Conditional Matching Preclusion for Hypercube-Like Interconnection Networks☆,☆☆

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Abstract

The *conditional matching preclusion number* of a graph with n vertices is the minimum number of edges whose deletion results in a graph without an isolated vertex that does not have a perfect matching if n is even, or an almost perfect matching if n is odd. We develop some general properties on conditional matching preclusion and then analyze the conditional matching preclusion numbers for some HL-graphs, hypercube-like interconnection networks.

Key words: Perfect matching, almost perfect matching, fault tolerance, edge fault, conditional fault, HL-graphs, restricted HL-graphs

1. Introduction

Given a graph G, a matching M in G is a set of pairwise non-adjacent edges; that is, no two edges share a common vertex. We say that a vertex is *matched* if it is incident to an edge in the matching. Otherwise the vertex is *unmatched*. A matching M of G with n vertices is called a *perfect matching* and an *almost perfect matching* if its size $|M|$ is equal to $n/2$ and $(n-1)/2$, respectively. A set F of edges in G is called a *matching preclusion set* if

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$G \backslash F$ has neither a perfect matching nor an almost perfect matching. The matching preclusion number of G, denoted by $mp(G)$, is the cardinality of a minimum matching preclusion set in G. If G has neither a perfect matching nor an almost perfect matching, then $mp(G) = 0$.

The matching preclusion problem was introduced by Brigham *et al.* in [1], and its application and related problems were addressed as follows. If $mp(G)$ is large, networks for which it is essential to have each node possess at any time a special partner will be robust in the event of link failures. Furthermore, the problem is related to two areas of study initiated by Harary: 'general and conditional connectivity' and 'changing and unchanging of invariants.' For details and references, refer to [1].

The matching preclusion numbers and the minimum matching preclusion sets were characterized for Petersen graph, complete graphs, complete bipartite graphs, and hypercubes in [1]. Cheng *et al.* in [3] found matching preclusion numbers and classified all the minimum matching preclusion sets for Cayley graphs generated by transpositions and (n, k)-star graphs. The same works for hypercube-like interconnection networks such as restricted HL-graphs and recursive circulant $G(2^m, 4)$ were done by Park in [6].

In a graph G with even number of vertices, the set of all edges incident to a single vertex forms a matching preclusion set, and thus $mp(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G. In the event of a random link failure, it is very unlikely that all of the links incident to a single vertex fail simultaneously. According to this motivation, Cheng *et al.* in [4] defined the *conditional matching preclusion number* of a graph G, denoted by $mp_1(G)$, as the minimum number of edges whose deletion leaves the resulting graph with no isolated vertices and without a perfect matching or an almost perfect matching. It was defined $mp_1(G) = 0$ if G has neither a perfect matching nor an almost perfect matching, or if G has no conditional matching preclusion set.

The conditional matching preclusion numbers and the minimum conditional matching preclusion sets for complete graphs, complete bipartite graphs, and hypercubes were studied in [4]. In this paper, we will develop some general properties on (conditional) matching preclusion and then analyze the conditional matching preclusion numbers for some HL-graphs [8], a class of hypercube-like interconnection networks. We will use standard terminology in graphs (see [2]). Throughout the paper, we deal with graphs having nonempty conditional matching preclusion sets, that is, graphs whose conditional matching preclusion numbers are nonzero.

When we are concerned with existence of a perfect matching or an almost perfect matching in $G \backslash F$ with some edge set F deleted from G, we will refer to the edge set F as an *edge fault set* or just as a *fault set* hereafter. Furthermore, if F does not contain all the edges incident to a single vertex, then F is said to be a *conditional* edge fault set or a *conditional* fault set. A conditional fault set F will be a conditional matching preclusion set if $G \backslash F$ has neither a perfect matching nor an almost perfect matching.

The *length* of a path refers to the number of vertices in the path. A path is called an *even path* if its length is even. Otherwise, it is called an *odd path*. We begin with a matching from a different standpoint. The matching, which is a set of pairwise non-adjacent edges, can be defined as a set of pairwise (vertex-)disjoint paths of length two. Furthermore, in view of vertex partition of a graph, the matching can be considered as a partition of the graph into pairwise disjoint paths having lengths of either two or one. Of course, an unmatched vertex corresponds to a path of length one.

We can observe that if a graph can be partitioned into all even paths, then the even paths can be further partitioned into paths of length two and thus the graph has a perfect matching. In a similar way, if a graph can be partitioned into even paths with only one exceptional odd path, then it has an almost perfect matching. For any edge fault set F with $|F| \leq f$ in a graph G, if the resultant graph $G \backslash F$ can be partitioned into even paths with at most one exceptional odd path, then the matching preclusion number of G is at least $f+1$. If all the fault sets F are taken from conditional fault sets, we can say that the conditional matching preclusion number of G is at least $f+1$. It can be summarized as follows.

Proposition 1. *For any fault set (resp. conditional fault set) F with $|F| \leq f$ in a graph G, if $G \backslash F$ can be partitioned into even paths with at most one exceptional odd path, then $G \backslash F$ has a perfect matching or an almost perfect matching and $mp(G) \geq f+1$ (resp. $mp_1(G) \geq f+1$).*

It was observed in [4] that a basic obstruction to a perfect matching under conditional fault situation in a graph with an even number of vertices is the existence of a path (u, w, v) of length three where the degrees of u and v are both one. This observation directly leads to the following proposition. For basic obstructions to an almost perfect matching in a graph with an odd number of vertices, refer to [4].

Proposition 2. *[4] Let G be a graph with an even number of vertices. Suppose every vertex in G has degree at least three. Then $mp_1(G)$ is at most the minimum of $d(u)+d(v)-2-g(u,v)$ over all pairs of vertices u and v joined by a path of length three, where $d(\cdot)$ is the degree function and $g(u,v)=1$ if u and v are adjacent and 0 otherwise.*

An *independent set* of a graph G is a set of pairwise non-adjacent vertices. The *independence number* $\alpha(G)$ of G is the size of a largest independent set of G. Obviously, it holds that if a graph G with n vertices has a perfect matching or an almost perfect matching, then $\alpha(G) \leq \lceil n/2 \rceil$. It can be used to obtain an upper bound on the matching preclusion number in such a way that for some fault set F, if the independence number of $G \backslash F$ is greater than $\lceil n/2 \rceil$, then $G \backslash F$ has no (almost) perfect matching, and thus the matching preclusion number of G is at most the cardinality of F. Similarly, we can also get an upper bound on the conditional matching preclusion number.

Proposition 3. *For some fault set F of a graph G with n vertices, if the independence number $\alpha(G \backslash F) > \lceil n/2 \rceil$, then F is a matching preclusion set and $mp(G) \leq |F|$. Furthermore, if F is a conditional fault set, then F is a conditional matching preclusion set and $mp_1(G) \leq |F|$.*

In the next section, we will investigate conditional matching preclusion for hypercube-like interconnection networks, especially restricted HL-graphs and bipartite HL-graphs. Concluding remarks on our problem for general HL-graphs will be addressed in Section 3.

2. Hypercube-Like Interconnection Networks

Given two graphs G_0 and G_1 with n vertices each, we denote by V_j and E_j the vertex set and edge set of G_j, $j = 0, 1$, respectively. Let $V_0 = \{v_1, v_2, \ldots, v_n\}$ and $V_1 = \{w_1, w_2, \ldots, w_n\}$. With respect to a permutation $P = (i_1, i_2, \ldots, i_n)$ of $\{1, 2, \ldots, n\}$, we can "merge" the two graphs into a graph $G_0 \oplus_P G_1$ with $2n$ vertices in such a way that the vertex set $V = V_0 \cup V_1$ and the edge set $E = E_0 \cup E_1 \cup E_2$, where $E_2 = \{(v_j, w_{i_j}) | 1 \leq j \leq n\}$. We denote by $G_0 \oplus G_1$ a graph obtained by merging G_0 and G_1 w.r.t. an arbitrary permutation P. Here, G_0 and G_1 are called *components* of $G_0 \oplus G_1$.

Vaidya *et al.* [8] introduced a class of hypercube-like interconnection networks, called *HL-graphs*, which can be defined by applying the $\oplus$ operation

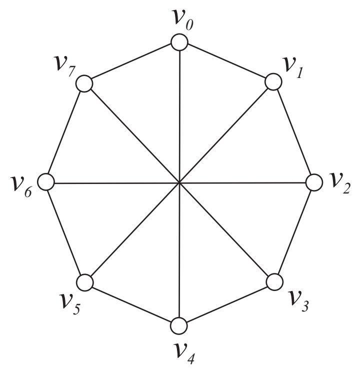


Figure 1: Recursive circulant $G(8,4)$.

repeatedly as follows: $HL_0 = \{K_1\}$; for $m \geq 1$, $HL_m = \{G_0 \oplus G_1 | G_0, G_1 \in HL_{m-1}\}$. Then, $HL_1 = \{K_2\}$; $HL_2 = \{C_4\}$; $HL_3 = \{Q_3,\ G(8,4)\}$. Here, C_4 is a cycle graph with 4 vertices, Q_3 is a 3-dimensional hypercube, and $G(8,4)$ is a recursive circulant shown in Figure 1, which is defined as follows: the vertex set is $\{v_i | 0 \leq i \leq 7\}$ and the edge set is $\{(v_i, v_j) | i+1$ or $i+4 \equiv j \pmod 8\}$. An arbitrary graph which belongs to HL_m is called an *m-dimensional HL-graph.*

Definition 1. A graph G is said to be *f-edge-fault perfectly matchable* if for any edge fault set F with $|F| \leq f$, $G \backslash F$ has a perfect matching. A graph G is said to be *conditional f-edge-fault perfectly matchable* if for any conditional edge fault set F with $|F| \leq f$, $G \backslash F$ has a perfect matching.

By the definition of an m-dimensional HL-graph, its edge set can be partitioned into m subsets, where each subset forms a perfect matching. This leads to the following proposition.

Proposition 4. *(a) Every m-dimensional HL-graph is $m-1$-edge-fault perfectly matchable. Its matching preclusion number is equal to the degree m.*
(b) Every m-dimensional HL-graph is conditional $m-1$-edge-fault perfectly matchable. Its conditional matching preclusion number is at least m.

Throughout this paper, a path in a graph is represented as a sequence of vertices. For a vertex v in $G_0 \oplus G_1$, we denote by $\bar{v}$ the *mate* of v, the vertex adjacent to v which is in a component different from the component in which v is contained. Let F be the set of faulty edges in $G_0 \oplus G_1$. F_0 and F_1 denote the sets of faulty edges in G_0 and G_1, respectively, and F_2 denotes the set of faulty edges joining vertices in G_0 and vertices in G_1, so that $F = F_0 \cup F_1 \cup F_2$. Let $f_0 = |F_0|$, $f_1 = |F_1|$, and $f_2 = |F_2|$.

2.1. Restricted HL-graphs

In [7], a subclass of nonbipartite HL-graphs, called *restricted HL-graphs*, was introduced and defined recursively as follows: $RHL_m = HL_m$ for $0 \leq m \leq 2$; $RHL_3 = HL_3 \backslash Q_3 = \{G(8,4)\}$; $RHL_m = \{G_0 \oplus G_1 | G_0, G_1 \in RHL_{m-1}\}$ for $m \geq 4$. A graph which belongs to RHL_m is called an *m-dimensional restricted HL-graph.* Many of the nonbipartite hypercube-like interconnection networks such as crossed cube, Möbius cube, twisted cube, multiply twisted cube, Mcube, generalized twisted cube, etc. proposed in the literature are restricted HL-graphs.

A graph G is called *f-fault hamiltonian* (resp. *f-fault hamiltonian-connected*) if there exists a hamiltonian cycle (resp. if each pair of vertices are joined by a hamiltonian path) in $G \backslash F$ for any set F of faulty elements (vertices and/or edges) with $|F| \leq f$. Fault-hamiltonicity of restricted HL-graphs was studied in [7] as follows.

Lemma 1. *[7] Every m-dimensional restricted HL-graph, $m \geq 3$, is $m-3$-fault hamiltonian-connected and $m-2$-fault hamiltonian.*

In this subsection, we will show that the conditional matching preclusion numbers of m-dimensional restricted HL-graphs are all $2m-2$ if $m \geq 5$, and will characterize 4-dimensional restricted HL-graphs whose conditional matching preclusion numbers are 6. We begin with conditional matching preclusion of the 3-dimensional restricted HL-graph $G(8,4)$ shown in Figure 1.

Lemma 2. $mp_1(G(8,4)) = 3$. *Furthermore, all of the eight minimum conditional matching preclusion sets are symmetric to* $\{(v_0, v_4), (v_0, v_1), (v_3, v_4)\}$.

Proof. It was shown in [6] that the minimum matching preclusion sets of $G(8,4)$ are either the sets of edges incident to a single vertex or the sets symmetric to $\{(v_0, v_4), (v_0, v_1), (v_3, v_4)\}$. The latter are conditional, and thus we have the lemma. □

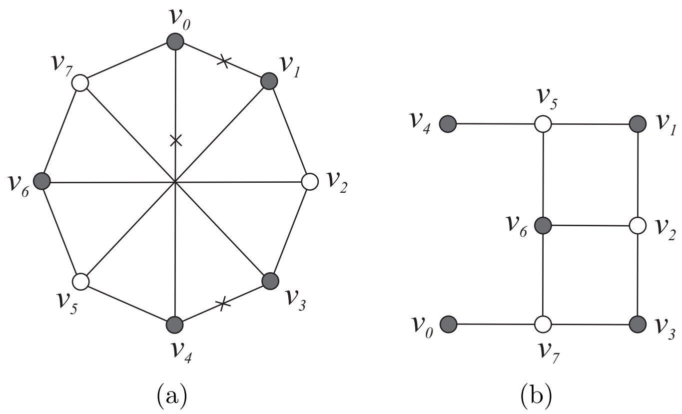


Figure 2: $G(8,4)$ with the minimum conditional matching preclusion set.

The graph $G(8,4)$ with the minimum conditional matching preclusion set $F = \{(v_0, v_4), (v_0, v_1), (v_3, v_4)\}$ is shown in Figure 2 (a) and (b). The symbol $\times$ on an edge in figure (a) indicates the edge is faulty, and the faulty edges are not shown in figure (b). $G(8,4)\backslash F$ becomes a bipartite graph with a set $\{v_0, v_1, v_3, v_4, v_6\}$ of five black vertices and a set $\{v_2, v_5, v_7\}$ of three white vertices as shown in the figure. It is straightforward to check that if we remove an arbitrary pair of black vertices in $G(8,4)\backslash F$, then the resultant graph always has a perfect matching.

Now, we investigate conditional matching preclusion sets of 4-dimensional restricted HL-graphs $G(8,4) \oplus G(8,4)$ and of higher dimensional restricted HL-graphs.

Theorem 1. *(a) Every* 4*-dimensional restricted HL-graph with a conditional fault set* F *with* $|F| \leq 5$ *has a perfect matching unless one component contains three faulty edges forming a conditional matching preclusion set of the component, the other component contains one faulty edge, and there is a single faulty edge between the two components.*
(b) Every m*-dimensional restricted HL-graph with* $m \geq 5$ *is conditional* $2m-3$*-edge-fault perfectly matchable.*

Proof. We let G be an m-dimensional restricted HL-graph with $m \geq 4$, which is isomorphic to $G_0 \oplus G_1$ for some $m-1$-dimensional restricted HL-graphs G_0 and G_1. The proof is by induction on m. Let F be a conditional fault set of size at most $2m-3$. It suffices to consider the case $|F| = 2m-3$.

If $f_2 = 0$, we are done since the set of edges joining $V(G_0)$ and $V(G_1)$ forms a perfect matching. Hereafter in this proof, we assume $f_2 \geq 1$. Furthermore, we assume w.l.o.g. $f_0 \geq f_1$. Then, we have $f_1 \leq m-2$ and F_1 is a conditional fault set of G_1. There are two cases.

Case 1: $f_0 \leq 2m - 5$.
Let us first consider the subcase when there exists a vertex x in G_0 such that all the edges in G_0 incident to x are faulty. We have $f_0 \geq m - 1$ and $f_1 \leq m-3$. Let $F'_0 = F_0 \cup \{x\} \setminus \{(x, v) | v \in V(G_0)\}$. Since $|F'_0| \leq (2m-5)+1-(m-1) = m-3$, by Lemma 1, there exists a hamiltonian cycle C_0 in $G_0 \setminus F'_0$. Moreover, $G_1 \setminus F_1$ also has a hamiltonian cycle C_1. Let the hamiltonian cycle C_1 be $(w_1, w_2, \ldots, w_{2^{m-1}})$ with $w_1 = \bar{x}$. There exists a vertex w_{2i}, $1 \leq i \leq 2^{m-2}$, such that $(w_{2i}, \bar{w_{2i}})$ is fault-free. The existence is due to the fact that there are 2^{m-2} candidates and at most $m - 2$ blocking elements (f_2 faulty edges). Note that $2^{m-2} > m - 2$ for any $m \geq 4$. Then, we have two even paths $(x, w_1, w_2, \ldots, w_{2i-1})$ of length $2i$ and $(C_0, w_{2i}, w_{2i+1}, \ldots, w_{2^{m-1}})$ of length $2^m - 2i$, which partition $V(G)$. Here, $(x, \bar{x})$ is fault-free since F is a conditional fault set. Thus, by Proposition 1, $G \setminus F$ has a perfect matching.

Now, we assume that no such vertex x exists in G_0, which implies that F_0 is a conditional fault set of G_0. Notice that F_1 is also a conditional fault set of G_1. If either $m \geq 6$ or $m = 5$ and G_0 (which is a 4-dimensional restricted HL-graph) with F_0 satisfies the sufficiency of (a), then, by induction hypothesis, $G_0 \setminus F_0$ and $G_1 \setminus F_1$ have perfect matchings M_0 and M_1, respectively. The union $M_0 \cup M_1$ is a desired perfect matching.

Let $m = 4$ first. If either $f_0 \leq 2$ or $f_0 = 3$ and F_0 is not a conditional matching preclusion set of G_0, then both G_0 and G_1 have perfect matchings and we are done. Assume $f_0 = 3$ and F_0 is a conditional matching preclusion set of G_0 as shown in Figure 2. We assume w.l.o.g. $F_0 = \{(v_0, v_4), (v_0, v_1), (v_3, v_4)\}$. Remember that if we delete an arbitrary pair of black vertices in G_0, the resultant graph has a perfect matching. If $f_1 = 0$, for some two black vertices x and y in G_0 such that $(x, \bar{x})$ and $(y, \bar{y})$ are fault-free, we have an even path (x, P_1, y), where P_1 is a hamiltonian path in G_1 joining $\bar{x}$ and $\bar{y}$. By Lemma 1, P_1 exists. $G_0 \setminus (F_0 \cup \{x, y\})$ has a perfect matching, and thus a perfect matching of $G \setminus F$ can be finished by dividing the even path into paths of length two. This completes the construction of perfect matchings when G with F satisfies the sufficiency of (a).

Finally, we assume $m = 5$ and G_0 with F_0 does not satisfy the sufficiency of (a). Note that G_0 is a 4-dimensional restricted HL-graph and $G_0 \setminus F_0$ may not have a perfect matching. Let G_0 be isomorphic to $G_{00} \oplus G_{01}$, where

G_{00} and G_{01} are 3-dimensional restricted HL-graphs. We assume that G_{00} has three faulty edges which form a conditional matching preclusion set of G_{00}. The construction of a perfect matching can be obtained similar to the previous case $m = 4$. There exist two black vertices x and y in G_{00} such that $(x, \bar{x})$ and $(y, \bar{y})$ are fault-free. Excluding x and y, $G_{00} \backslash F_0$ has a perfect matching. $G_{01} \backslash F_0$ also has a perfect matching. The union of two perfect matchings forms a perfecting matching of $G_0 \backslash (F_0 \cup \{x, y\})$. Since $f_1 \leq 1$, there exists a hamiltonian path P_1 in $G_1 \backslash F_1$ joining $\bar{x}$ and $\bar{y}$. The even path (x, P_1, y) can be partitioned into paths of length two, thus the construction is completed.

Case 2: $f_0 = 2m - 4$ and $f_2 = 1$ ($f_1 = 0$).
We are to pick up a faulty edge (x, y) in G_0 which satisfies the following two conditions simultaneously:

(i) If (x, y) is regarded as a *virtual* fault-free edge, $G_0 \backslash F_0$ has a perfect matching. In precise words, $G_0 \backslash F_0'$ has a perfect matching, where $F_0' = F_0 \backslash (x, y)$.
(ii) Both $(x, \bar{x})$ and $(y, \bar{y})$ are fault-free.

If such a faulty edge (x, y) exists, a perfect matching in $G \backslash F$ can be constructed in a simple manner as follows. When (x, y) is not contained in the perfect matching M_0 of $G_0 \backslash F_0'$, the union of M_0 and a perfect matching M_1 of G_1 will do. Otherwise, we construct an even path (x, P_1, y), where P_1 is a hamiltonian path in G_1 joining $\bar{x}$ and $\bar{y}$, and then divide it into a set M' of pairwise disjoint paths of length two. Obviously, $(M_0 \backslash (x, y)) \cup M'$ is a desired perfect matching.

It remains to show that there exists a faulty edge (x, y) which satisfies both conditions (i) and (ii). Let $m \geq 6$ first. If there exists a vertex z such that all the edges in G_0 incident to z are faulty, let (x, y) be an edge incident to z which satisfies the condition (ii). The edge (x, y) exists since $f_2 = 1$ and $(z, \bar{z})$ is fault-free. Remember F is a conditional fault set. If no such vertex z exists, let (x, y) be an arbitrary faulty edge satisfying the condition (ii). Then, letting $F_0' = F_0 \backslash (x, y)$, F_0' is a conditional fault set in G_0 of size $2m - 5$. Notice that every vertex other than z has a fault-free edge incident to it; suppose otherwise, f_0 should be at least $2m - 3$, which is a contradiction. By induction hypothesis, $G_0 \backslash F_0'$ has a perfect matching and thus the condition (i) is also satisfied.

For $m = 4$ or 5, it is not sufficient to show that F_0' is a conditional fault set. Let $m = 4$ ($f_0 = 4$) now. If there exists a vertex z such that all the edges in G_0 incident to z are faulty, say $z = v_0$ and (v_0, v_1), (v_0, v_4), and (v_0, v_7) are faulty, let

$$(x, y) = \begin{cases} (v_0, v_4) & \text{if } (v_4, \bar{v_4}) \text{ is fault-free;} \\ (v_0, v_1) & \text{if } (v_4, \bar{v_4}) \text{ is faulty and } (v_3, v_4) \text{ is faulty;} \\ (v_0, v_7) & \text{otherwise.} \end{cases}$$

Then $F_0' = F_0 \backslash (x, y)$ is a conditional fault set and not a conditional matching preclusion set. Thus, $G_0 \backslash F_0'$ has a perfect matching and the two conditions are satisfied. When there exists no such vertex z, we claim that among the four subsets with cardinality three of F_0, at most one is a conditional matching preclusion set of G_0. The proof is direct from the fact that $f_0 = 4$ and any two conditional matching preclusion sets of cardinality three share at most one edge. If there exists a subset forming a conditional matching preclusion set, then let (x, y) be an edge in the subset satisfying the condition (ii); otherwise, let (x, y) be an arbitrary faulty edge satisfying the condition (ii). Then, (x, y) is a faulty edge satisfying both conditions (i) and (ii).

Finally, let $m = 5$ ($f_0 = 6$). Let G_0 be isomorphic to $G_{00} \oplus G_{01}$ for 3-dimensional restricted HL-graphs G_{00} and G_{01}. Let f_{00} and f_{01} denote the numbers of faulty edges in G_{00} and G_{01}, respectively. Assume w.l.o.g. $f_{00} \geq f_{01}$. If $f_{00} = 4$, we pick up a faulty edge (x, y) in G_{00} for two subcases depending on whether or not there exists a vertex z such that all the edges in G_0 incident to z are faulty, in the same way as the above case $m = 4$ so that, letting F_{00} be the set of faulty edges in G_{00}, $F_{00} \backslash (x, y)$ is a conditional fault set and not a conditional matching preclusion set of G_{00}. Obviously, $F_0' = F_0 \backslash (x, y)$ is a conditional fault set of G_0, too. Thus, $G_0 \backslash F_0'$ has a perfect matching. Hereafter in this proof, we assume $f_{00} \neq 4$. If there exists a vertex z (in G_{00}) such that all the edges in G_0 incident to z are faulty, we pick up an edge (x, y) incident to z which satisfies the condition (ii). Then, $F_0' = F_0 \backslash (x, y)$ is a conditional fault set of G_0. Moreover, it is straightforward to check that G_0 with fault set F_0' satisfies the sufficiency of (a). By induction hypothesis, $G_0 \backslash F_0'$ has a perfect matching. We assume no such vertex z exists from now on, and thus we need not check if F_0' is a conditional fault set. If $f_{00} = 3$ and F_{00} forms a conditional matching preclusion set of G_{00}, we pick up a faulty edge (x, y) in G_{00} satisfying the condition (ii). For all the other cases, we pick up an arbitrary faulty edge (x, y) satisfying the condition (ii). It is easy to see that G_0 with fault set F_0' satisfies the sufficiency of (a). Thus, $G_0 \backslash F_0'$ has a perfect matching. This completes the proof. □

Due to Proposition 2 and the fact that no HL-graph contains a cycle of length three, we have the following.

Corollary 1. *For any m-dimensional restricted HL-graph G with $m \geq 5$, $mp_1(G) = 2m - 2$.*

It would be a natural question to ask if the sufficient condition given in Theorem 1(a) is also a necessary one. The rest of this subsection is devoted to characterizing the minimum conditional matching preclusion sets of 4-dimensional restricted HL-graphs $G(8,4) \oplus G(8,4)$. As a result, it will be noticed later that some 4-dimensional restricted HL-graphs have conditional matching preclusion number 6 while the others have 5.

We begin with a hamiltonian property of $G(8,4)$ with a single faulty edge, which will be utilized later.

Lemma 3. *For any single edge fault (x,y) in $G(8,4)$, $G(8,4)\backslash(x,y)$ has a hamiltonian path between every pair of vertices $s \in \{x,y\}$ and $t(\neq s)$.*

Proof. The proof is by an immediate inspection. □

Let G be a 4-dimensional restricted HL-graph isomorphic to $G_0 \oplus G_1$, where G_0 and G_1 are isomorphic to $G(8,4)$. To represent which component a vertex is contained in, we assume $V(G_0) = \{v_0, v_1, \ldots, v_7\}$ and $V(G_1) = \{w_0, w_1, \ldots, w_7\}$. Furthermore, we assume that v_i is adjacent to v_{i+1} and v_{i+4}, and w_i is adjacent to w_{i+1} and w_{i+4} for every $0 \leq i < 8$. Here, all arithmetic on the indices of vertices will be assumed to be done modulo 8.

We assume that G with a conditional fault set F of cardinality five does not satisfy the sufficiency of Theorem 1(a), that is, F_0 is a minimum conditional matching preclusion set of G_0, $f_1 = 1$, and $f_2 = 1$. Without loss of generality, let $F_0 = \{(v_0, v_4), (v_0, v_1), (v_3, v_4)\}$. We denote by B_0 the set of five black vertices $\{v_0, v_1, v_3, v_4, v_6\}$ in G_0 and by W_0 the set of three white vertices $\{v_2, v_5, v_7\}$ as shown in Figure 2. Remember that for any pair of black vertices x and y in G_0, $G_0\backslash(F_0 \cup \{x,y\})$ has a perfect matching. Let us consider the case first when the faulty edge in G_1 is a *diagonal* edge (w_i, w_{i+4}) for some i, say (w_0, w_4).

Lemma 4. *If $F_1 = \{(w_0, w_4)\}$, $G\backslash F$ is perfectly matchable.*

Proof. First, if for some black vertex x in G_0, $(x, \bar{x})$ is fault-free and $\bar{x}$ is either w_0 or w_4, then for some black vertex y in G_0 such that $(y, \bar{y})$ is fault-free, there exists a hamiltonian path P_1 in $G_1 \backslash F_1$ joining $\bar{x}$ and $\bar{y}$ by Lemma 3. From a perfect matching in $G_0 \backslash (F_0 \cup \{x, y\})$ and an even path (x, P_1, y), a perfect matching in $G \backslash F$ can be obtained. Second, if for some pair of black vertices x and y, both $(x, \bar{x})$ and $(y, \bar{y})$ are fault-free and $\bar{x} = w_i$ and $\bar{y} = w_{i+1}$ for some $0 \leq i < 8$, then a perfect matching of $G \backslash F$ can be constructed similarly by using a hamiltonian path P_1 in $G_1 \backslash F_1$ joining w_i and w_{i+1}. Notice that P_1 is obtained from a hamiltonian cycle $(w_0, w_1, \ldots, w_7)$ by deleting an edge (w_i, w_{i+1}). Finally, for the remaining case, there exists a black vertex x such that $(x, \bar{x})$ is faulty, and $\{\bar{y} | y \in B_0 \backslash x\} = \{w_1, w_3, w_5, w_7\}$. We observe that $G_1 \backslash (F_1 \cup \{w_1, w_3\})$ has a perfect matching $M_1 = \{(w_0, w_7), (w_2, w_6), (w_4, w_5)\}$. Then, letting M_0 be a perfect matching in $G_0 \backslash (F_0 \cup \{\bar{w_1}, \bar{w_3}\})$, the union $M_0 \cup M_1 \cup \{(w_1, \bar{w_1}), (w_3, \bar{w_3})\}$ is a perfect matching of $G \backslash F$. Therefore, we conclude that $G \backslash F$ is perfectly matchable. □

Now, let the faulty edge in G_1 be a *boundary* edge (w_i, w_{i+1}) for some i, say (w_0, w_7).

Lemma 5. *If $F_1 = \{(w_0, w_7)\}$, $G \backslash F$ is perfectly matchable unless there exists a black vertex x in G_0 such that $(x, \bar{x})$ is faulty and $\{\bar{y} | y \in B_0 \backslash x\} = \{w_1, w_3, w_4, w_6\}$.*

Proof. If there exists a black vertex x such that $(x, \bar{x})$ is fault-free and $\bar{x}$ is either w_0 or w_7, then there exists a hamiltonian path in $G_1 \backslash F_1$ from $\bar{x}$ to any other vertex by Lemma 3. In a very similar way to the first case of Lemma 4, we can construct a perfect matching in $G \backslash F$. Suppose otherwise. There exists a hamiltonian cycle $C_1 = (w_0, w_1, w_2, w_3, w_7, w_6, w_5, w_4)$ in $G_1 \backslash F_1$, and thus between any pair of vertices a and b such that (a, b) is an edge of C_1, there exists a hamiltonian path in $G_1 \backslash F_1$ joining the pair. If there exists a pair of black vertices x and y in G_0 such that both $(x, \bar{x})$ and $(y, \bar{y})$ are fault-free and $(\bar{x}, \bar{y})$ is an edge of C_1, then we have an even path (x, P_1, y), where $P_1 = C_1 \backslash (\bar{x}, \bar{y})$. Thus, a perfect matching can be obtained from a perfect matching of $G_0 \backslash (F_0 \cup \{x, y\})$ and the even path. It remains the case exactly when the sufficiency of the lemma is not satisfied. Thus, the proof is completed. □

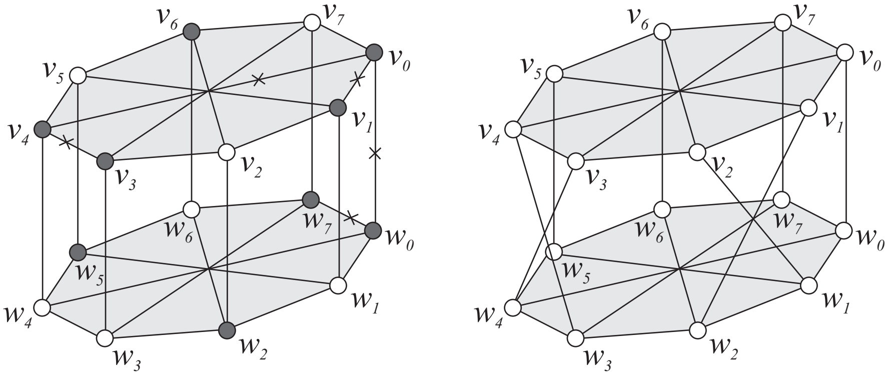
Suppose $F_1 = \{(w_0, w_7)\}$ and the sufficiency of Lemma 5 is not satisfied. For convenience, we will refer to the vertices $\{w_1, w_3, w_4, w_6\}$ as white vertices and the vertices $\{w_0, w_2, w_5, w_7\}$ as black vertices. Then, $G_1 \backslash F_1$ has a unique edge joining vertices of the same color, (w_3, w_4). Since $\{\bar{y} | y \in B_0 \backslash x\} = \{w_1, w_3, w_4, w_6\}$ for the unique black vertex x in G_0 such that $(x, \bar{x})$ is faulty, we have $\{\bar{z} | z \in W_0\} \subset \{w_0, w_2, w_5, w_7\}$. Thus, all the fault-free edges between G_0 and G_1 join pairs of vertices with different colors each other. Therefore, $G \backslash (F \cup \{(w_3, w_4)\})$ is a bipartite graph. The set of black vertices in $G \backslash F$ forms an independent set of size nine, which implies, by Proposition 3, $G \backslash F$ has no perfect matching. Eventually, we reach a necessary and sufficient condition. It is summarized in the following.

Lemma 6. *Given a 4-dimensional restricted HL-graph G and a conditional fault set F of G with $|F| \leq 5$, $G \backslash F$ has no perfect matching if and only if $F_0 = \{(v_i, v_{i+4}), (v_i, v_{i+1}), (v_{i+3}, v_{i+4})\}$ for some i, $F_1 = \{(w_j, w_{j-1})\}$ for some j, and there exists a vertex x in $B_0 = \{v_i, v_{i+1}, v_{i+3}, v_{i+4}, v_{i+6}\}$ such that $(x, \bar{x}) \in F_2$ and $\{\bar{y} | y \in B_0 \backslash x\} = \{w_{j+1}, w_{j+3}, w_{j+4}, w_{j+6}\}$.*

Next step will be characterization of 4-dimensional restricted HL-graphs which are conditional 5-edge-fault perfectly matchable. It can be derived directly from Lemma 6 as follows.

Theorem 2. *A 4-dimensional restricted HL-graph G is conditional 5-edge-fault perfectly matchable if and only if for any i and any vertex x in $B_0^i = \{v_i, v_{i+1}, v_{i+3}, v_{i+4}, v_{i+6}\}$, the set $\{\bar{y} | y \in B_0^i \backslash x\}$ is not equal to $\{w_{j+1}, w_{j+3}, w_{j+4}, w_{j+6}\}$ for any j.*

Of course, there exists a 4-dimensional restricted HL-graph which does not satisfy the condition of Theorem 2 and thus is not conditional 5-edge-fault perfectly matchable. The graph $G_0 \oplus_I G_1$ for an identity permutation $I = (0, 1, 2, 3, 4, 5, 6, 7)$, which is shown in Figure 3(a), is such a graph. It can be defined as the product $G(8, 4) \times K_2$, where K_2 is a complete graph with two vertices. Discover a conditional matching preclusion set $F = \{(v_0, v_4), (v_0, v_1), (v_3, v_4), (w_0, w_7), (v_0, w_0)\}$ of size five. Also, there exists a 4-dimensional restricted HL-graph which satisfies the condition of Theorem 2 and thus is conditional 5-edge-fault perfectly matchable. For example, the graph $G_0 \oplus_P G_1$ for $P = (0, 2, 1, 4, 3, 5, 6, 7)$ shown in Figure 3(b) is such a graph. For any i, $\{\bar{y} | y \in B_0^i\}$ is symmetric to either $\{w_0, w_1, w_2, w_4, w_5\}$ or $\{w_0, w_1, w_2, w_4, w_6\}$, and thus no such vertex x in B_0^i exists.



(a) $G_0 \oplus_I G_1$ (b) $G_0 \oplus_P G_1$

Figure 3: 4-dimensional restricted HL-graphs.

2.2. Bipartite HL-graphs

A bipartite graph is called *equitable* if it has a proper bicoloring such that both color sets have the same cardinality. Every bipartite HL-graph is equitable. It can be proved easily by induction. We assume that an m-dimensional bipartite HL-graph has 2^{m-1} black and 2^{m-1} white vertices and no pair of black and white vertices are joined by an edge. In this subsection, we will show that every m-dimensional bipartite HL-graph with $m \geq 2$ is conditional $2m-3$-edge-fault perfectly matchable.

For our purpose, we first construct a perfect matching in an m-dimensional bipartite HL-graph with at most m faults, whereas the fault set contains a pair of black and white vertices.

Lemma 7. *Let G be an m-dimensional bipartite HL-graph with $m \geq 2$. Then, for any hybrid fault set F' containing a single black vertex, single white vertex, and at most $m-2$ edges, $G \backslash F'$ has a perfect matching.*

Proof. We denote by u and v the black and white faulty vertices in G, respectively. It is assumed w.l.o.g. that the number of faulty edges in G is $m-2$. The proof is by induction on m. For $m = 2$, G is isomorphic to C_4 and the lemma holds true. Assume $m \geq 3$. There exist two $m-1$-dimensional bipartite HL-graphs G_0 and G_1 such that G is isomorphic to $G_0 \oplus G_1$. As usual, F_i denotes the *edge* fault set of G_i, $i = 0, 1$, and $f_i = |F_i|$. There are two cases.

Case 1: $f_0, f_1 \leq m-3$.
When both u and v are contained in one component, say G_0, the union of a perfect matching M_0 of $G_0 \backslash (F_0 \cup \{u, v\})$ and a perfect matching M_1 in $G_1 \backslash F_1$ is a desired matching. The existence of M_0 is due to induction hypothesis and the existence of M_1 is due to Proposition 4. When u is contained in one component, say G_0, and v is contained in the other component G_1, we first pick up an edge $(x, \bar{x})$ such that x is a vertex in G_0 having a different color from u and $(x, \bar{x})$ is fault-free. The picking up is always possible since we have 2^{m-2} candidates and at most $m-2$ blocking elements (faulty edges). Obviously, $2^{m-2} > m-2$ for any $m \geq 3$. Then, we find a perfect matching M_0 in $G_0 \backslash (F_0 \cup \{u, x\})$ and a perfect matching M_1 in $G_1 \backslash (F_1 \cup \{v, \bar{x}\})$. We have a perfect matching $M_0 \cup M_1 \cup \{(x, \bar{x})\}$ in $G \backslash F$.

Case 2: $f_0 = m-2$.
There is no faulty edge outside G_0. When both u and v are contained in G_0, we pick up a faulty edge (x, y) in G_0. Letting (x, y) be a *virtual* fault-free edge, we find a perfect matching M_0 in $G_0 \backslash (F'_0 \cup \{u, v\})$, where $F'_0 = F_0 \backslash (x, y)$. If $(x, y) \notin M_0$, the union of M_0 and a perfect matching M_1 in G_1 is a desired matching. Otherwise, letting M_1 be a perfect matching in $G_1 \backslash \{\bar{x}, \bar{y}\}$, we have a desired matching $(M_0 \backslash (x, y)) \cup M_1 \cup \{(x, \bar{x}), (y, \bar{y})\}$. When one of u and v, say u, is contained in G_0 and v is contained in G_1, for some faulty edge (x, y) in G_0 with x being different in color from u, we find a perfect matching M_0 in $G_0 \backslash (F'_0 \cup \{u, x\})$, where $F'_0 = F_0 \backslash (x, y)$. Letting M_1 be a perfect matching in $G_1 \backslash \{v, \bar{x}\}$, we have a desired matching $M_0 \cup M_1 \cup \{(x, \bar{x})\}$. Finally when both u and v are contained in G_1, the union of a perfect matching M_0 in $G_0 \backslash F_0$ and a perfect matching M_1 in $G_1 \backslash \{u, v\}$ is a desired matching. □

Now, we are ready to consider conditional matching preclusion of bipartite HL-graphs.

Theorem 3. *Every m-dimensional bipartite HL-graph with $m \geq 2$ is conditional $2m-3$-edge-fault perfectly matchable.*

PROOF. The proof is by induction on m. For $m = 2$, the theorem clearly holds. Let $m \geq 3$ and G denote an m-dimensional bipartite HL-graph isomorphic to $G_0 \oplus G_1$ for some $m-1$-dimensional bipartite HL-graphs G_0 and G_1. Let F denote a conditional edge fault set with $|F| \leq 2m-3$. We will show $G \backslash F$ has a perfect matching. For our purpose, it is assumed $|F| = 2m-3$.

If $f_2 = 0$, we are done since the set of edges between G_0 and G_1 forms a perfect matching. Thus, we assume $f_2 \geq 1$ hereafter. Furthermore, we assume w.l.o.g. $f_0 \geq f_1$. Then, $f_1 \leq m-2$.

Case 1: $f_0 \leq 2m-5$.

If F_0 is a conditional fault set of G_0, the union of perfect matchings M_0 of $G_0 \backslash F_0$ and M_1 of $G_1 \backslash F_1$ is indeed a perfect matching of $G \backslash F$. Suppose otherwise, there exists a vertex x in G_0 such that all the edges in G_0 incident to x are faulty. We assume w.l.o.g. x is a white vertex. There exists a black vertex y in G_0 such that $(y, \bar{y})$ is fault free, since the number 2^{m-2} of candidates is greater than the upper bound $m-2$ on the number of blocking elements for any $m \geq 3$. Then, by Lemma 7, there exists a perfect matching M_0 in $G_0 \backslash (F_0' \cup \{x, y\})$, where $F_0' = F_0 \backslash \{(x, v) | v \in V(G_0)\}$. Note that F_0' has at most $(2m-5)-(m-1) = m-4$ faulty edges. Furthermore, a perfect matching M_1 of $G_1 \backslash (F_1 \cup \{\bar{x}, \bar{y}\})$ also exists by Lemma 7 since $f_1 = f - f_0 - f_2 \leq (2m-3)-(m-1)-1 = m-3$. The union $M_0 \cup M_1 \cup \{(x, \bar{x}), (y, \bar{y})\}$ is a desired perfect matching.

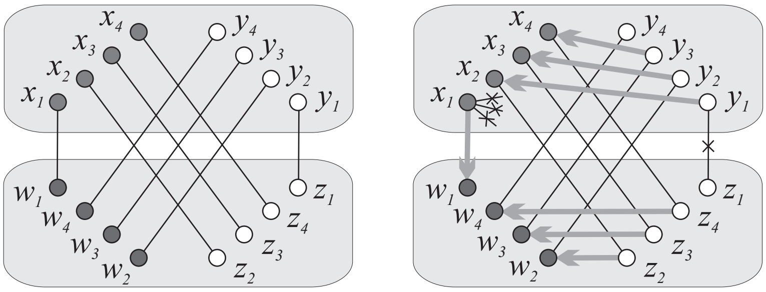
Case 2: $f_0 = 2m-4$ and $f_2 = 1$ ($f_1 = 0$).

We are to pick up a faulty edge (x, y) in G_0 such that (i) $F_0' \equiv F_0 \backslash (x, y)$ is a conditional fault set and (ii) both $(x, \bar{x})$ and $(y, \bar{y})$ are fault-free. If there exists a vertex z such that all the edges in G_0 incident to z are faulty, (x, y) will be an arbitrary edge incident to z satisfying condition (ii). Such a vertex z is unique, if any. Otherwise, (x, y) will be an arbitrary faulty edge in G_0 satisfying condition (ii). By induction hypothesis, $G_0 \backslash F_0'$ has a perfect matching M_0. If $(x, y) \notin M_0$, the union of M_0 and a perfect matching M_1 in G_1 will do. If $(x, y) \in M_0$, letting M_1 be a perfect matching of $G_1 \backslash \{\bar{x}, \bar{y}\}$, the union $(M_0 \backslash (x, y)) \cup M_1 \cup \{(x, \bar{x}), (y, \bar{y})\}$ is a desired matching. The existence of M_1 is due to Lemma 7. Thus, we have the theorem. □

Corollary 2. *For any m-dimensional bipartite HL-graph G with $m \geq 3$, $mp_1(G) = 2m-2$.*

3. Concluding Remarks

In this paper, the conditional matching preclusion numbers for both m-dimensional restricted HL-graphs with $m \geq 5$ and m-dimensional bipartite HL-graphs with $m \geq 3$ were determined to be $2m-2$. Every m-dimensional HL-graph, by definition, has an edge partition into m perfect matchings. Thus, one might expect that Theorem 1 and Theorem 3 can be extended to



(a) (b)

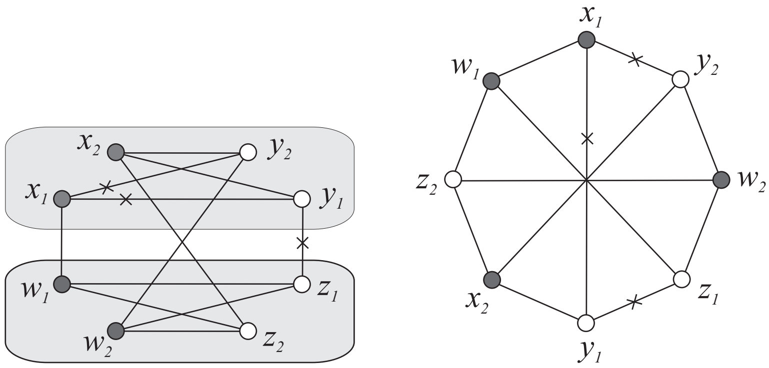
Figure 4: The graph $G_0 \oplus_\Pi G_1$.

general HL-graphs so that for some constant m_0, every m-dimensional HL-graph with $m \geq m_0$ is conditional $2m-3$-edge-fault perfectly matchable.

Unfortunately, this is not the case as shown below. Let G_0 and G_1 be arbitrary $m-1$-dimensional bipartite HL-graphs for $m \geq 3$. We let $\{x_1, x_2, \ldots, x_q\}$ and $\{y_1, y_2, \ldots, y_q\}$ be the sets of black and white vertices in G_0, respectively, and let $\{w_1, w_2, \ldots, w_q\}$ and $\{z_1, z_2, \ldots, z_q\}$ be the sets of black and white vertices in G_1, where $q = 2^{m-2}$. There exists a permutation Π between $V(G_0)$ and $V(G_1)$ such that in the graph $G_0 \oplus_\Pi G_1$, $\bar{x_1} = w_1$, $\bar{x_i} = z_i$ for every $2 \leq i \leq q$, $\bar{y_1} = z_1$, and $\bar{y_j} = w_j$ for every $2 \leq j \leq q$. See Figure 4(a). The graph $G_0 \oplus_\Pi G_1$ is 'near' bipartite in a sense that if we delete two edges (x_1, w_1) and (y_1, z_1), then the resultant graph becomes bipartite. In other words, its *bipartization number* [5] is only two.

Observation 1. *For any $m \geq 3$, the m-dimensional HL-graph $G_0 \oplus_\Pi G_1$ is not conditional m-edge-fault perfectly matchable.*

PROOF. We denote by G the graph $G_0 \oplus_\Pi G_1$ and let F be a conditional fault set of size m that contains all the edges in G_0 incident to x_1 and the edge (y_1, z_1). See Figure 4(b). Suppose, for a contradiction, $G \backslash F$ has a perfect matching M. The edge (x_1, w_1) is included in M. Since (y_1, z_1) is not included in M, y_1 should be matched to a black vertex in G_0, say x_2. Then, since $(x_2, z_2) \notin M$, z_2 should be matched to a black vertex in G_1, say w_2. And then, since $(w_2, y_2) \notin M$, y_2 should be matched to a black vertex in G_0, say x_3. This process continues until we find a vertex v to which y_q is matched. At that time, however, w_q and all the black vertices in G_0 were already matched. Thus, no such vertex v exists. This is a contradiction. $\square$



(a) (b)

Figure 5: The coincidence.

The above Observation 1 indicates that the lower bound m on the conditional matching preclusion number of an m-dimensional HL-graph given in Proposition 4(b) is the best possible. It seems worth pointing out that the conditional matching preclusion set F presented in Observation 1 for $m = 3$ coincides with the set given in Lemma 2, as shown in Figure 5. The conditional matching preclusion number of the graph $G_0 \oplus_\Pi G_1$ is m, which is not greater than and equal to its matching preclusion number. This motivates the study of conditional matching preclusion for general HL-graphs and study of graphs G with $mp_1(G) = mp(G) > 0$ and their relationship to something like bipartization.

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