then, $|F| \ge 4|B|$. If $k_1 \ge 4$ or $k_1 = 3$ & $k_2 \ge 4$, then $|F| \ge 4 \cdot 4^{n-2} = 4^{n-1} > 8n - 9 \ge \gamma_4(G)$, which is a contradiction. If $k_1 = k_2 = 3$, then $|F| \ge 4 \cdot 3^{n-2} \ge 8n - 12 = \gamma_4(G)$, which is also a contradiction. The proof of (a) is completed.

To prove (b), suppose $R_W \cap B = \emptyset$, i.e., $B \in \bar{\mathbb{W}}$, for some four-vertex subset W. Then, $X \in \bar{\mathbb{W}}$ for each neighboring supernode, X, of B, because $B \cap F = \emptyset$ and there is a perfect matching in $G\langle X \cup B\rangle$ between X and B. Thus, $|\bar{\mathbb{W}}| \ge 5$. Also, $|\mathbb{W}| \ge 4$ from Lemma 17. Then, by Lemma 15(d), there exists a matching of size four, $\{(X_j, Y_j) : 1 \le j \le 4\}$, between $\mathbb{W}$ and $\bar{\mathbb{W}}$, where $X_j \in \mathbb{W}$ and $Y_j \in \bar{\mathbb{W}}$ for each j. For $1 \le j \le 4$, we count the vertices of $(X_j \cup Y_j) \cap F$ in the same way as we did to derive inequality (1). Let x be a vertex in $X_j \cap R_W$. Because of $Y_j \in \bar{\mathbb{W}}$, the neighbor in Y_j of x should be a vertex of F, and for each neighbor y in X_j of x, we have $y \in F$ or $y' \in F$ where y' is the neighbor in Y_j of y. Therefore each $X_j \cup Y_j$ contains at least $1 + (2n - 4)$ vertices of F. Thus, $\bigcup_{1\le j\le 4}(X_j \cup Y_j)$ contains at least $8n - 12$ vertices of F in total.

We need to find a few more vertices of F outside $\bigcup_j(X_j \cup Y_j)$. Consider a neighboring supernode Y of $\{X_j : 1 \le j \le 4\}$, if any, such that $Y \ne Y_j$ for $1 \le j \le 4$. We claim $Y \cap F \ne \emptyset$. Suppose otherwise. Then, $Y \in \mathbb{W}$ since there is an edge (y, x) of G where $y \in Y$ and $x \in X_j \cap R_W$ for some j. However, $Y \in \bar{\mathbb{W}}$ by (a) of this lemma and the fact of $B \in \bar{\mathbb{W}}$. This is a contradiction. The claim is thus proved. The number of such supernodes Y is at least $\gamma_4(\mathbb{G}) - 4$, where $\gamma_4(\mathbb{G})$ depends on k_1 and k_2 as shown in Theorem 4. Therefore, $|F| \ge (8n - 12) + (\gamma_4(\mathbb{G}) - 4)$. If $k_1 \ge 4$, then $|F| \ge (8n-12)+(7-4) = 8n-9 = \gamma_4(G)$, which is a contradiction. If $k_1 = 3$ & $k_2 \ge 4$, then $|F| \ge (8n-12)+(6-4) = 8n - 10 = \gamma_4(G)$, which is a contradiction; if $k_1 = k_2 = 3$, then $|F| \ge (8n - 12) + (4 - 4) = 8n - 12 = \gamma_4(G)$, which is also a contradiction. Thus, the lemma is proved. □

The discussions so far about the 3-super-connectivity of a higher-dimensional torus network can be summarized as follows.

Theorem 8. *Let G be $T(k_1, k_2, \ldots, k_n)$ where $n \ge 3$. Then, $\kappa_s^3(G) = \gamma_4(G)$.*

Proof. In each of two cases, depending on whether $|\mathbb{Z}| \le 3$ or $|\mathbb{Z}| \ge 4$, we have $R_W \cap B \ne \emptyset$ for every four-vertex subset W of $G \setminus F$ by Lemmas 16, 17, and 18. Thus, the theorem follows. □

From Lemma 14 and Theorem 8, the 2- and 1-super-connectivities of higher-dimensional torus networks can be derived.

Theorem 9. *Let G be $T(k_1, k_2, \ldots, k_n)$ where $n \ge 3$. Then, $\kappa_s^2(G) = \gamma_3(G)$ and $\kappa_s^1(G) = \gamma_2(G)$.*

Proof. From Lemmas 8, 9 and Theorem 4, $\gamma_4(G) \ge \gamma_3(G) \ge \gamma_2(G)$. Thus, by Theorem 8 and Lemma 14, $\kappa_s^2(G) \ge \gamma_3(G)$ and $\kappa_s^1(G) \ge \gamma_2(G)$. To prove $\kappa_s^2(G) \le \gamma_3(G)$ and $\kappa_s^1(G) \le \gamma_2(G)$, the counting argument in the proof of Lemma 11 will be employed. For a minimum-neighborhood set S of order three of G where $|N_G(S)| = \gamma_3(G)$, we have $|V(G)| - (|S| + |N_G(S)|) = \prod_{j=1}^{n} k_j - (3 + \gamma_3(G)) \ge 3^n - (3 + (6n - 5)) \ge |S| = 3$, proving $\kappa_s^2(G) \le \gamma_3(G)$. For a minimum-neighborhood set S of order two of G where $|N_G(S)| = \gamma_2(G)$, we have $|V(G)| - (|S| + |N_G(S)|) = \prod_{j=1}^{n} k_j - (2 + \gamma_2(G)) \ge 3^n - (2 + (4n - 2)) \ge |S| = 2$, proving $\kappa_s^1(G) \le \gamma_2(G)$. □

4.3. Conditional Diagnosability under the PMC Model

Let G be a two or higher-dimensional torus network. In this subsection, it will be shown that the conditional diagnosability, $t_c(G)$, of G is equal to $\gamma'_4(G) + 1$, excluding the three small torus networks $T(3, 3)$, $T(3, 4)$, and $T(4, 4)$. For the exceptional torus networks, their conditional diagnosabilities turned out to be $\lceil |V(G)|/2\rceil - 1$. The general approach proposed in Section 3 will be applied to the analysis of $t_c(G)$ where Theorems 2 and 3 for the upper and lower bounds on $t_c(G)$ are used. All the graph invariants needed are ready from Sections 4.1 and 4.2: $\gamma_p(G)$ for $1 \le p \le 3$, $\gamma'_4(G)$, and $\kappa_s^3(G)$.

Theorem 10. *Let G be an n-dimensional torus $T(k_1, k_2, \ldots, k_n)$ where $n \ge 2$. Then,*

$$t_c(G) = \begin{cases} \lceil |V(G)|/2\rceil - 1 & \textit{if } (n, k_1, k_2) \in \{(2, 3, 3), (2, 3, 4), (2, 4, 4)\}, \\ \gamma'_4(G) + 1 & \textit{otherwise.} \end{cases}$$

PROOF. We prove the theorem in two cases.

Case 1: $G \in \{T(3,3), T(3,4), T(4,4)\}$.

To prove $t_c(G) \le \lceil |V(G)|/2 \rceil - 1$, it suffices to provide two conditional fault sets F_1, F_2 with size of at most $\lceil |V(G)|/2 \rceil$, which are indistinguishable, i.e., are consistent with some syndrome σ. Let $F_1 = V_1 \cup \{v_{2,1}, v_{2,2}\}$ and $F_2 = V(G) \setminus F_1$ if $G \in \{T(3,3), T(3,4)\}$; let $F_1 = V_1 \cup V_2$ and $F_2 = V(G) \setminus F_1$ if $G = T(4,4)$. Then, $|F_1|, |F_2| \le \lceil |V(G)|/2 \rceil$, and F_1 and F_2 are conditional sets. Consider a syndrome σ such that for each test (u, v),

$$\sigma(u,v) = \begin{cases} 0 & \text{if } u, v \in F_j \text{ for some } j \in \{1,2\}, \\ 1 & \text{otherwise.} \end{cases}$$

Then, F_1 and F_2 are both consistent with σ. Therefore, $t_c(G) \le \lceil |V(G)|/2 \rceil - 1$.

To show $t_c(G) \ge \lceil |V(G)|/2 \rceil - 1$, Theorem 3 will be applied for $q = 1$ and $t = \lceil |V(G)|/2 \rceil - 1$. It holds true that $t \le \gamma'_4(G) + 1$ since by Theorem 5, $\gamma'_4(T(3,3)) = 4$, $\gamma'_4(T(3,4)) = 6$, and $\gamma'_4(T(4,4)) = 8$. It remains to confirm that the three conditions of Theorem 3 are all satisfied. Condition (i) of $\lceil |V(G)|/2 \rceil \ge t + 1$ is obvious. Condition (ii) of $\gamma_p(G) \ge p$ for every $p \le 3$ can be verified easily from Lemmas 8 and 9. The last condition (iii) of $\kappa_s^3(G) \ge t - 2$ is valid, since $\kappa_s^3(T(3,3)) = 8$, $\kappa_s^3(T(3,4)) = 6$, and $\kappa_s^3(T(4,4)) = 6$ by Theorem 6. Therefore, $t_c(G) \ge t = \lceil |V(G)|/2 \rceil - 1$.

Case 2: $G \notin \{T(3,3), T(3,4), T(4,4)\}$.

Utilizing Theorem 2 and Remark 2, it will be proved that $t_c(G) \le \gamma'_4(G) + 1$ first. We recycle the minimum-neighborhood cycle of order four, $C = (x, y, z, w)$, of G defined in Section 4.1, where $x = v_{1,1,1,\ldots,1}$, $y = v_{1,2,1,\ldots,1}$, $z = v_{2,2,1,\ldots,1}$, and $w = v_{2,1,1,\ldots,1}$. Let $F_1 = N_G(V(C)) \cup \{x, y\}$ and $F_2 = N_G(V(C)) \cup \{z, w\}$. It remains to check that F_1 and F_2 are both conditional sets, i.e., $N_G(u) \nsubseteq F_1$ and $N_G(u) \nsubseteq F_2$ for every $u \in V(G)$. Imagine the two-dimensional skeleton $\mathbb{G}$ of G, where $x \in V_{1,1}$, $y \in V_{1,2}$, $z \in V_{2,2}$, and $w \in V_{2,1}$. Note that if $n = 2$, each supernode of $\mathbb{G}$ is a singleton, i.e., $V_{1,1} = \{x\}$, $V_{1,2} = \{y\}$, etc. Firstly, suppose $u \in V_{1,1} \cup V_{1,2} \cup V_{2,1} \cup V_{2,2}$, say, $u \in V_{1,1}$ without loss of generality. If $u = x$, then $y \notin F_2$, $w \notin F_1$, and $y, w \in N_G(u)$. If $u \ne x$, the neighbor in V_{1,k_2} of u is not included in $F_1 \cup F_2$. In either case, $N_G(u) \nsubseteq F_1$ and $N_G(u) \nsubseteq F_2$. Secondly, suppose $u \in V_{i,j}$ for some $(i, j) \notin \{(1,1), (1,2), (2,1), (2,2)\}$. If $n \ge 3$, then there are at least two vertices in $N_{G_{i,j}}(u)$. Moreover, at least one of them is not included in $F_1 \cup F_2$ since $|V_{i,j} \cap (F_1 \cup F_2)| \le 1$. If $n = 2$, then $k_2 \ge 5$ by the assumption of Case 2. Therefore there exists a neighboring supernode, $V_{i',j'}$, of $V_{i,j}$ such that $V_{i',j'} \cap (F_1 \cup F_2) = \emptyset$, which implies $u' \notin F_1 \cup F_2$ for the neighbor $u' \in V_{i',j'}$ of u. Thus, $N_G(u) \nsubseteq F_1$ and $N_G(u) \nsubseteq F_2$, proving that F_1 and F_2 are conditional sets.

To prove $t_c(G) \ge \gamma'_4(G) + 1$, Theorem 3 will be applied for $q = 1$ and $t = \gamma'_4(G) + 1$. It suffices to check that the three conditions of Theorem 3 are all satisfied. For condition (i), it will be shown that $\lceil |V(G)|/2 \rceil \ge \gamma'_4(G) + 2$ in four cases. If $n = 2$ & $k_1 \ge 4$ ($k_2 \ge 5$), then $\lceil |V(G)|/2 \rceil \ge \lceil (4 \cdot 5)/2 \rceil = 10 = (8n - 8) + 2 = \gamma'_4(G) + 2$. If $n = 2$ & $k_1 = 3$ ($k_2 \ge 5$), then $\lceil |V(G)|/2 \rceil \ge \lceil (3 \cdot 5)/2 \rceil = 8 = (8n - 10) + 2 = \gamma'_4(G) + 2$. If $n \ge 3$ & $(k_1, k_2) \ne (3,3)$, then $\lceil |V(G)|/2 \rceil \ge \lceil 3 \cdot 4^{n-1}/2 \rceil \ge (8n - 8) + 2 \ge \gamma'_4(G) + 2$. If $n \ge 3$ & $(k_1, k_2) = (3,3)$, then $\lceil |V(G)|/2 \rceil \ge \lceil 3^n/2 \rceil \ge (8n - 12) + 2 = \gamma'_4(G) + 2$. Thus, condition (i) is satisfied. Condition (ii) is obvious from Lemmas 8 and 9. Finally, regarding condition (iii), it will be shown that $\kappa_s^3(G) \ge t - 2 = \gamma'_4(G) - 1$. Since $\kappa_s^3(G) = \gamma_4(G)$ by Theorems 6 and 8, it suffices to show $\gamma_4(G) \ge \gamma'_4(G) - 1$, which is a direct consequence of Theorems 4 and 5. This completes the entire proof. □

Corollary 1. *Let G be an n-dimensional torus $T(k_1, \ldots, k_n)$ where $n \ge 2$. Then,*

$$t_c(G) = \begin{cases} 4 & \text{if } (n, k_1, k_2) = (2,3,3), \\ 5 & \text{if } (n, k_1, k_2) = (2,3,4), \\ 7 & \text{if } (n, k_1, k_2) = (2,4,4), \\ 8n - 7 & \text{if } k_1 \ge 4,\ (n, k_1, k_2) \ne (2,4,4), \\ 8n - 9 & \text{if } k_1 = 3 \text{ \& } k_2 \ge 4,\ (n, k_1, k_2) \ne (2,3,4), \\ 8n - 11 & \text{if } k_1 = k_2 = 3,\ (n, k_1, k_2) \ne (2,3,3). \end{cases}$$

A *k-ary n-cube* is defined as a Cartesian product of n cycles of length k, $C_k \times C_k \times \cdots \times C_k$. Directly from Corollary 1, the conditional diagnosability of a k-ary n-cube for every possible n and k can be obtained, as shown below. This is an extension of the work of Chang *et al.* [5], where the conditional diagnosability of a k-ary n-cube for $n, k \ge 4$ was determined.

Corollary 2. *Let G be a k-ary n-cube where $n \geq 2$ and $k \geq 3$. Then,*

$$t_c(G) = \begin{cases} 4 & \textit{if } n = 2 \ \& \ k = 3, \\ 7 & \textit{if } n = 2 \ \& \ k = 4, \\ 9 & \textit{if } n = 2 \ \& \ k \geq 5, \\ 8n - 7 & \textit{if } n \geq 3 \ \& \ k \geq 4, \\ 8n - 11 & \textit{if } n \geq 3 \ \& \ k = 3. \end{cases}$$

5. Conclusion

A general technique was suggested for finding the conditional diagnosability of interconnection networks under the PMC model. This technique is based on several graph invariants, including the girth, the size of the minimum-neighborhood set/cycle of order p for some p, and the r-super-connectivity for some r. More specifically, to determine the conditional diagnosability of a graph G of girth $g(G)$, where $4q - 4 < g(G) \leq 4q$ for some integer q, we need to analyze $\gamma_p(G)$ for $1 \leq p < 4q$, $\gamma'_{4q}(G)$, and the $(4q-1)$-super-connectivity of G, $\kappa_s^{4q-1}(G)$. The proposed technique was applied to two or higher-dimensional torus networks, and their conditional diagnosabilities as well as the aforementioned graph invariants were completely established without exception. This technique is expected to be applicable to many interconnection graphs, especially those whose girths are not too big, so as to determine their conditional diagnosabilities. Fortunately, the girth of an interconnection graph is usually small if it possesses a recursive structure or it can be defined recursively.

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