Disjoint Path Covers in Cubes of Connected Graphs

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Abstract

Given a graph G, and two vertex sets S and T of size k each, a many-to-many k-disjoint path cover of G joining S and T is a collection of k disjoint paths between S and T that cover every vertex of G. It is classified as *paired* if each vertex of S must be joined to a designated vertex of T, or *unpaired* if there is no such constraint. In this article, we first present a necessary and sufficient condition for the cube of a connected graph to have a paired 2-disjoint path cover. Then, a corresponding condition for the unpaired type of 2-disjoint path cover problem is immediately derived. It is also shown that these results can easily be extended to determine if the cube of a connected graph has a hamiltonian path from a given vertex to another vertex that passes through a prescribed edge.

Keywords: Disjoint path cover, strong hamiltonicity, hamiltonian path, prescribed edge, cube of graph.

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1. INTRODUCTION

1.1. Problem specification

Given an undirected graph G, a *path cover* is a set of paths in G where every vertex in $V(G)$ is covered by at least one path. Of special interest is the *vertex-disjoint path cover*, or simply called *disjoint path cover*, which is one with an additional constraint that every vertex, possibly except for

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terminal vertices, must belong to one and only one path. The disjoint path cover made of k paths is called the *k-disjoint path cover* (*k-DPC* for short).

Given two disjoint terminal vertex sets $S = \{s_1, s_2, \ldots, s_k\}$ and $T = \{t_1, t_2, \ldots, t_k\}$ of G, each representing k sources and sinks, the *many-to-many k-DPC* is a disjoint path cover each of whose paths joins a pair of source and sink. The disjoint path cover is regarded as *paired* if every source s_i must be matched with a specific sink t_i. On the other hand, it is called *unpaired* if any permutation of sinks may be mapped bijectively to sources. A graph G is called *paired (resp. unpaired) k-coverable* if $2k \leq |V(G)|$ and there always exists a paired (resp. unpaired) k-DPC for any S and T. The k-DPC has two simpler variants. One is the *one-to-many k-DPC*, whose paths join a single source to k distinct sinks. The other is the *one-to-one k-DPC*, whose paths always start from a single source and end up in a single sink.

The existence of a disjoint path cover in a graph is closely related to the concept of vertex connectivity: Menger's theorem states the connectivity of a graph in terms of the number of disjoint paths joining two distinct vertices, whereas the Fan Lemma states the connectivity of a graph in terms of the number of disjoint paths joining a vertex to a set of vertices [2]. Moreover, it can be shown that a graph is k-connected if and only if it has k disjoint paths joining two arbitrary vertex sets of size k each, in which a vertex that belongs to both sets is counted as a valid path. When a graph does not have a disjoint path cover of desired kind, it is natural to consider an augmented graph with higher connectivity. A simple way of increasing the connectivity is to raise a graph to a power: Given a positive integer d, the d-th power G^d of G is defined as a graph with the same vertex set $V(G)$ and an edge set that is augmented in such a way that two vertices of G^d are adjacent if and only if there exists a path of length at most d in G joining them. In particular, the graph G^2 is called the *square* of G, while G^3 is said to be the *cube* of G.

This paper aims to investigate the structures of the cubes of connected graphs in the point of disjoint path covers. First, we show a necessary and sufficient condition for the cube G^3 of a connected graph G with $|V(G)| \geq 4$ to have a paired 2-DPC joining two arbitrary disjoint vertex sets $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$. Then, the corresponding condition for the existence of an unpaired 2-DPC is immediately derived. In addition, we establish a necessary and sufficient condition under which the cube of a connected graph has an s-t hamiltonian path passing through a prescribed edge e for an arbitrary triple of s, t, and e.

1.2. Disjoint path covers

The disjoint path cover problem has been studied for several classes of graphs: hypercubes [6, 9, 14, 16], recursive circulants [18, 19, 25, 26], and hypercube-like graphs [25, 26]. The structure of the cubes of connected graphs was investigated with respect to single-source 3-disjoint path covers [24]. The problem was also investigated in view of a full utilization of nodes in interconnection networks [25]. Its intractability was shown that deciding the existence of a one-to-one, one-to-many, or many-to-many k-DPC in a general graph, joining arbitrary sets of sources and sinks, is NP-complete for all $k \geq 1$ [25, 26].

The method for finding a disjoint path cover can easily be used for finding a hamiltonian path (or cycle) due to its natural relation to the hamiltonicity of graph. For instance, a hamiltonian path between two distinct vertices in a graph G is in fact a 1-DPC of G joining the vertices. An s-t hamiltonian path in G that passes an arbitrary sequence of k pairwise nonadjacent edges $((x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k))$ in the specified order always exists for any distinct s and t with $s \neq x_i, y_i$ and $t \neq x_i, y_i$ $(1 \leq i \leq k)$ if G is paired $(k+1)$-coverable [25]. A simpler, s-t hamiltonian path that passes a prescribed edge (x, y) with $\{s, t\} \cap \{x, y\} = \emptyset$ can also be found by solving the corresponding unpaired or paired 2-DPC problem [26]. While the unpaired version would be easier to tackle than the paired one, the difference is that the direction between x and y in the path may not be enforced through the unpaired 2-DPC. For more discussion on the hamiltonian paths (or cycles) passing through prescribed edges, refer to, for example, [3, 8].

1.3. Strong hamiltonian properties

The cube of a connected graph with at least four vertices is 1-hamiltonian, i.e., it is hamiltonian and remains so after the removal of any one vertex, as Chartrand and Kapoor showed [5]. Sekanina [29] and Karaganis [17] independently proved that the cube of a connected graph is hamiltonian-connected. Whether the cube is 1-hamiltonian-connected, i.e., it still remains hamiltonian-connected after the removal of any one vertex, was characterized for trees by Lesniak [21] and for connected graphs by Schaar [28]. Characterizations of connected graphs whose cubes are p-hamiltonian for $p \leq 3$ were also made in [20, 28], and strong hamiltonian properties of the cube of a 2-edge connected graph were studied in [23].

On the other hand, the hamiltonicity of the square of a graph was investigated by several researchers. Fleischner proved that the square of every 2-connected graph is hamiltonian [11] (for an alternative proof, refer to [13] or [22]). In fact, the square of a 2-connected graph is both hamiltonian-connected and 1-hamiltonian provided that its order is at least four [4]. These works were followed by several results on the hamiltonicity of the square graphs, in particular, by Abderrezzak *et al.* [1], Chia *et al.* [7], and Ekstein [10]. Interesting results on pancyclicity and panconnectivity of the square of a connected graph were given in [12]. For the square of a tree T, Harary and Schwenk showed that T^2 is hamiltonian if and only if T is a caterpillar [15]. Recently, Radoszewski and Rytter proved that T^2 has a hamiltonian path if and only if T is a horsetail [27].

1.4. Fundamental concepts and notation

Before proceeding to the main results, we summarize some fundamental terminologies on the graph connectivity that are frequently used in this article (refer to Figure 1 for an illustration of them). An edge of a graph G is called a *bridge*, or *cut-edge*, if its removal increases the number of connected components. Trivially, an edge is a bridge if and only if it is not contained in a cycle. A bridge is said to be *nontrivial* if none of its two end-vertices is of degree one. A vertex of G is called a *pure bridge vertex* if each of its incident edges is a nontrivial bridge. Furthermore, a set of three pairwise adjacent vertices, each having a degree of at least three, is called a *pure bridge triangle* if every edge that is incident with exactly one of the triangular vertices is a nontrivial bridge. In addition to these terminologies, we introduce another one:

Definition 1. *A set of two adjacent vertices is called a* pure bridge pair *if both vertices are pure bridge vertices.*

Next, we present two fundamental theorems about the hamiltonian properties of the cubes of connected graphs that play important roles in deriving our results.

Theorem 1 (Sekanina [29] and Karaganis [17]). *The cube of every connected graph is hamiltonian-connected.*

Theorem 2 (Schaar [28]). *Given three vertices s, t, and v_f of a connected graph G, there exists an s-t hamiltonian path in $G^3 \setminus v_f$ if and only if*

- $\{s, t, v_f\} \nsubseteq N[v]$ *for any pure bridge vertex v of G, and*
- $\{s, t, v_f\}$ *does not form a pure bridge triangle of G.*

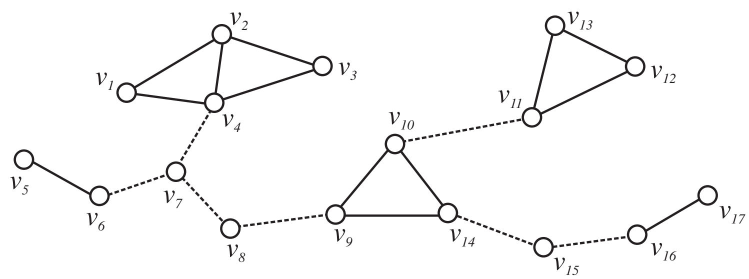


Figure 1: In this connected graph, the seven nontrivial bridges are marked in dotted lines. There are three pure bridge vertices v_7, v_8, and v_{15}, one pure bridge triangle $\{v_9, v_{10}, v_{14}\}$, and one pure bridge pair $\{v_7, v_8\}$.

In this article, $N_G(v)$, or $N(v)$ if the graph G is clear in the context, represents the open neighborhood of a vertex $v \in V(G)$, i.e. $N_G(v) = \{u \in V(G) : (u, v) \in E(G)\}$, whereas $N_G[v]$, or $N[v]$, denotes the closed neighborhood of v, i.e. $N_G[v] = N_G(v) \cup \{v\}$. These neighborhood definitions are naturally extended to vertex sets, and will be used frequently in this article: $N_G(X) = \bigcup_{v \in X} N_G(v) \setminus X$ and $N_G[X] = N_G(X) \cup X$ for $X \subseteq V(G)$. In addition, the usual notations, $\delta_G(v)$ and $d_G(u, v)$, are employed to indicate the degree of vertex v in G and the distance between vertices u and v in G, respectively. Finally, we will call a vertex *free* if it does not belong to any terminal vertex set.

2. MANY-TO-MANY TWO-DISJOINT PATH COVERS

In this section, we derive, in parallel, the exact conditions for the cube of a connected graph G with $|V(G)| \geq 4$ to have a paired and unpaired 2-DPC, respectively, joining two arbitrary disjoint vertex sets $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$. First, let us discuss the necessary conditions.

Lemma 1 (Necessity for unpaired 2-DPC). *Let $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$ be terminal sets of a connected graph G. If G^3 has an unpaired 2-DPC joining S and T, then $\{s_1, t_1, s_2, t_2\} \not\subseteq N_G[v]$ for any pure bridge vertex v of G*

Proof. Suppose to the contrary $\{s_1, s_2, t_1, t_2\} \subseteq N_G[v]$ for some pure bridge vertex v. Let $N_G(v) = \{v_1, v_2, \ldots, v_d\}$ with $d = \delta_G(v)$. Also let $W = \{w_i : 1 \leq i \leq d\}$, where w_i is a neighbor of v_i in G other than v, implying $(w_i, w_{i'}) \notin E(G^3)$ for any pair of w_i and $w_{i'}$ in W. Let P_1

and P_2 be two paths of the unpaired 2-DPC of G^3, defined as sequences of vertices. We denote by P'_1 and P'_2 the longest subsequences of P_1 and P_2, respectively, whose elements, not necessarily contiguous to each other, are contained in $N_G[v] \cup W$. Then, (i) P'_1 and P'_2 altogether have $d+1$ vertices in $N_G[v]$ and d vertices in W, and (ii) their first and last vertices are all terminals contained in $N_G[v]$. Thus, at least one of P'_1 and P'_2, say P'_1, should have two consecutive vertices, w_i and w_j, of W. This indicates that, for $P_1 = (x_1, x_2, \ldots, x_p\,(= w_i), \ldots, x_q\,(= w_j), \ldots, x_m)$, every vertex x_k with $p \leq k \leq q$ is not contained in $N_G[v]$. Since w_i and w_j are respectively contained in different connected components of $G \setminus v$, there should be two consecutive vertices x_{k^*} and x_{k^*+1} of P_1, for $p \leq k^* \leq q-1$, that are contained in different connected components of $G \setminus v$. This is, however, impossible since $d_G(x_{k^*}, x_{k^*+1}) \geq 4$. This completes the proof. □

Obviously, the condition above is also necessary for G^3 to have a paired 2-DPC joining S and T.

Lemma 2 (Necessity for paired 2-DPC). *Let $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$ be terminal sets of a connected graph G. If G^3 has a paired 2-DPC joining S and T, then $\{s_i, t_i\}$ is not a pure bridge pair of G such that $\{s_{3-i}, t_{3-i}\} \subseteq N_G(\{s_i, t_i\})$ for each $i = 1, 2$.*

Proof. Suppose for a contradiction that, for some i, say, $i = 2$, $\{s_2, t_2\}$ is a pure bridge pair such that $\{s_1, t_1\} \subseteq N_G(\{s_2, t_2\})$. Due to Lemma 1, s_1 and t_1 must reside in the different connected components of $G \setminus (s_2, t_2)$, allowing us to assume w.l.o.g. that there exists a path (s_1, s_2, t_2, t_1) in G, as shown in Figure 2. Now, for each neighbor $x_i\,(\neq t_2)$ of s_2 with $1 \leq i \leq p\,(= \delta_G(s_2) - 1)$ and $x_1 = s_1$, x_i has another neighbor x'_i in G other than s_2. Similarly, for each neighbor $y_j\,(\neq s_2)$ of t_2 with $1 \leq j \leq q\,(= \delta_G(t_2) - 1)$ and $y_1 = t_1$, y_j has another neighbor y'_j in G other than t_2. Let X, X', Y, and Y' with $|X| = |X'| = p$ and $|Y| = |Y'| = q$ be the sets of vertices made of x_i, x'_i, y_j, and y'_j, respectively. For the vertex set X_i (resp. Y_j) of a connected component of $G \setminus \{s_2, t_2\}$ containing x_i (resp. y_j), $1 \leq i \leq p$, $1 \leq j \leq q$, we can easily see that, letting $X'_i = X_i \setminus x_i$ and $Y'_j = Y_j \setminus y_j$, the distance in G between X'_i and Y'_j, the distance between X'_i and $X'_{i'}$ for $i \neq i'$, and the distance between Y'_j and $Y'_{j'}$ for $j \neq j'$ are all greater than three.

For the s_1-t_1 path P_1 and the s_2-t_2 path P_2 in the paired 2-DPC, define further that P'_1 and P'_2 are the longest subsequences of P_1 and P_2, respectively, whose elements, not necessarily contiguous to each other, are contained in $X \cup X' \cup Y \cup Y' \cup \{s_2, t_2\}$. Then, (i) P'_1 and P'_2 altogether pass

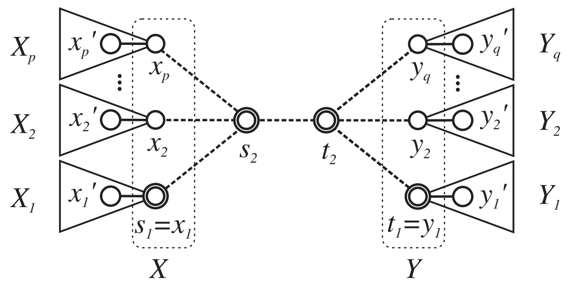


Figure 2: A contradictory situation in the proof of Lemma 2.

through the $p+q+2$ vertices in $B = X \cup Y \cup \{s_2, t_2\}$ and the $p+q$ vertices in $W = X' \cup Y'$, and (ii) their end-vertices, four in total, are contained in B. Similar to the proof of Lemma 1, we can reason that the two subsequences of vertices may not have two consecutive vertices from W, indicating that the vertices of P_1' and P_2' should alternate in B and W. Observe in particular that s_1 and t_1, the two end-vertices of P_1, are contained in X and Y, respectively. The fact that the distance in G between X and $\bigcup_j Y_j'$, and also between Y and $\bigcup_i X_i'$, is four, however, means that s_1 may not reach t_1 in $G^3 \setminus \{s_2, t_2\}$ without passing through the two consecutive vertices from B (one from X and the next from Y), implying the path P_1 is impossible to have. This contradiction completes the proof. $\square$

Interestingly enough, these two necessary conditions together form the sufficient condition for the cube of a connected graph to have a paired 2-DPC as follows.

Theorem 3 (Paired 2-DPC). *Let $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$ be terminal sets of a connected graph G. The cube G^3 has a paired 2-DPC joining S and T if and only if*

- ***C1:*** $\{s_1, s_2, t_1, t_2\} \not\subseteq N_G[v]$ *for any pure bridge vertex v of G, and*
- ***C2:*** $\{s_i, t_i\}$ *is not a pure bridge pair of G such that* $\{s_{3-i}, t_{3-i}\} \subseteq N_G(\{s_i, t_i\})$ *for each $i = 1, 2$.*

Proof. The necessity part is due to Lemmas 1 and 2. The sufficiency part proceeds by induction on the number of vertices of G, $|V(G)|$, in which we assume $|V(G)| \geq 5$ as the case of $|V(G)| = 4$ is trivial. Given the four terminals, we can always select a terminal z such that the remaining terminals $\{s_1, s_2, t_1, t_2\} \setminus z$ reside in the same connected component of $G \setminus z$. (Suppose that removing a terminal from G separates the other terminals into different connected components. Then, at least one of them must contain a single terminal, allowing us to choose that terminal.) If there are more than one such terminals, we select a terminal, named as s_1 w.l.o.g., as follows: choose a terminal with a minimum number of neighbors that are both terminals and pure bridge vertices. Now, let us denote by H the connected component of $G \setminus s_1$ containing all s_2, t_1, and t_2, and by H' the subgraph of G induced by $V(G) \setminus V(H)$, in which all the edges between the two disjoint subgraphs H and H' are incident to s_1. In addition, define a vertex set W as follows:

$$W = \begin{cases} \{w \in V(H) : d_G(s_1, w) \leq 2\} & \text{if } |V(H')| \geq 2, \\ \{w \in V(H) : d_G(s_1, w) \leq 3\} & \text{if } |V(H')| = 1, \text{ i.e. } V(H') = \{s_1\}. \end{cases}$$

In this proof, we consider two major cases in which a free vertex in W is to be considered as a terminal instead of s_1. Suppose first that there exists a free vertex s'_1 in W such that $\{s'_1, s_2\}$ and $\{t_1, t_2\}$ satisfy the two conditions C1 and C2 with respect to H. Then, by the induction hypothesis, there exists a paired 2-DPC in H^3, composed of an s'_1-t_1 path and an s_2-t_2 path. Consider an s_1-v hamiltonian path in H'^3 for some neighbor v of s_1 in H' if $|V(H')| \geq 2$, which is guaranteed to exist by Theorem 1, or a one-vertex path s_1 if $|V(H')| = 1$. Notice that v and s'_1 are adjacent to each other in G^3 due to the definition of W. Hence, by combining the s_1-v and s'_1-t_1 paths via the edge (v, s'_1), we can build a paired 2-DPC joining S and T, completing the proof of the first major case.

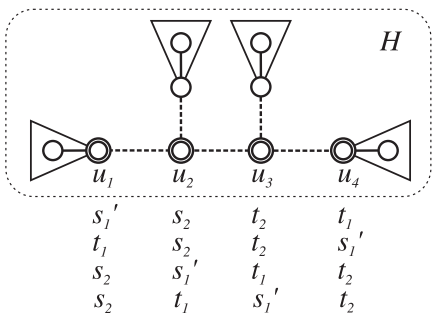
Next, consider the second major case, where, for every free vertex s'_1 in W, if any, $\{s'_1, s_2\}$ and $\{t_1, t_2\}$ violate at least one of the two conditions with respect to H. There are three cases.

Case 1: *For some free vertex $s'_1 \in W$, $\{s'_1, s_2\}$ and $\{t_1, t_2\}$ violate C2 but satisfy C1 with respect to H.* In this case, there exists an induced path (u_1, u_2, u_3, u_4) in H, where $\{u_2, u_3\}$, either $\{s'_1, t_1\}$ or $\{s_2, t_2\}$, is a pure bridge pair of H, and $\{u_1, u_4\} = \{s'_1, s_2, t_1, t_2\} \setminus \{u_2, u_3\}$ (see Figure 3(a) for possible permutations). While each edge of G from s_1 to H may have the other end-vertex in one of the three disjoint sets $V(H) \setminus N_H[\{u_2, u_3\}]$, $N_H(\{u_2, u_3\})$, and $\{u_2, u_3\}$, we claim that $(s_1, v) \notin E(G)$ for every $v \in V(H) \setminus \{u_2, u_3\}$. Due to the hypothesis of the second major case, it holds true that $(s_1, v) \notin E(G)$ for $v \in V(H) \setminus N_H[\{u_2, u_3\}]$ since $\{v, s_2\}$ and $\{t_1, t_2\}$ obviously satisfy both C1 and C2 with respect to H. Similarly, $(s_1, v) \notin E(G)$ for $v \in N_H(\{u_2, u_3\})$ since for some $v' \in V(H) \setminus N_H[\{u_2, u_3\}]$ adjacent to v, $\{v', s_2\}$ and $\{t_1, t_2\}$ obviously satisfy both C1 and C2 with respect to H. Thus, the claim is proved.

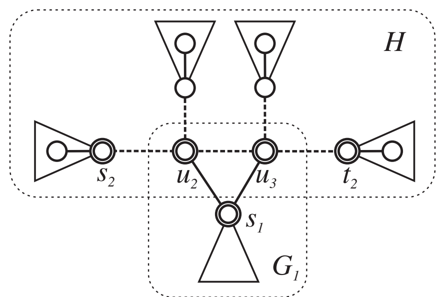
Now, it remains that $(s_1, u_2) \in E(G)$, $(s_1, u_3) \in E(G)$, or both. First, in the case where $\{u_1, u_4\} = \{s_2, t_2\}$, as illustrated in Figure 3(b), there exists an s_2-t_2 hamiltonian path P_2 in $H^3 \setminus t_1$ by Theorem 2. There also exists an s_1-t_1 hamiltonian path P_1 in $G_1^3 \setminus s_1'$ by Theorem 2 again, where G_1 is the subgraph of G induced by $V(H') \cup \{u_2, u_3\}$. The two paths P_1 and P_2, then, form a paired 2-DPC of G^3. Second, if $\{u_2, u_3\} = \{s_2, t_2\}$ and there are edges from s_1 to both u_2 and u_3, we consider the two connected components of $H \setminus (u_2, u_3)$, called H_1 and H_2, where $u_2 \in V(H_1)$ and $u_3 \in V(H_2)$. If $u_1 = s_1'$ and $u_4 = t_1$ (refer to Figure 3(c)), we can build a paired 2-DPC of G^3 by finding an s_1-t_1 hamiltonian path P_1 in $G_1^3 \setminus u_3$ with G_1, the subgraph induced by $V(H_2) \cup \{s_1\}$, and an s_2-t_2 hamiltonian path P_2 in $G_2^3 \setminus s_1$ with G_2, the subgraph induced by $V(H_1) \cup V(H') \cup \{u_3\}$, whose existence is guaranteed by Theorem 2. In the other case of $u_1 = t_1$ and $u_4 = s_1'$, a paired 2-DPC of G^3 can be constructed symmetrically.

Third, suppose that $\{u_2, u_3\} = \{s_2, t_2\}$ and there is exactly one edge from s_1 to either u_2 or u_3. Consider the case of $(s_1, u_2) \in E(G)$ first. In this case, $\delta_G(s_1) = 1$. Otherwise, that is, if $\delta_G(s_1) \geq 2$, (s_1, u_2) should be a nontrivial bridge of G. Then, contrary to the hypothesis of the theorem, $\{s_1, s_2\}$ and $\{t_1, t_2\}$ would violate the condition C2 with respect to G. Now, let G_1 and G_2 be the subgraphs of G induced by $V(H_1) \cup \{s_1\}$ and by $V(H_2) \cup \{u_2\}$, respectively, where, again, H_1 and H_2 are the two connected components of $H \setminus (u_2, u_3)$ such that $u_2 \in V(H_1)$ and $u_3 \in V(H_2)$. In the case where $u_4 = t_1$ (refer to Figure 3(d)), there exists an s_2-t_2 hamiltonian path in $G_2^3 \setminus t_1$. Also, by merging an s_1-s_1' hamiltonian path in $G_1^3 \setminus u_2$ and a one-vertex path t_1 via the edge (s_1', t_1) in G^3, we can build an s_1-t_1 path P_1 covering the remaining vertices, showing the existence of a paired 2-DPC of G^3. For the case of $u_1 = t_1$, an s_1-t_1 hamiltonian path in $G_1^3 \setminus u_2$ and an s_2-t_2 hamiltonian path in G_2^3 together form a paired 2-DPC. Lastly, when $(s_1, u_3) \in E(G)$, a symmetric argument also leads to the existence of a paired 2-DPC of G^3.

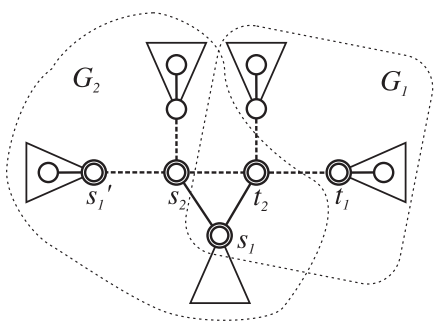
Case 2: *For some free vertex $s_1' \in W$, $\{s_1', s_2\}$ and $\{t_1, t_2\}$ violate C1 with respect to H.* Here, we may assume that, for every free vertex $s_1' \in W$, $\{s_1', s_2\}$ and $\{t_1, t_2\}$ violate C1 since the case of only C2 being broken has been dealt with in Case 1. Since C1 is violated, $\{t_1, s_2, t_2\} \subseteq N_H[v]$ for some pure bridge vertex v of H. Such v should be unique since (i) at least two of t_1, s_2, and t_2 are distance two apart, and (ii) an extra candidate for v would form a cycle. Thus, we can say that $\{s_1', t_1, s_2, t_2\} \subseteq N_H[v]$ for every free vertex s_1' in W. Notice that there cannot be an edge from s_1 to any vertex $u \neq v$ because the existence of such an edge implies that there is another



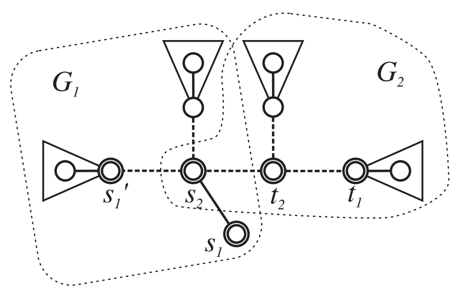
(a) $\{s'_1, s_2\}$ and $\{t_1, t_2\}$ violate C2 but satisfy C1.



(b) $\{u_1, u_4\} = \{s_2, t_2\}$.



(c) $\{u_2, u_3\} = \{s_2, t_2\}$ and $(s_1, u_2), (s_1, u_3) \in E(G)$.



(d) $\{u_2, u_3\} = \{s_2, t_2\}$ and $(s_1, u_2) \in E(G)$.

Figure 3: An illustration for the proof of Case 1 of Theorem 3.

free vertex in W that is not in $N_H[v]$, contradicting to our assumption.

Hence, v must be a unique neighbor of s_1 in H, indicating that (s_1, v) is a bridge of G. Now, if $\delta_G(s_1) \geq 2$, (s_1, v) is nontrivial, and v becomes a pure bridge vertex of G such that $\{s_1, t_1, s_2, t_2\} \subseteq N_G[v]$, contradicting the hypothesis of the theorem. If $\delta_G(s_1) = 1$, then there must exist a free vertex in W, which is at distance three from s_1, but is not in $N_H[v]$, also contradicting to our assumption. Hence, Case 2 is impossible.

Case 3: *There exists no free vertex in W.* Suppose $t_1 \in W$. Consider an s_1-w hamiltonian path in H'^3 for some neighbor w of s_1 in H' if $|V(H')| \geq 2$, which is guaranteed to exist by Theorem 1, or a one-vertex path s_1 if $|V(H')| = 1$. Then, appending t_1 to the path results in an s_1-t_1 path. If $H^3 \setminus t_1$ has an s_2-t_2 hamiltonian path, these two paths form a paired 2-DPC of G^3, and we are done. Suppose otherwise. By Theorem 2, (i) there exists a pure bridge vertex v in H such that $\{t_1, s_2, t_2\} \subseteq N_H[v]$, or (ii) $\{t_1, s_2, t_2\}$ forms a pure bridge triangle in H. The case (ii) is impossible since any neighbor u of s_1 in W would be free itself (if $u \notin \{t_1, s_2, t_2\}$), or would have a free neighbor in W (if $u \in \{t_1, s_2, t_2\}$), which, in both possibilities, contradicts the assumption of Case 3. In the case (i), v is a terminal because, otherwise, every neighbor u of s_1 in W would be free itself (if $u \notin \{t_1, s_2, t_2\}$), or would have a free neighbor v in W (if $u \in \{t_1, s_2, t_2\}$). Furthermore, v is the unique neighbor of s_1 in H similarly because, otherwise, every neighbor u of s_1 in W other than v would be free itself, or would have a free neighbor in W. Now, if $\delta_G(s_1) \geq 2$, (s_1, v) is a nontrivial bridge of G and thus v is a pure bridge vertex of G such that $\{s_1, t_1, s_2, t_2\} \subseteq N_G[v]$, which contradicts to the hypothesis of the theorem. If $\delta_G(s_1) = 1$, there must exist a free vertex in W, which is at distance two from v in H, contradicting our assumption.

Finally, suppose $t_1 \notin W$. By the definition of W, either (i) $|V(H')| \geq 2$ and $d_G(s_1, t_1) \geq 3$, or (ii) $|V(H')| = 1$ and $d_G(s_1, t_1) \geq 4$. The case (ii) is simply impossible since there are at least three vertices in W, at most two of which may be terminals. This suggests the existence of a free vertex in W, leading to a contradiction. In the case (i), it is obvious that W is made of s_2 and t_2, where one of them, x, is adjacent to both s_1 and the other one y. Notice that x is a pure bridge vertex of G. Furthermore, $(t_1, y) \in E(G)$ and y is a pure bridge vertex of G because, otherwise, the vertex t_1 would have been selected as s_1 in the beginning. This implies that (s_1, x, y, t_1) is an induced path of G such that $\{x, y\}$, i.e. $\{s_2, t_2\}$ is a pure bridge pair and $\{s_1, t_1\} \subseteq N_G(\{s_2, t_2\})$, which contradicts to the hypothesis of the theorem. This completes the entire proof. $\square$

Corollary 1. *For a connected graph G with four or more vertices, G^3 is*

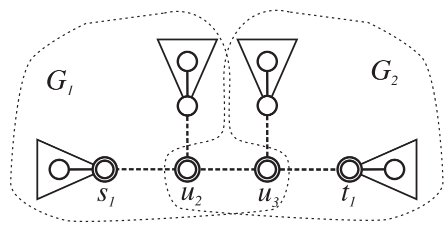


Figure 4: An illustration of the proof of Theorem 4. Under the assumption $\{u_2, u_3\} = \{s_2, t_2\}$, this figure represents the first case where the induced path (u_1, u_2, u_3, u_4) is either (s_1, s_2, t_2, t_1) or (s_1, t_2, s_2, t_1). The other case of (t_1, s_2, t_2, s_1) or (t_1, t_2, s_2, s_1) can be handled symmetrically.

paired 2-coverable if and only if G has neither a pure bridge vertex of degree at least three nor a pure bridge pair.

Now, the necessary and sufficient condition for the existence of an unpaired 2-DPC of the cube of a connected graph is easily derived.

Theorem 4 (Unpaired 2-DPC). *Let $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$ be terminal sets of a connected graph G. The cube G^3 has an unpaired 2-DPC joining S and T if and only if $\{s_1, s_2, t_1, t_2\} \not\subseteq N_G[v]$ for any pure bridge vertex v of G.*

Proof. The necessity part is due to Lemma 1. For the proof of the sufficient condition, it suffices to handle the situation where the second condition of Theorem 3, C2, is broken and the first condition C1 is satisfied. Suppose that the four terminals form an induced path (u_1, u_2, u_3, u_4) such that $\{u_2, u_3\}$, say, $\{s_2, t_2\}$ w.l.o.g. is a pure bridge pair. Let H_1 and H_2 be the two connected components of $G \setminus (u_2, u_3)$ such that $u_2 \in V(H_1)$ and $u_3 \in V(H_2)$. There are two cases to be considered: $u_1 = s_1$ & $u_4 = t_1$ as shown in Figure 4, and $u_1 = t_1$ & $u_4 = s_1$. In the first case, we can find an s_1-t_2 hamiltonian path in $G_1^3 \setminus s_2$ by Theorem 2, where G_1 is the subgraph of G induced by $V(H_1) \cup \{u_3\}$. Furthermore, an s_2-t_1 hamiltonian path in $G_2^3 \setminus t_2$ also exists by the same reason, where G_2 is the subgraph induced by $V(H_2) \cup \{u_2\}$, forming an unpaired 2-DPC of G^3 along with the s_1-t_2 path. A symmetric argument shows the existence of an unpaired 2-DPC for the second case of $u_1 = t_1$ & $u_4 = s_1$, too. This completes the proof. □

Corollary 2. *For a connected graph G with four or more vertices, G^3 is unpaired 2-coverable if and only if G has no pure bridge vertex of degree at least three.*

3. HAMILTONIAN PATH THROUGH A PRESCRIBED EDGE

In this section, we are concerned to know when the cube of a connected graph has a hamiltonian path between given source and sink that passes through a certain edge. First, given two distinct vertices s and t and a prescribed edge (x, y) of a connected graph H with three or more vertices, the following observations are easily made.

1. If $\{s,t\} \cap \{x,y\} = \emptyset$, it is straightforward that H has an s-t hamiltonian path that passes through (x, y) if and only if H has an *unpaired* 2-*DPC* joining $S = \{s,t\}$ and $T = \{x,y\}$. Note that the order of the vertices x and y on the s-t hamiltonian path may vary depending on the paths of the unpaired 2-DPC.
2. If $\{s,t\} = \{x,y\}$, there exists no such hamiltonian path.
3. If $|\{s,t\} \cap \{x,y\}| = 1$, say, $s = x$ and $t \neq y$ w.l.o.g., H has an s-t hamiltonian path passing through (x, y) if and only if $H \setminus s$ has a y-t hamiltonian path.

These facts naturally lead to the next theorem.

Theorem 5. *Let G be a connected graph with three or more vertices. Given s, $t \in V(G)$ with $s \neq t$, and $(x, y) \in E(G^3)$ such that $\{s,t\} \neq \{x,y\}$, the cube G^3 has an s-t hamiltonian path passing through the prescribed edge (x, y) if and only if*

- *if $\{s,t\} \cap \{x,y\} = \emptyset$, then $\{s,t,x,y\} \not\subseteq N_G[v]$ for any pure bridge vertex v of G, and*
- *if $\{s,t\} \cap \{x,y\} \neq \emptyset$, assuming $s = x$ and $t \neq y$, then $\{s,t,y\} \not\subseteq N_G[v]$ for any pure bridge vertex v of G and $\{s,t,y\}$ does not form a pure bridge triangle of G.*

PROOF. The proof is a direct consequence of Theorem 4 and Theorem 2. □

Corollary 3. *Let G be a connected graph with three or more vertices. The cube G^3 has an s-t hamiltonian path passing through (x, y) for any distinct vertices s and t, and a prescribed edge (x, y) of G^3 such that $\{s,t\} \neq \{x,y\}$ if and only if there exists neither a pure bridge vertex nor a pure bridge triangle in G.*

Remark 1. From Theorem 2, we deduce that the condition given in Corollary 3 can be rephrased as "G^3 is 1-*hamiltonian-connected*, i.e. $G^3 \setminus v_f$ has an s-t hamiltonian path for any three vertices s, t, and v_f."

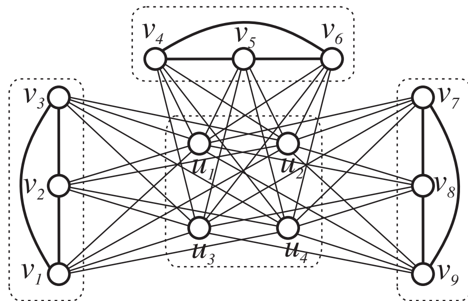


Figure 5: An example graph H that reveals that Corollary 4 does not hold for an arbitrary connected graph. Here, the subgraph of H induced by $\{u_1, u_2, u_3, u_4\}$ is edgeless, whereas those induced by $\{v_1, v_2, v_3\}$, $\{v_4, v_5, v_6\}$, and $\{v_7, v_8, v_9\}$, respectively, are all complete graphs. Furthermore, $(u_i, v_j) \in E(H)$ for all $1 \leq i \leq 4$ and $1 \leq j \leq 9$.

The next corollary is derived directly from Theorems 4 and 5.

Corollary 4. *Let G be a connected graph with four or more vertices. The cube G^3 has an s-t hamiltonian path passing through (x, y) for any distinct vertices s and t, and a prescribed edge (x, y) of G^3 such that $\{s, t\} \cap \{x, y\} = \emptyset$ if and only if G^3 is unpaired 2-coverable.*

Remark 2. If an arbitrary graph H with four or more vertices is unpaired 2-coverable, then H has an s-t hamiltonian path passing through a prescribed edge (x, y) for any s, $t \in V(H)$ with $s \neq t$, and $(x, y) \in E(H)$ such that $\{s, t\} \cap \{x, y\} = \emptyset$. However, unlike the cube of a connected graph, the converse is not always true. For instance, take a look at the graph illustrated in Figure 5. We can see by an immediate inspection that the graph has no unpaired 2-DPC joining $\{u_1, u_2\}$ and $\{u_3, u_4\}$, while it always has an s-t hamiltonian path that passes through (x, y) for any s, t, and (x, y) such that $\{s, t\} \cap \{x, y\} = \emptyset$.

So far, the order in which the two end-vertices of prescribed edge (x, y) are encountered during traversal of an s-t hamiltonian path starting from s has not been important. We may further require that x is visited before y, where such a path is now described as an s-t hamiltonian path passing through an arc $\langle x, y\rangle$ (note that we still consider an undirected graph).

Given a connected graph H with three or more vertices, we can make the following observations similarly as before:

1. If $\{s, t\} \cap \{x, y\} = \emptyset$, H has an s-t hamiltonian path passing through $\langle x, y\rangle$ if and only if H has a *paired* 2-*DPC* joining the (s, x) and (y, t)

pairs. Here, the paired 2-DPC is composed of an s-x path and a y-t path.
2. If either $\{s, t\} = \{x, y\}$, $s = y$, or $t = x$, there exists no such hamiltonian path.
3. If $s = x$ and $t \neq y$, H has an s-t hamiltonian path passing through $\langle x, y\rangle$ if and only if $H \setminus s$ has a y-t hamiltonian path. Similarly, if $t = y$ and $s \neq x$, such a hamiltonian path exists if and only if $H \setminus t$ has an s-x hamiltonian path.

From these observations, a counterpart to Theorem 5 follows immediately.

Theorem 6. *Let G be a connected graph with three or more vertices. Given s, $t \in V(G)$, and a prescribed arc $\langle x, y\rangle$ from $(x, y) \in E(G^3)$ such that $\{s, t\} \neq \{x, y\}$, $s \neq y$, and $t \neq x$, G^3 has an s-t hamiltonian path passing through $\langle x, y\rangle$ if and only if*

- *if $\{s, t\} \cap \{x, y\} = \emptyset$, then $\{s, t, x, y\} \not\subseteq N_G[v]$ for any pure bridge vertex v of G, and $\{s, x\}$ and $\{y, t\}$, respectively, are not a pure bridge pair of G such that $\{y, t\} \subseteq N_G(\{s, x\})$ and $\{s, x\} \subseteq N_G(\{y, t\})$,*
- *if $s = x$ and $t \neq y$, then $\{s, t, y\} \not\subseteq N_G[v]$ for any pure bridge vertex v of G and $\{s, t, y\}$ does not form a pure bridge triangle of G, and*
- *if $t = y$ and $s \neq x$, then $\{s, t, x\} \not\subseteq N_G[v]$ for any pure bridge vertex v of G and $\{s, t, x\}$ does not form a pure bridge triangle of G.*

PROOF. The proof is a direct consequence of Theorem 3 and Theorem 2. □

We also have counterpart corollaries for this stronger problem. The first one is as follows:

Corollary 5. *Let G be a connected graph with three or more vertices. The cube G^3 has an s-t hamiltonian path passing through $\langle x, y\rangle$ for any distinct vertices s and t, and a prescribed arc $\langle x, y\rangle$ from an edge (x, y) of G^3 such that $\{s, t\} \neq \{x, y\}$, $s \neq y$, and $t \neq x$ if and only if there exists neither a pure bridge vertex nor a pure bridge triangle in G.*

Note that the condition of this corollary is the same as that of Corollary 3, indicating that, whether the prescribed edge is directed or not, the absence of a pure bridge vertex and a pure bridge triangle in G is the exact condition for a wanted hamiltonian path to exist in G^3. Next, the following corollary can be deduced directly from Theorems 3 and 6.

Corollary 6. *Let G be a connected graph with four or more vertices. The cube G^3 has an s-t hamiltonian path passing through $\langle x, y\rangle$ for any distinct vertices s and t, and a prescribed arc $\langle x, y\rangle$ from an edge (x, y) of G^3 such that $\{s, t\} \cap \{x, y\} = \emptyset$ if and only if G^3 is paired 2-coverable.*

Remark 3. If an arbitrary graph H with four or more vertices is paired 2-coverable, then H has an s-t hamiltonian path passing through a prescribed arc $\langle x, y\rangle$ for any s, $t \in V(H)$ with $s \neq t$, and $(x, y) \in E(H)$ such that $\{s, t\} \cap \{x, y\} = \emptyset$. Again, the converse is not always true for H as can also be seen from the graph in Figure 5.

4. CONCLUDING REMARKS

Deciding if the cube of a connected graph contains a paired or unpaired 2-disjoint path cover for arbitrarily given sources and sinks is an interesting problem in graph theory. In this paper, we have presented the necessary and sufficient conditions for the two 2-disjoint path cover problems. We have also characterized exactly when the cube of a connected graph has a hamiltonian path from a given vertex to another vertex passing through a prescribed edge. All the proofs given for the four main theorems in this paper are constructive themselves, hence, can be used effectively to design an algorithm to find the respective 2-disjoint path covers or hamiltonian paths. Recall that the pure bridge vertices, pure bridge triangles, and pure bridge pairs can all be computed in linear time with respect to the size of the graph, based on the well-known biconnected component algorithm. Therefore, all the exact conditions for the four theorems can be checked efficiently in linear time.

Unlike the cube of a connected graph that allows easy recursive construction, generating a paired or unpaired 2-disjoint path cover in the square of a 2-connected graph remains a challenging problem. We hope that the following two conjectures could initiate future research.

Conjecture 1. *The square of every 2-connected graph with four or more vertices is unpaired 2-coverable.*

Conjecture 2. *Let $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$ be terminal sets of a 2-connected graph G with four or more vertices. The square G^2 has no paired 2-DPC joining S and T if and only if*

- *G is isomorphic to an even cycle $(v_0, v_1, \ldots, v_{|V(G)|-1})$, and*
- *S and T are such that $\{s_1, t_1\} = \{v_0, v_q\}$ and $\{s_2, t_2\} = \{v_p, v_r\}$ for some even integers p, q, and r with $0 < p < q < r$.*

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