

# Disjoint Path Covers in Recursive Circulants $G(2^m, 4)$ with Faulty Elements<sup>☆</sup>

Sook-Yeon Kim<sup>a</sup>, Jae-Ha Lee<sup>b</sup>, Jung-Heum Park<sup>c,\*</sup>

<sup>a</sup>*Department of Computer Engineering, Hankyong National University, Ansung 456-749, Korea*

<sup>b</sup>*Computer Engineering Department, Konkuk University, Seoul 143-701, Korea*

<sup>c</sup>*School of Computer Science and Information Engineering, The Catholic University of Korea, Bucheon 420-743, Korea*

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## Abstract

A  $k$ -disjoint path cover of a graph is defined as a set of  $k$  internally vertex-disjoint paths connecting given sources and sinks in such a way that every vertex of the graph is covered by a path in the set. In this paper, we analyze the  $k$ -disjoint path cover of recursive circulant  $G(2^m, 4)$  under the condition that at most  $f$  faulty vertices and/or edges are removed. It is shown that when  $m \geq 3$ ,  $G(2^m, 4)$  has a  $k$ -disjoint path cover (of one-to-one type) joining any pair of two distinct source and sink for arbitrary  $f$  and  $k \geq 2$  subject to  $f + k \leq m$ . In addition, it is proven that when  $m \geq 5$ ,  $G(2^m, 4)$  has a  $k$ -disjoint path cover (of unpaired many-to-many type) joining any two disjoint sets of  $k$  sources and  $k$  sinks for arbitrary  $f$  and  $k \geq 2$  satisfying  $f + k \leq m - 1$ , in which sources and sinks are freely matched. In particular, the mentioned bounds  $f + k \leq m$  and  $f + k \leq m - 1$  of the two cases are shown to be optimal.

*Keywords:* Disjoint paths, fault-hamiltonicity, graph theory, embedding, linear arrays, fault tolerance, interconnection networks.

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\*Corresponding author

*Email addresses:* sookyeon@hknu.ac.kr (Sook-Yeon Kim), jaehalee@konkuk.ac.kr (Jae-Ha Lee), j.h.park@catholic.ac.kr (Jung-Heum Park)

## 1. Introduction

An interconnection network is frequently modeled as a graph, where vertices and edges respectively represent nodes and communication links in the network. One of the several key problems in the study of interconnection networks is to detect (vertex-)disjoint paths that abstract the routing between nodes and the embedding of linear arrays. Such vertex-disjoint paths can be viewed as parallel routes that indicate data communication between nodes. A *k-disjoint path cover* (*k-DPC* for short) of a graph is a set of  $k$  internally disjoint paths that altogether cover every vertex of the graph. The *k-disjoint path cover* problem, originated from the community of interconnection networks, is intended to search for a way of fully utilizing nodes for efficient communications [25]. When a graph contains faulty elements, whether vertices or edges, its *k-disjoint path cover* naturally means a *k-disjoint path cover* of the graph with the faulty elements deleted.

The problem of finding such *k-disjoint path covers* can be classified into three kinds according to the source and sink configuration: one-to-one, one-to-many, and many-to-many. The *one-to-one* class considers disjoint path covers joining a single pair of source  $s$  and sink  $t$ , while the *one-to-many* class deals with disjoint path covers joining a single source  $s$  and a set of  $k$  distinct sinks  $t_1, t_2, \dots, t_k$ . Obviously, the paths of one-to-one *k-DPC*, also known as *k\*-container* [5, 28], have common vertices only at their source and sink, while those of one-to-many *k-DPC* overlap only at their source.

The *many-to-many* class, on the other hand, considers disjoint path covers between a set of  $k$  sources  $s_1, s_2, \dots, s_k$  and another set of  $k$  sinks  $t_1, t_2, \dots, t_k$ , where any many-to-many *k-DPC* of graph partitions its vertex set into  $k$  paths. The problems in this class are further subdivided into two subclasses: paired and unpaired. In the *paired* type problem, each source  $s_i$  is required to be paired to a designated sink  $t_i$ . In the *unpaired* type problem, on the other hand, the sources and sinks are allowed to be freely mapped. In other words, source  $s_i$  can be freely matched to sink  $t_{\sigma_i}$  under an arbitrary permutation  $\sigma$  on  $\{1, 2, \dots, k\}$ .

Several types of graphs have already been studied on their disjoint path covers. One-to-one covers were analyzed for recursive circulants [19, 28] and hypercubes with faulty edges [5]. In [20], one-to-many covers were constructed for hypercube-like interconnection networks with faulty elements. Furthermore, for a class of nonbipartite hypercube-like interconnection networks, called *restricted HL-graphs*, having faulty elements, paired disjoint path covers [25, 26] and unpaired disjoint path covers [21] were built. In [13], all  $m$ -dimensional crossed cubes, twisted cubes, and Möbius cubes with

$m \geq 5$  were shown to have a paired 2-DPC whose paths are of equal length.

The disjoint path cover problem has also been studied for some bipartite graphs. Paired disjoint path covers were investigated for hypercubes [11] and hypercubes with faulty vertices [8]. Unpaired disjoint path covers were considered for hypercubes with faulty edges [6] and bipartite graphs obtained by adding edges to hypercubes [7]. Interestingly, it was proven to be all NP-complete to determine if, for any fixed  $k \geq 1$ , there exists either a one-to-one  $k$ -DPC, a one-to-many  $k$ -DPC, or a many-to-many  $k$ -DPC, whether paired or unpaired, in general graphs [25, 26].

Before turning to the next section, we briefly go over the definitions of key notions. First of all, throughout this paper, we assume that the source and sink sets  $S$  and  $T$  of graph  $G$  are disjoint to each other and both belong to  $V(G) \setminus F$ , where  $V(G)$  and  $F$  represent the vertex set and a fault set of  $G$ , respectively. Sometimes, the sources and sinks, generally called *terminals*, are assumed to be fixed, but in our work, we deal with a stronger case where  $k$ -disjoint path covers are sought for graphs with arbitrary faulty elements and source/sink sets.

**Definition 1.** (a) *A graph  $G$  is called  $f$ -fault one-to-one  $k$ -disjoint path coverable if  $f + 2 \leq |V(G)|$  and for any fault set  $F$  with  $|F| \leq f$ ,  $G$  has a one-to-one  $k$ -DPC joining an arbitrary pair of source  $s$  and sink  $t$  in  $G \setminus F$  subject to  $s \neq t$ .*

(b) *A graph  $G$  is called  $f$ -fault one-to-many  $k$ -disjoint path coverable if  $f + k + 1 \leq |V(G)|$  and for any fault set  $F$  with  $|F| \leq f$ ,  $G$  has a one-to-many  $k$ -DPC joining an arbitrary source  $s$  and an arbitrary set  $T$  of  $k$  sinks in  $G \setminus F$  subject to  $s \notin T$ .*

(c) *A graph  $G$  is called  $f$ -fault unpaired (resp. paired) many-to-many  $k$ -disjoint path coverable if  $f + 2k \leq |V(G)|$  and for any fault set  $F$  with  $|F| \leq f$ ,  $G$  has an unpaired (resp. paired)  $k$ -DPC joining an arbitrary set  $S$  of  $k$  sources and another arbitrary set  $T$  of  $k$  sinks in  $G \setminus F$  subject to  $S \cap T = \emptyset$ .*

This paper's interest is to investigate the construction of the disjoint path covers in recursive circulants. The recursive circulant  $G(N, d)$ ,  $d \geq 2$ , proposed in [23], is a graph with a vertex set  $V = \{v_0, v_1, v_2, \dots, v_{N-1}\}$  and an edge set  $E = \{(v_i, v_j) : i + d^k \equiv j \pmod{N} \text{ for some } k, 0 \leq k \leq \lceil \log_d N \rceil - 1\}$ . In other words,  $G(N, d)$  is a circulant graph with  $N$  vertices and jumps of powers of  $d$ ,  $d^0, d^1, \dots, d^{\lceil \log_d N \rceil - 1}$ , which can also be defined as a Cayley graph of the cyclic group  $\mathbb{Z}_N$  with the generating set  $\{d^0, d^1, \dots, d^{\lceil \log_d N \rceil - 1}\}$ . Examples of  $G(N, d)$  are shown in Figure 1.

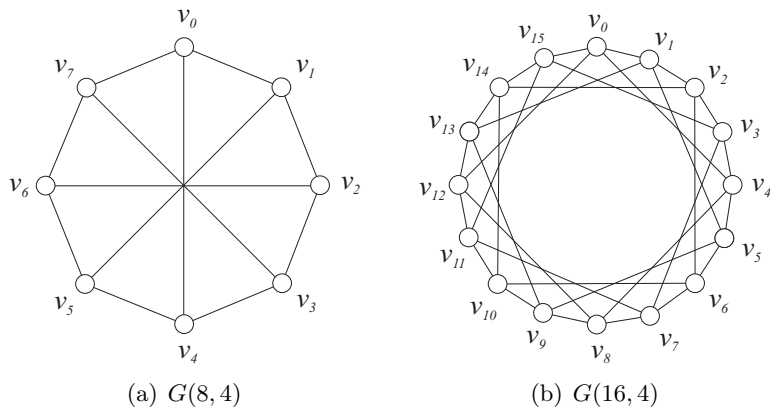


Figure 1: Recursive circulants.

In this article, we focus on the recursive circulant  $G(N, d)$  with  $N = 2^m$  and  $d = 4$ . Such recursive circulant  $G(2^m, 4)$  of degree  $m$  compares favorably to hypercube  $Q_m$ . While retaining attractive properties of the hypercube such as node-symmetry, recursive structure, maximum connectivity, etc., it achieves a noticeable improvement in diameter [23] as well as includes a complete binary tree with  $2^m - 1$  vertices as a subgraph [15]. Many results on recursive circulants are found in the literature, regarding, say hamiltonian decomposition [3, 10, 16, 18], panconnectivity and pancyclicity [1, 2, 22], independent spanning trees [29], maximum induced subgraph [30], chromatic number [17], parallel routing [12], recognition problem [9], edge forwarding index and bisection width [10], etc.

In the previous works, it has been shown that  $G(2^m, 4)$ ,  $m \geq 3$ , is (0-fault) one-to-one  $k$ -disjoint path coverable for any  $1 \leq k \leq m$  [19], is  $f$ -fault one-to-many  $k$ -disjoint path coverable for any  $f$  and  $k \geq 2$  with  $f+k \leq m-1$  [20], and is  $f$ -fault paired many-to-many  $k$ -disjoint path coverable for any  $f$  and  $k \geq 2$  with  $f+2k \leq m$  [26]. In addition to these results, we will show that  $G(2^m, 4)$ ,  $m \geq 3$ , is  $f$ -fault one-to-one  $k$ -disjoint path coverable for any  $f$  and  $k \geq 2$  with  $f+k \leq m$ , and  $G(2^m, 4)$ ,  $m \geq 5$ , is  $f$ -fault *unpaired* many-to-many  $k$ -disjoint path coverable for any  $f$  and  $k \geq 2$  with  $f+k \leq m-1$ . The bound  $f+k \leq m$  achieved for a one-to-one  $k$ -DPC problem is proven optimal based upon the necessary condition shown in Lemma 7. The bound  $f+k \leq m-1$  established for an unpaired  $k$ -DPC problem is also found optimal due to the necessary condition derived in [26].

This paper is organized as follows. In the next section, we discuss the

recursive structure and fault-hamiltonicity of recursive circulant and recursive circulant-like graphs. By utilizing the recursive structure, one-to-many DPC's, one-to-one DPC's, and unpaired many-to-many DPC's of recursive circulant and recursive circulant-like graphs are constructed in Sections 3, 4, and 5, respectively. Finally, concluding remarks of the paper are given in Section 6.

## 2. Recursive Structures

Before discussing the recursive structure of recursive circulants, we define a simple graph construction operation. For two graphs  $H_0$  and  $H_1$  with the same number of vertices, consider a bijection  $f$  between the vertex sets  $V(H_0)$  and  $V(H_1)$ . We denote by  $H_0 \oplus H_1$  the graph obtained by joining the vertices of  $H_0$  and  $H_1$  using edges  $(v, f(v))$  for all  $v \in V(H_0)$ . Given  $H_0 \oplus H_1$ ,  $H_0$  and  $H_1$  are called *components*, and  $f(v)$  for  $v \in V(H_0)$  and  $f^{-1}(v)$  for  $v \in V(H_1)$  are both represented by  $\bar{v}$  for short.

The recursive circulant  $G(N, d)$  has a recursive structure when  $N = cd^m$ ,  $1 \leq c < d$  [23], based upon the following property.

**Property 1.** [23] *Given  $G(cd^m, d)$  with  $m \geq 1$ , consider a vertex subset  $V_i$  such that  $V_i = \{v_j : j \equiv i \pmod{d}\}$ . Then the subgraph  $G_i$  induced by  $V_i$  is isomorphic to  $G(cd^{m-1}, d)$  for all  $i = 0, 1, \dots, d-1$ .*

When  $m \geq 1$ ,  $G(cd^m, d)$  can be recursively constructed using  $d$  copies of  $G(cd^{m-1}, d)$ , which we denote by  $G_i(V_i, E_i)$ ,  $0 \leq i < d$ , with  $V_i = \{v_0^i, v_1^i, \dots, v_{cd^{m-1}-1}^i\}$ . Here,  $G_i$  is isomorphic to  $G(cd^{m-1}, d)$  with regard to a bijection mapping  $v_j^i$  to  $v_j$ . Let  $v_j^i$  be relabeled by  $v_{jd+i}$  for convenience. Then  $G(cd^m, d)$  can be built by defining the vertex set  $V$  as  $\bigcup_{0 \leq i < d} V_i$ , and the edge set  $E$  as  $\bigcup_{0 \leq i < d} E_i \cup X$ , where  $X = \{(v_j, v_{j'}) : j+1 \equiv j' \pmod{cd^m}\}$ .

Recursive circulant  $G(2^m, 4)$ , a special case of  $G(cd^m, d)$ , consists of four *components*  $G_0, G_1, G_2$ , and  $G_3$  each of which is isomorphic to  $G(2^{m-2}, 4)$  when  $m \geq 2$  (see Figure 2 to understand how  $G(32, 4)$  is built from the four copies of  $G(8, 4)$ ). It is notable that the subgraph induced by vertices in  $G_i$  and  $G_{(i+1) \bmod 4}$  for any  $i = 0, 1, 2, 3$ , is isomorphic to the product  $G(2^{m-2}, 4) \times K_2$  of  $G(2^{m-2}, 4)$  and  $K_2$ , where  $K_2$  is a complete graph with two vertices. Let  $H_0$  and  $H_1$  be the subgraphs induced by  $V(G_0) \cup V(G_1)$  and by  $V(G_2) \cup V(G_3)$ , respectively. Then, the graph can be expressed as  $H_0 \oplus H_1$ , where  $H_0$  and  $H_1$  are isomorphic to  $G(2^{m-2}, 4) \times K_2$ .

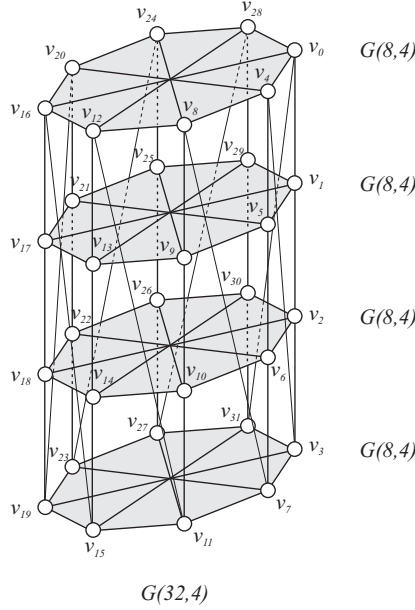


Figure 2: Recursive structure of  $G(32, 4)$ .

Now, consider  $d$  copies  $G_0, G_1, \dots, G_{d-1}$  of a graph  $G$  having  $n$  vertices. If we apply the graph constructor  $\oplus$  to each pair  $G_i$  and  $G_{(i+1) \bmod d}$ ,  $0 \leq i < d$ , we obtain a graph with  $nd$  vertices. This graph, which is said to be obtained through the *cycle-based recursive construction*, will be denoted as  $G \otimes C_d$ . Here,  $C_d$  represents a cycle graph with  $d$  vertices. In the following discussion, we respectively denote by  $v^+$  and  $v^-$  the vertices of  $G_{(i+1) \bmod d}$  and  $G_{(i-1) \bmod d}$  that are adjacent to  $v$  in  $G_i$ . Then, recursive circulant  $G(2^m, 4)$  can also be expressed in terms of  $G(2^{m-2}, 4) \otimes C_4$  as well as  $[G(2^{m-2}, 4) \times K_2] \oplus [G(2^{m-2}, 4) \times K_2]$ . It can be observed that any graph representable as  $G \otimes C_4$  is also representable as  $[G_0 \oplus G_1] \oplus [G_2 \oplus G_3]$  for some  $G_i$ 's isomorphic to  $G$ , although the converse does not always hold.

In general,  $G(2^m, 4)$  cannot be obtained from a single operation  $\oplus$  on two recursive circulants. In other words, an arbitrary  $G(2^m, 4)$  is not always representable as  $H_0 \oplus H_1$  of two graphs  $H_0$  and  $H_1$  that are isomorphic to  $G(2^{m-1}, 4)$ . This implies that, when we want to recursively construct a disjoint path cover in  $G(2^m, 4)$ , we cannot utilize the disjoint path coverability of  $G(2^{m-1}, 4)$ . On the other hand, we can still utilize the disjoint path coverability of  $G(2^{m-2}, 4)$ , which undesirably provokes a large number of cases. Thus, we introduce a class of nonbipartite graphs containing  $G(2^m, 4)$  in

order to take advantage of simple recursive structure. An arbitrary higher dimensional graph (with a unique exception) may be represented as  $H_0 \oplus H_1$  for two lower dimensional graphs  $H_0$  and  $H_1$  in the class.

**Definition 2.** *A class of graphs, called RC-like graphs or RCL-graphs for short, is defined as follows:*

- $RCL_3 = \{G(8, 4)\};$
- $RCL_4 = \{G(16, 4), G(8, 4) \times K_2\};$
- $RCL_m = \{G(2^m, 4), G(2^{m-1}, 4) \times K_2, G(2^{m-2}, 4) \times C_4\}$  for  $m \geq 5$ .

Here, a graph that belongs to  $RCL_m$  for some  $m \geq 3$  is called an  $m$ -dimensional RC-like graph.

For convenience, we define a superclass of RC-like graphs, called the *expanded RC-like graphs*, as  $RCL_m^e = \{G(2^m, 4), G(2^{m-1}, 4) \times K_2, G(2^{m-2}, 4) \times C_4\}$  for  $m \geq 3$ . Notice that the graph  $G(4, 4) \times C_4$  in  $RCL_4^e$  does not belong to  $RCL_4$ . Also,  $G(4, 4) \times K_2$  and  $G(2, 4) \times C_4$  in  $RCL_3^e$  do not exist in  $RCL_3$ . These three graphs are bipartite, while all the graphs in the class of RC-like graphs are nonbipartite since each of them contains a subgraph isomorphic to  $G(8, 4)$  or  $G(16, 4)$ . Now, we have a small lemma:

**Lemma 1.** (a) *Every RC-like graph is nonbipartite.*  
(b) *Every  $m$ -dimensional RC-like graph  $G^m$  is made of  $2^m$  vertices of degree  $m$ .*

Since each of the two graphs  $G(2^m, 4)$  and  $G(2^{m-2}, 4) \times C_4$  in  $RCL_m^e$  has four components  $G_0, G_1, G_2,$  and  $G_3$ , which are respectively isomorphic to  $G(2^{m-2}, 4)$ , they can be represented in the form of  $G(2^{m-2}, 4) \otimes C_4$ . Let  $H_0$  and  $H_1$  be the subgraphs induced by  $V(G_0) \cup V(G_1)$  and by  $V(G_2) \cup V(G_3)$ , respectively. Then, the two graphs can also be expressed as  $H_0 \oplus H_1$ , where  $H_0$  and  $H_1$  are isomorphic to  $G(2^{m-2}, 4) \times K_2$ .

Let's take a look at the third graph  $G(2^{m-1}, 4) \times K_2$  in  $RCL_m^e$  more carefully. It also has a recursive structure, which is derived from the recursive structure of  $G(2^{m-1}, 4)$ . Again, it has four components  $G_0, G_1, G_2,$  and  $G_3$ , isomorphic to  $G(2^{m-3}, 4) \times K_2$ . Thus, the graph can be expressed as  $[G(2^{m-3}, 4) \times K_2] \otimes C_4$ . If we define  $H_0$  and  $H_1$  as in the above paragraph,  $H_0$  and  $H_1$  are isomorphic to  $[G(2^{m-3}, 4) \times K_2] \times K_2$ , which is, in fact, isomorphic to  $G(2^{m-3}, 4) \times C_4$ . Therefore, the graph can also be represented as  $H_0 \oplus H_1$ , where  $H_0$  and  $H_1$  are isomorphic to  $G(2^{m-3}, 4) \times C_4$ . This observation leads to the next lemma.

**Lemma 2.** (a) For  $m \geq 5$ , every  $m$ -dimensional RC-like graph  $G^m$  except for  $G(16, 4) \times K_2$  can be expressed as  $G \otimes C_4$ , where the four components  $G_0, G_1, G_2$ , and  $G_3$  are isomorphic to a graph  $G$  in  $RCL_{m-2}$ . Furthermore, the graph can also be expressed as  $H_0 \oplus H_1$ , where  $H_0$  and  $H_1$  are the subgraphs induced by  $V(G_0) \cup V(G_1)$  and by  $V(G_2) \cup V(G_3)$ , respectively, and both of them are isomorphic to a graph in  $RCL_{m-1}$ .

(b) For  $m \geq 4$ , every  $m$ -dimensional RC-like graph  $G^m$  except for  $G(16, 4)$  can be expressed as  $H_0 \oplus H_1$ , where the two components  $H_0$  and  $H_1$  are isomorphic to a graph in  $RCL_{m-1}$ .

The last but not the least property of RC-like graphs we discuss in this preliminary section is the fault-hamiltonicity. A graph  $G$  is called  $f$ -fault hamiltonian (resp.  $f$ -fault hamiltonian-connected) if there exists a hamiltonian cycle (resp. if each pair of vertices are joined by a hamiltonian path) in  $G \setminus F$  for any set  $F$  of faulty elements with  $|F| \leq f$ . It is worth mentioning that a graph  $G$  is  $f$ -fault (either one-to-one, one-to-many, or many-to-many) 1-disjoint path coverable if and only if  $G$  is  $f$ -fault hamiltonian-connected. In the following, let  $\delta(G)$  denote the minimum degree of a graph  $G$ .

**Lemma 3.** For  $m \geq 3$ , every  $m$ -dimensional RC-like graph is  $(m - 3)$ -fault hamiltonian-connected and  $(m - 2)$ -fault hamiltonian.

PROOF. It has been proven that (i) the graph  $G(2^m, 4)$  with  $m \geq 3$  is  $(m - 3)$ -fault hamiltonian-connected and  $(m - 2)$ -fault hamiltonian [24, 27], and that (ii) if a graph  $G$  is  $(\delta(G) - 3)$ -fault hamiltonian-connected and  $(\delta(G) - 2)$ -fault hamiltonian, then  $G \times K_2$  is  $(\delta(G) - 2)$ -fault hamiltonian-connected and  $(\delta(G) - 1)$ -fault hamiltonian [24]. Clearly, the proof of this lemma is a direct consequence of these two facts. Recall that  $G \times C_4$  is isomorphic to  $[G \times K_2] \times K_2$ .  $\square$

### 3. One-to-Many Disjoint Path Covers

In this section, we will consider the problem of constructing one-to-many DPC's in RC-like graphs with faulty elements. The construction will be utilized when we build one-to-one DPC's in the graphs. The problem on recursive circulant  $G(2^m, 4)$  was studied in [20] as follows.

**Lemma 4.** [20]  $G(2^m, 4)$ ,  $m \geq 3$ , is  $f$ -fault one-to-many  $k$ -disjoint path coverable for any  $f$  and  $k \geq 2$  subject to  $f + k \leq m - 1$ .



It is worthy of remark that the bound  $f+k \leq m-1$  achieved in Lemma 4 is optimal due to the following necessary condition given in [14]. We denote by  $\kappa(G)$  the connectivity of a graph  $G$ .

**Lemma 5.** [14] *If a graph  $G$  is  $f$ -fault one-to-many  $k$ -disjoint path coverable, then  $\kappa(G) \geq f+k$ . Furthermore, if  $G$  has  $f+k+2$  or more vertices, then  $\kappa(G) \geq f+k+1$ .*

To construct one-to-many DPC's in RC-like graphs, we begin by pointing out the fact in [20] that a graph  $G$  is  $f$ -fault one-to-many 2-disjoint path coverable if and only if  $G$  is  $f$ -fault one-to-many 1-disjoint path coverable, which is equivalent to that  $G$  is  $f$ -fault hamiltonian-connected. By utilizing fault-hamiltonicity of RC-like graphs given in Lemma 3, an  $f$ -fault one-to-many  $k$ -DPC for  $k=1,2$  can be constructed when  $f \leq m-3$ . It has been shown in [20] that an  $f$ -fault one-to-many  $k$ -DPC in  $H_0 \oplus H_1$  can be recursively constructed from  $f$ -fault one-to-many  $(k-1)$ -DPC and fault-hamiltonicity of  $H_i$ ,  $i=0,1$ , as follows.

**Lemma 6.** [20] *For  $f \geq 0$  and  $k \geq 3$ , let  $H_i$  be a graph with  $n$  vertices satisfying the following three conditions,  $i=0,1$ .*

- (a)  $H_i$  is  $f$ -fault one-to-many  $(k-1)$ -disjoint path coverable.
- (b)  $H_i$  is  $(f+k-3)$ -fault hamiltonian-connected (2-disjoint path coverable).
- (c)  $H_i$  is  $(f+k-2)$ -fault hamiltonian.

*Then,  $H_0 \oplus H_1$  is  $f$ -fault one-to-many  $k$ -disjoint path coverable.*

Lemmas 3 and 6 lead to one-to-many disjoint path coverability of RC-like graphs as follows.

**Theorem 1.** *Every  $m$ -dimensional RC-like graph  $G^m$ ,  $m \geq 3$ , is  $f$ -fault one-to-many  $k$ -disjoint path coverable for any  $f$  and  $k \geq 2$  subject to  $f+k \leq m-1$ .*

PROOF. The proof is by induction on  $m$ . Due to Lemma 4, it suffices to consider  $G(2^{m-1}, 4) \times K_2$  with  $m \geq 4$  and  $G(2^{m-2}, 4) \times C_4$  with  $m \geq 5$ . Let  $H_0 \oplus H_1$  be one of these two graphs, where  $H_0$  and  $H_1$  are isomorphic to either  $G(2^{m-1}, 4)$  or  $G(2^{m-2}, 4) \times K_2$ . If  $k=2$ , then  $f \leq m-3$  and by Lemma 3,  $H_0 \oplus H_1$  is  $f$ -fault one-to-many 2-disjoint path coverable. Assume  $k \geq 3$ . Since  $f+k \leq m-1$ , each  $H_i$  is (i)  $f$ -fault one-to-many  $(k-1)$ -disjoint path coverable by induction hypothesis, (ii)  $(f+k-3)$ -fault hamiltonian-connected by Lemma 3, and (iii)  $(f+k-2)$ -fault hamiltonian by Lemma 3. Thus, by Lemma 6,  $H_0 \oplus H_1$  is  $f$ -fault one-to-many  $k$ -disjoint path coverable. This completes the proof.  $\square$

Of course, the bound  $f + k \leq m - 1$  achieved in Theorem 1 is optimal due to Lemma 5.

#### 4. One-to-One Disjoint Path Covers

We begin with a necessary condition for a graph to be  $f$ -fault one-to-one  $k$ -disjoint path coverable.

**Lemma 7.** *If a graph  $G$  is  $f$ -fault one-to-one  $k$ -disjoint path coverable, then  $\kappa(G) \geq f + k$ .*

PROOF. According to Menger's theorem (see ref. [4]), a graph  $G$  is  $k$ -connected if and only if for every pair of source  $s$  and sink  $t$ ,  $G$  has  $k$  internally disjoint paths of type one-to-one joining them. A one-to-one  $k$ -disjoint path coverable graph should be  $k$ -connected, and thus the lemma follows.  $\square$

We are going to construct  $f$ -fault one-to-one  $k$ -disjoint path covers in  $m$ -dimensional RC-like graphs for any  $f$  and  $k \geq 2$  satisfying the optimal bound  $f + k \leq m$  of Lemma 7. That is, we will establish the following theorem.

**Theorem 2.** *Every  $m$ -dimensional RC-like graph  $G^m$ ,  $m \geq 3$ , is  $f$ -fault one-to-one  $k$ -disjoint path coverable for any  $f$  and  $k \geq 2$  subject to  $f + k \leq m$ .*

A graph  $G$  is  $f$ -fault one-to-one 2-disjoint path coverable if and only if  $G$  is  $f$ -fault hamiltonian. Thus, to prove Theorem 2, we can assume that

$$k \geq 3$$

due to Lemma 3. A path in a graph is represented as a sequence of vertices. An  $s$ - $t$  path refers to a path from vertex  $s$  to  $t$ , and an  $s$ -path refers to a path whose starting vertex is  $s$ .

##### 4.1. Proof of Theorem 2 when $f = 0$

The one-to-one DPC problem in fault-free  $G(2^m, 4)$  was studied in [19] as follows. We denote by  $P(l)$  a graph isomorphic to a path having  $l$  vertices. In  $G(2^{m-2}, 4) \times P(l)$  with  $l \geq 2$ , each component is isomorphic to  $G(2^{m-2}, 4)$  and referred to  $G_0, G_1, \dots, G_{l-1}$ .

**Lemma 8.** [19] (a)  $G(2^m, 4)$  with  $m \geq 3$  is one-to-one  $k$ -disjoint path coverable for any  $1 \leq k \leq m$ .

(b)  $G(2^{m-1}, 4) \times P(l)$  with  $m \geq 4$  and  $l \geq 2$  has a one-to-one  $k$ -DPC joining any source  $s$  in  $G_0$  and sink  $t$  in  $G_{l-1}$  for any  $1 \leq k \leq m$ .

Now, let us consider one-to-one disjoint path coverability of  $G(2^{m-1}, 4) \times K_2$  and  $G(2^{m-2}, 4) \times C_4$ .

**Lemma 9.**  $G(2^{m-1}, 4) \times K_2$  with  $m \geq 4$  is one-to-one  $k$ -disjoint path coverable for any  $3 \leq k \leq m$ .

PROOF. Let  $G_0$  and  $G_1$  be components isomorphic to  $G(2^{m-1}, 4)$ . If  $s \in V(G_0)$  and  $t \in V(G_1)$ , then by Lemma 8(b), there exists a one-to-one  $k$ -DPC joining  $s$  and  $t$ . Now let  $s, t \in V(G_0)$ . We first construct a one-to-one  $(k-1)$ -DPC in  $G_0$  by Lemma 8(a), and then path  $P_k = (s, \bar{s}, P_h, \bar{t}, t)$  is added to the DPC, where  $P_h$  is an  $\bar{s}$ - $\bar{t}$  hamiltonian path in  $G_1$ . Thus, we have the lemma.  $\square$

**Lemma 10.**  $G(2^{m-2}, 4) \times C_4$  with  $m \geq 5$  is one-to-one  $k$ -disjoint path coverable for any  $3 \leq k \leq m$ .

PROOF. Let  $G_0, G_1, G_2,$  and  $G_3$  be the four components isomorphic to  $G(2^{m-2}, 4)$ . We assume  $s \in V(G_0)$  and let  $t \in V(G_i)$ . We assume w.l.o.g.  $i = 0, 1,$  or  $2$ . If  $i = 0$ , we first find a one-to-one  $(k-2)$ -DPC in  $G_0$ , and then add two paths  $P_{k-1} = (s, s^+, P_h^1, t^+, t)$  and  $P_k = (s, s^-, P_h^2, t^-, t)$  to the DPC, where  $P_h^1$  is a hamiltonian path in  $G_1$  joining  $s^+$  and  $t^+$ , and  $P_h^2$  is a hamiltonian path in the subgraph  $H_1$  induced by  $V(G_2) \cup V(G_3)$  joining  $s^-$  and  $t^-$ . It can be easily seen that  $H_1$  is hamiltonian-connected since both  $G_2$  and  $G_3$  are hamiltonian-connected. If  $i = 1, 2$ , we first find a one-to-one  $(k-1)$ -DPC in the subgraph induced by  $V(G_0) \cup \dots \cup V(G_i)$  joining  $s$  and  $t$  by Lemma 8(b). For the subcase of either  $i = 1$  or  $i = 2$  and  $s^- \neq t^+$ , we add path  $P_k = (s, s^-, P_h, t^+, t)$  to the DPC, where  $P_h$  is a hamiltonian path in the subgraph induced  $V(G_{i+1}) \cup \dots \cup V(G_3)$  joining  $s^-$  and  $t^+$ .

Finally, let  $i = 2$  and  $s^- = t^+$ . In this subcase, let the last path  $P_k = (s, s^-, t)$ . To cover the vertices in  $G_3$  other than  $s^-$ , we are going to pick up an edge  $(x, y) \in E(G_2) \cup E(G_0)$  on some path  $P_j$  in the DPC such that  $G_3 \setminus s^-$  has a hamiltonian path  $P_h$  joining  $x^+$  and  $y^+$  when  $x, y \in V(G_2)$  or joining  $x^-$  and  $y^-$  when  $x, y \in V(G_0)$ . And then, the edge  $(x, y)$  on  $P_j$  is replaced with  $(x, x^+, P_h, y^+, y)$  or  $(x, x^-, P_h, y^-, y)$ , resulting in a new path  $P'_j$ . If  $m \geq 6$ , an arbitrarily edge  $(x, y)$  in  $G_2$  or in  $G_0$  such that  $\{x, y\} \cap \{s, t\} = \emptyset$  is acceptable since  $G_3$  is  $(m-5)$ -fault hamiltonian-connected. Let  $m = 5$ .



Suppose otherwise. Then, regarding  $(s, t)$  as a virtual fault allows us to find  $(f + 1)$ -fault one-to-one  $(k - 1)$ -DPC and to add the path  $(s, t)$  to the DPC, resulting in an  $f$ -fault  $k$ -DPC. We also assume that when  $f = 1$  and  $k = m - 1$ ,

$$F \neq \{(s, t)\} \text{ and } F \neq \{v_f\} \text{ for any } v_f \text{ with } (s, v_f), (t, v_f) \in E(G^m).$$

Suppose otherwise. Then, regarding the faulty element as a virtual fault-free element allows us to find a 0-fault one-to-one  $m$ -DPC (by the algorithm in Subsection 4.1) and to remove the path either  $(s, t)$  or  $(s, v_f, t)$  passing through the faulty element from the DPC, resulting in a 1-fault  $(m - 1)$ -DPC.

The proof will proceed by induction on  $m$ . Recall that every  $m$ -dimensional RC-like graph  $G^m$ ,  $m \geq 4$ , except for  $G(16, 4)$  can be expressed as  $H_0 \oplus H_1$ , where  $H_0, H_1 \in RCL_{m-1}$  by Lemma 2(b). To construct an  $f$ -fault one-to-one  $k$ -DPC in  $G^m$ , the recursive structure of  $H_0 \oplus H_1$  will be utilized. For the exception  $G(16, 4)$ , a computer program for finding 1-fault 3-DPC for given a fault and a pair of  $s$  and  $t$  was written in C language. The validity of the following lemma was checked by the program.

**Lemma 11.**  $G(16, 4)$  is 1-fault one-to-one 3-disjoint path coverable.

From now on, let  $G^m$  be expressed as  $H_0 \oplus H_1$ .  $F_0$  and  $F_1$  denote the sets of faulty elements in  $H_0$  and  $H_1$ , respectively, and  $F_2$  denotes the set of faulty edges joining vertices in  $H_0$  and vertices in  $H_1$ , so that  $F = F_0 \cup F_1 \cup F_2$ . Let  $f_0 = |F_0|$ ,  $f_1 = |F_1|$ , and  $f_2 = |F_2|$ . Since a one-to-one  $k$ -DPC in  $H_0 \oplus H_1$  with a virtual fault set  $F \cup F'$ , where  $F'$  is a set of arbitrary  $f - |F|$  fault-free edges, is also a one-to-one  $k$ -DPC in  $H_0 \oplus H_1$  with the fault set  $F$ , we assume  $|F| = f$ .

Remember that each  $H_i$  is  $(m - 4)$ -fault hamiltonian-connected and  $(m - 3)$ -fault hamiltonian by Lemma 3. A vertex  $v$  is called *free* if  $v$  is fault-free and not a terminal. An edge  $(v, w)$  is called *free* if  $v$  and  $w$  are free and  $(v, w) \notin F$ . There are three cases.

**Case 1:**  $s, t \in V(H_0)$  and  $f_1 + f_2 = 0$  ( $f_0 = f$ ).

We first present a procedure for constructing a one-to-one DPC for this case, and then show that the procedure is correct.

**Procedure DPC-A( $H_0 \oplus H_1, s, t, F$ )**

*/\*  $s, t \in V(H_0)$  and  $f_1 + f_2 = 0$  ( $f_0 = f$ ). \*/*

1. Regarding a faulty element  $\alpha$  as a *virtual* fault-free element, find an  $(f_0 - 1)$ -fault  $k$ -DPC in  $H_0$ .

2. When some path  $P_i$  in the DPC passes through  $\alpha$ , let  $P_i = (s, P_s, x, \alpha, y, P_t, t)$  if  $\alpha$  is a vertex; let  $P_i = (s, P_s, x, y, P_t, t)$  if  $\alpha$  is an edge  $(x, y)$ . Here,  $P_s$  and  $P_t$  are path segments of  $P_i$ . When no path in the DPC passes through  $\alpha$ , pick up an arbitrarily path  $P_i = (s, P_s, x, y, P_t, t)$  in the DPC.
3. Replace  $P_i$  with  $P'_i = (s, P_s, x, \bar{x}, P_h, \bar{y}, y, P_t, t)$ , where  $P_h$  is a hamiltonian path in  $H_1$  between  $\bar{x}$  and  $\bar{y}$ .

**Lemma 12.** *When  $s, t \in V(H_0)$  and  $f_1 + f_2 = 0$  ( $f_0 = f$ ), Procedure DPC-A constructs an  $f$ -fault one-to-one  $k$ -DPC for any  $m \geq 4$ .*

PROOF. The  $(f_0 - 1)$ -fault  $k$ -DPC in Step 1 exists since  $(f_0 - 1) + k = f + k - 1 = m - 1$ . The  $\bar{x}$ - $\bar{y}$  hamiltonian-path in Step 3 exists due to Lemma 3. Thus, Procedure DPC-A can always be applied.  $\square$

**Case 2:**  $s \in V(H_0)$  and  $t \in V(H_1)$ .

In this case, it is assumed that  $f_0 \geq f_1$ .

**Procedure DPC-B( $H_0 \oplus H_1, s, t, F$ )**

*/\*  $s \in V(H_0), t \in V(H_1)$ , and  $f_0 \geq f_1$ . \*/*

1. Let  $z = \bar{t}$  if  $(t, \bar{t}), \bar{t} \notin F$ ; otherwise, let  $z$  be a free vertex in  $H_0$  such that  $(z, \bar{z}), \bar{z} \notin F$ . Find an  $f_0$ -fault  $s$ - $z$  hamiltonian path in  $H_0$ .
2. Pick up  $k - 1$  distinct vertices  $z_1, z_2, \dots, z_{k-1}$  on the hamiltonian path such that for each  $i$ ,  $(s, z_i) \in E(G^m) \setminus F$  and  $\bar{x}_i, (x_i, \bar{x}_i) \notin F$ , where  $x_i$  is the vertex on the hamiltonian path that precedes  $z_i$ .
3. If  $z = \bar{t}$ , find  $f_1$ -fault *one-to-many*  $(k-1)$ -DPC in  $H_1$  joining  $\{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{k-1}\}$  and  $t$ ; if  $z \neq \bar{t}$ , find  $f_1$ -fault *one-to-many*  $k$ -DPC in  $H_1$  joining  $\{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{k-1}, \bar{z}\}$  and  $t$ ;
4. Merge the hamiltonian path and the one-to-many DPC with edges  $(z, \bar{z})$  and  $(x_i, \bar{x}_i)$ ,  $1 \leq i \leq k - 1$ . Discard edges  $(x_i, z_i)$  for all  $i$  with  $x_i \neq s$ .

**Lemma 13.** *When  $s \in V(H_0)$  and  $t \in V(H_1)$ , Procedure DPC-B constructs an  $f$ -fault one-to-one  $k$ -DPC for any  $m \geq 4$  unless (a)  $k = 3$  and  $f_0 = m - 3$  or (b)  $f_0 + f_2 = 1$  and one of  $\bar{t}$  or  $(t, \bar{t})$  is faulty.*

PROOF. The hamiltonian path in Step 1 exists if  $f_0 \leq m - 4$ . Thus, it exists unless  $k = 3$  and  $f_1 + f_2 = 0$  ( $f_0 = m - 3$ ) since  $f_0 = f - (f_1 + f_2) = m - k - (f_1 + f_2)$ . Notice  $k \geq 3$ . All the vertices adjacent to  $s$  are candidates for  $z_i$ 's

of Step 2. Each faulty element may block at most one candidate. There are  $m - 1$  candidates and at most  $f$  blocking elements, and thus the number of nonblocked candidates is at least  $m - 1 - f = k - 1$ . Thus, we can always pick up  $k - 1$  vertices  $z_1, z_2, \dots, z_{k-1}$  on the hamiltonian path. When  $z = \bar{t}$ , the  $f_1$ -fault one-to-many  $(k - 1)$ -DPC in Step 3 exists if  $f_1 + (k - 1) \leq m - 2$  by Theorem 1. By the assumption of  $f \geq 1$ , we have  $f_0 + f_2 \geq 1$ . Recall  $f_0 \geq f_1$ . Then,  $f_1 + (k - 1) = f - (f_0 + f_2) + (k - 1) = m - (f_0 + f_2) - 1 \leq m - 2$ . Similarly, we can see that when  $z \neq \bar{t}$  ( $\bar{t}$  or  $(t, \bar{t})$  is faulty), the  $f_1$ -fault one-to-many  $k$ -DPC in Step 3 exists unless  $f_0 + f_2 = 1$ . This completes the proof.  $\square$

The two exceptional cases (a) and (b) of Lemma 13 are considered in the following two lemmas.

**Lemma 14.** *When  $s \in V(H_0)$ ,  $t \in V(H_1)$ ,  $k = 3$ , and  $f_0 = m - 3$ , there exists an  $f$ -fault one-to-one  $k$ -DPC in  $H_0 \oplus H_1$  for any  $m \geq 4$ .*

PROOF. There exists a hamiltonian cycle  $C_h$  in  $H_0 \setminus F_0$ . When  $m \geq 5$ , let  $(x, y)$  be an edge on  $C_h$  such that  $x, y \neq s$  and  $\bar{x}, \bar{y} \neq t$ . A one-to-many 3-DPC in  $H_1$  joining  $\{\bar{s}, \bar{x}, \bar{y}\}$  and  $t$  is merged with  $C_h$  to obtain a one-to-one 3-DPC in  $H_0 \oplus H_1$ . Let  $m = 4$  and  $H_0 \oplus H_1$  be isomorphic to  $G(8, 4) \times K_2$ . If  $\bar{t} \notin F$ , we pick up an edge  $(x, y)$  on  $C_h$  such that  $x = \bar{t}$  and  $y \neq s$ . A one-to-many 2-DPC in  $H_1$  between  $\{\bar{s}, \bar{y}\}$  and  $t$  is merged with  $C_h$  for our purpose. The last subcase of  $\bar{t} \in F$  is deferred to Lemma 33 in Appendix.  $\square$

**Lemma 15.** *When  $s \in V(H_0)$ ,  $t \in V(H_1)$ ,  $f_0 + f_2 = 1$ , and one of  $\bar{t}$  or  $(t, \bar{t})$  is faulty, there exists an  $f$ -fault one-to-one  $k$ -DPC in  $H_0 \oplus H_1$  for any  $m \geq 4$ .*

PROOF. If  $(t, \bar{t})$  is faulty, then  $f_0 = 0$  and thus  $F = \{(t, \bar{t})\}$ . By the assumption of  $F \neq \{(s, t)\}$ , we have  $\bar{t} \neq s$  and  $(s, \bar{s}) \notin F$ . It suffices to switch  $H_0$  and  $H_1$  and apply Procedure DPC-B( $H_1 \oplus H_0, t, s, F$ ). Let  $\bar{t}$  be faulty. If  $f_1 = 1$  and  $\bar{s} \notin F$ , similar to the previous case, it suffices to switch  $H_0$  and  $H_1$  and apply Procedure DPC-B( $H_1 \oplus H_0, t, s, F$ ). Hereafter in this proof, we assume  $F = \{\bar{t}\}$  or  $F = \{\bar{t}, \bar{s}\}$ . When (i)  $m \geq 6$  or (ii)  $m = 5$  and  $H_0 \oplus H_1$  is isomorphic to  $G(32, 4)$  or  $G(8, 4) \times C_4$ , we let  $G_0, G_1, G_2$ , and  $G_3$  be the four components of the graph such that  $H_0$  and  $H_1$  are the subgraphs induced by  $V(G_0) \cup V(G_1)$  and by  $V(G_2) \cup V(G_3)$ , respectively. Note that all  $G_i$ 's are isomorphic to a graph in  $RCL_{m-2}$ , and that the subgraph

induced by  $V(G_i) \cup V(G_{(i+1) \bmod 4})$  for each  $i = 0, 1, 2, 3$ , is isomorphic to a graph in  $RCL_{m-1}$ . Assume w.l.o.g.  $t \in V(G_2)$ . Then,  $\bar{t} \in V(G_1)$ .

If  $s \in V(G_1)$ , it suffices to apply Procedure DPC-A( $H'_0 \oplus H'_1, s, t, F$ ), where  $H'_0$  and  $H'_1$  are the subgraphs induced by  $V(G_1) \cup V(G_2)$  and by  $V(G_3) \cup V(G_0)$ , respectively. Let  $s \in V(G_0)$ . If  $(s, \bar{t}) \notin E(G^m)$ , then it suffices to apply Procedure DPC-B( $H'_0 \oplus H'_1, t, s, F$ ). When  $F = \{\bar{t}\}$ , by the assumption of  $F \neq \{v_f\}$  for any  $v_f$  with  $(s, v_f), (t, v_f) \in E(G^m)$ , we always have  $(s, \bar{t}) \notin E(G^m)$  and thus we are done. If  $F = \{\bar{t}, \bar{s}\}$  and  $(t, \bar{s}) \notin E(G^m)$ , it suffices to apply Procedure DPC-B( $H'_1 \oplus H'_0, s, t, F$ ). Finally, if  $F = \{\bar{t}, \bar{s}\}$  and  $(s, \bar{t}), (t, \bar{s}) \in E(G^m)$ , then regarding  $\bar{s}$  and  $\bar{t}$  as virtual fault-free vertices, it suffices to find 0-fault one-to-one  $m$ -DPC and remove the two paths  $(s, \bar{s}, t)$  and  $(s, \bar{t}, t)$  from the DPC.

The case when  $m = 5$  and  $H_0 \oplus H_1$  is isomorphic to  $G(16, 4) \times K_2$  is deferred to Lemma 34 in Appendix. The case when  $m = 4$  and  $H_0 \oplus H_1$  is isomorphic to  $G(8, 4) \times K_2$  is also deferred to Lemma 33. This completes the proof.  $\square$

**Case 3:**  $s, t \in V(H_0)$  and  $f_1 + f_2 \geq 1$ .

**Procedure DPC-C**( $H_0 \oplus H_1, s, t, F$ )

*/\*  $s, t \in V(H_0)$  and  $f_1 + f_2 \geq 1$ . \*/*

1. Find an  $f_0$ -fault one-to-one  $k$ -DPC in  $H_0$ .
2. For some edge  $(x, y)$  on a path  $P_i$  in the DPC such that  $x, (x, \bar{x}), y$ , and  $(y, \bar{y})$  are all fault-free,  $(x, y)$  is replaced with  $(x, \bar{x}, P_h, \bar{y}, y)$ , where  $P_h$  is a hamiltonian path in  $G_1 \setminus F_1$  between  $\bar{x}$  and  $\bar{y}$ .

**Lemma 16.** *When  $s, t \in V(H_0)$  and  $f_1 + f_2 \geq 1$ , Procedure DPC-C constructs an  $f$ -fault one-to-one  $k$ -DPC for any  $m \geq 4$  unless  $k = 3$  and  $f_1 = m - 3$ .*

PROOF. The  $f_0$ -fault one-to-one  $k$ -DPC in Step 1 exists since  $f_0 + k = f - (f_1 + f_2) + k = m - (f_1 + f_2) \leq m - 1$ . The  $\bar{x}$ - $\bar{y}$  hamiltonian path in Step 2 exists if  $f_1 = f - (f_0 + f_2) = m - k - (f_0 + f_2) \leq m - 4$ . That is, it exists unless  $k = 3$  and  $f_0 = f_2 = 0$  ( $f_1 = m - 3$ ). Thus, we have the lemma.  $\square$

**Lemma 17.** *When  $s, t \in V(H_0)$ ,  $k = 3$ , and  $f_1 = m - 3$ , there exists an  $f$ -fault one-to-one  $k$ -DPC in  $H_0 \oplus H_1$  for any  $m \geq 4$ .*



PROOF. Let us consider the case  $m \geq 6$  first. There exists a free vertex  $x$  in  $H_0$  adjacent to  $s$  such that  $\bar{x} \notin F$ . Since  $H_1$  is  $(m-3)$ -fault hamiltonian, there exists a fault-free vertex  $y$  in  $H_1$  such that  $\bar{y} \neq s$  and  $\bar{x}$  and  $y$  are joined by a hamiltonian path in  $H_1 \setminus F_1$ . Let  $z$  be a free vertex in  $H_0$  adjacent to  $s$  such that  $z \neq x, \bar{y}$ . Regarding  $x$  as a *virtual* fault, we find a one-to-many 3-DPC joining  $\{s, z, \bar{y}\}$  and  $t$  if  $\bar{y} \neq t$ ; otherwise, we find a one-to-many 2-DPC joining  $\{s, z\}$  and  $t$ . The one-to-many DPC in  $H_0$  and the hamiltonian path in  $H_1 \setminus F_1$  are merged with edges  $(s, z)$ ,  $(s, x)$ ,  $(x, \bar{x})$ , and  $(y, \bar{y})$ , resulting in a desired one-to-one 3-DPC joining  $s$  and  $t$ .

Second, let  $m = 5$  and  $F \neq \{\bar{s}, \bar{t}\}$ . Assume  $\bar{s} \notin F$ . Similar to the case  $m \geq 6$ , a one-to-one 3-DPC can be obtained by merging a hamiltonian path in  $H_1 \setminus F_1$  between  $\bar{s}$  and a fault-free vertex  $y$  such that  $\bar{y} \neq t$  and a one-to-many 3-DPC in  $H_0$  between  $\{s, z, \bar{y}\}$  and  $t$ , where  $z$  is a free vertex adjacent to  $s$  in  $H_0$  such that  $z \neq \bar{y}$ . Now, let  $m = 5$ ,  $F = \{\bar{s}, \bar{t}\}$ , and  $H_0 \oplus H_1$  be isomorphic to  $G(32, 4)$  or  $G(8, 4) \times C_4$ . As in the proof of Lemma 15, this graph has four components  $G_0, G_1, G_2$ , and  $G_3$  such that  $H_0$  and  $H_1$  are the subgraphs induced by  $V(G_0) \cup V(G_1)$  and by  $V(G_2) \cup V(G_3)$ , respectively. If both  $s$  and  $t$  are contained in the same component, say  $G_1$ , it suffices to apply Procedure DPC-A( $H'_0 \oplus H'_1, s, t, F$ ), where  $H'_0$  and  $H'_1$  are the subgraphs induced by  $V(G_1) \cup V(G_2)$  and by  $V(G_3) \cup V(G_0)$ , respectively. If  $s$  and  $t$  are contained in different components, say  $s \in V(G_1)$  and  $t \in V(G_0)$ , it suffices to apply Procedure DPC-B( $H'_0 \oplus H'_1, s, t, F$ ). The case when  $m = 5$ ,  $F = \{\bar{s}, \bar{t}\}$ , and  $H_0 \oplus H_1$  is isomorphic to  $G(16, 4) \times K_2$  is deferred to Lemma 32 in Appendix. The last case of  $m = 4$  is also deferred to Lemma 31.  $\square$

## 5. Unpaired Many-to-Many Disjoint Path Covers

In terms of connectivity and the minimum degree, necessary conditions for a graph to be  $f$ -fault unpaired many-to-many  $k$ -disjoint path coverable were derived in [26] as follows.

**Lemma 18.** [26] *Let  $G$  be an  $f$ -fault unpaired many-to-many  $k(\geq 2)$ -disjoint path coverable graph. Then,  $\kappa(G) \geq f + k$ . Furthermore, if  $G$  has  $f + 2k + 1$  or more vertices, then  $\delta(G) \geq f + k + 1$ .*

In this section, we will construct  $f$ -fault unpaired  $k$ -disjoint path covers in  $m$ -dimensional RC-like graphs with  $m \geq 5$  for any  $f$  and  $k \geq 2$  satisfying the optimal bound  $f + k \leq m - 1$  given in Lemma 18. That is, we will establish the following theorem.

**Theorem 3.** *Every  $m$ -dimensional RC-like graph  $G^m$ ,  $m \geq 5$ , is  $f$ -fault unpaired many-to-many  $k$ -disjoint path coverable for any  $f \geq 0$  and  $k \geq 2$  subject to  $f + k \leq m - 1$ .*

The 4-dimensional RC-like graphs are not 0-fault unpaired 3-disjoint path coverable. However, they are 0-fault unpaired 2-disjoint path coverable, which is a direct consequence of a result in [26] that every  $m$ -dimensional RC-like graph,  $m \geq 4$ , is  $f$ -fault *paired* many-to-many  $k$ -disjoint path coverable for any  $f$  and  $k \geq 2$  with  $f + 2k \leq m$ . Notice that a paired many-to-many  $k$ -disjoint path coverable graph is always unpaired  $k$ -disjoint path coverable.

**Lemma 19.** *Every  $G^4$  is 0-fault unpaired 2-disjoint path coverable.*

The proof of Theorem 3 will proceed by induction on  $m$ . For the base case of  $m = 5$ , we obtained the following Lemma 20 from a computer program that exhaustively searched out  $f$ -fault unpaired  $k$ -DPC's for any  $f \geq 0$  and  $k \geq 2$  satisfying  $f + k \leq 4$ .

**Lemma 20.** *Every  $G^5$  is  $f$ -fault unpaired  $k$ -disjoint path coverable for any  $f \geq 0$  and  $k \geq 2$  with  $f + k \leq 4$ .*

Let  $m \geq 6$ , and recall that  $G^m$  is isomorphic to  $H_0 \oplus H_1$  for some  $H_0, H_1 \in RCL_{m-1}$ . We will construct an  $f$ -fault unpaired  $k$ -DPC for any given set  $S$  of  $k$  sources and set  $T$  of  $k$  sinks in  $G^m$  having at most  $f$  faulty elements such that  $f + k \leq m - 1$ . An unpaired  $k$ -DPC with a fault set  $F$  is also an unpaired  $k$ -DPC with a virtual fault set  $F \cup F'$ , where  $F'$  is a set of arbitrary  $m - 1 - k - |F|$  fault-free edges. As a result, it can be assumed that

$$f = |F| \text{ and } f + k = m - 1.$$

We denote by  $S_i$  and  $T_i$  the sets of sources and sinks in  $H_i$ ,  $i = 0, 1$ , respectively. We assume w.l.o.g. that  $|S_0| \geq |T_0|$  and  $|S_1| \leq |T_1|$ . We let  $k_0 = |T_0|$ ,  $k_1 = |S_1|$ , and  $k_2 = k - (k_0 + k_1)$ . Then,  $H_0$  has  $k_0 + k_2$  sources and  $k_0$  sinks, and  $H_1$  has  $k_1$  sources and  $k_1 + k_2$  sinks. We assume that  $S_0 = \{s_i : 1 \leq i \leq k_0 + k_2\}$ ,  $S_1 = \{s_i : k_0 + k_2 < i \leq k\}$ ,  $T_0 = \{t_j : 1 \leq j \leq k_0\}$ , and  $T_1 = \{t_j : k_0 < j \leq k\}$ . Furthermore, we also assume w.l.o.g. that

$$k_0 \geq k_1, \text{ and if } k_0 = k_1, f_0 \geq f_1.$$

Hereafter in this section, an unpaired  $k$ -DPC in a graph  $G$  with fault set  $F$  joining  $S$  and  $T$  is denoted by  $k$ -DPC[ $S, T|G, F$ ]. We have three cases. Remember  $k \geq 2$ .

**Case 1:**  $k_1 \geq 1$  or  $f_0 \leq f - 1$ .

We first present a basic procedure for constructing an unpaired DPC in this case.

**Procedure DPC-D( $H_0 \oplus H_1, S, T, F$ )**

*/\*  $k_1 \geq 1$  or  $f_0 \leq f - 1$ . \*/*

1. Pick up  $k_2$  free edges joining vertices in  $H_0$  and vertices in  $H_1$ . Let  $X_0$  be the set of endvertices of the free edges in  $H_0$  and  $X_1$  be in  $H_1$ .
2. Find a  $(k_0 + k_2)$ -DPC[ $S_0, T_0 \cup X_0 | H_0, F_0$ ].
3. Case  $k_1 + k_2 \geq 1$ :
  - (a) Find a  $(k_1 + k_2)$ -DPC[ $S_1 \cup X_1, T_1 | H_1, F_1$ ].
  - (b) Merge the two DPC's with the  $k_2$  free edges.
4. Case  $k_1 + k_2 = 0$ :
  - (a) Let  $(x, y)$  be an edge on some path in the  $(k_0 + k_2)$ -DPC such that all the  $\bar{x}$ ,  $(x, \bar{x})$ ,  $\bar{y}$ , and  $(y, \bar{y})$  are fault-free.
  - (b) Find a hamiltonian path joining  $\bar{x}$  and  $\bar{y}$  in  $H_1 \setminus F_1$ .
  - (c) Merge the  $(k_0 + k_2)$ -DPC and the hamiltonian path with the edges  $(x, \bar{x})$  and  $(y, \bar{y})$ . Discard the edge  $(x, y)$ .

**Lemma 21.** *When  $k_1 \geq 1$  or  $f_0 \leq f - 1$ , Procedure DPC-D constructs an  $f$ -fault unpaired  $k$ -DPC for any  $m \geq 6$  unless (a)  $k_0 = 1$ ,  $k_1 = 1$ , and  $f_0 = m - 3$ , (b)  $k_0 = 1$ ,  $k_2 = 1$ , and  $f_1 = m - 3$ , or (c)  $k_0 = 2$  and  $f_1 = m - 3$ .*

PROOF. For Step 1, we have  $2^{m-1}$  candidate edges and  $f + 2k$  blocking elements ( $f$  faults and  $2k$  terminals). The number of nonblocked candidates is at least  $2^{m-1} - (f + 2k) \geq 2^{m-1} - 2(m-1) > m > k_2$  for any  $m \geq 6$ . Thus, it is possible to pick up  $k_2$  free edges. Since  $f_0 + (k_0 + k_2) = f_0 + (k - k_1) \leq f + k - 1 = m - 2$ , by induction hypothesis, the  $(k_0 + k_2)$ -DPC in Step 2 exists when  $k_0 + k_2 \geq 2$ . If  $k_0 + k_2 = 1$ , the  $(k_0 + k_2)$ -DPC is indeed a hamiltonian path, and it exists, by Lemma 3, when  $f_0 \leq m - 4$ . Thus, the  $(k_0 + k_2)$ -DPC in Step 2 exists unless  $f_0 = m - 3$  ( $k = 2$ ) and  $k_0 + k_2 = 1$  ( $k_1 = 1$ ), or equivalently, unless the exceptional case (a). For Step 3, note that  $f_1 + (k_1 + k_2) = f_1 + (k - k_0) \leq f + k - 1 = m - 2$ . Recall the assumption that  $k_0 \geq k_1$ , and that if  $k_0 = k_1$ ,  $f_0 \geq f_1$ . If  $k_1 + k_2 \geq 2$ , the  $(k_1 + k_2)$ -DPC exists. If  $k_1 + k_2 = 1$ , the  $(k_1 + k_2)$ -DPC exists unless  $f_1 = m - 3$ . Thus, the  $(k_1 + k_2)$ -DPC in Step 3 exists unless  $f_1 = m - 3$  ( $k = 2$ ) and  $k_1 + k_2 = 1$  ( $k_0 = 1$ ), i.e., unless the exceptional case (b). Finally, the hamiltonian path in Step 4(b) exists unless  $f_1 = m - 3$ . That is, it exists unless  $f_1 = m - 3$  ( $k = 2$ ) and  $k_1 + k_2 = 0$  ( $k_0 = 2$ ), i.e., unless the exceptional case (c). This completes the proof.  $\square$

The three exceptional cases (a), (b), and (c) of Lemma 21 are considered in the following three lemmas.

**Lemma 22.** *When  $k_0 = 1$ ,  $k_1 = 1$ , and  $f_0 = m - 3$ , there exists an  $f$ -fault unpaired  $k$ -DPC in  $H_0 \oplus H_1$  for any  $m \geq 6$ .*

PROOF. There exists a hamiltonian cycle  $C_h$  in  $H_0 \setminus F_0$  by Lemma 3. Let  $C_h = (s_1, P_a, t_1, P_b)$  for some subpaths  $P_a$  and  $P_b$ . We assume w.l.o.g. the length of  $P_a$  is at least that of  $P_b$ . Let  $P_a = (x, P'_a, y)$ . Then,  $x \neq y$ . If  $\{\bar{x}, \bar{y}\} \cap \{s_2, t_2\} = \emptyset$ , it suffices to find 2-DPC $[\{s_2, t_2\}, \{\bar{x}, \bar{y}\} | H_1, \emptyset]$  and merge  $C_h$  and the 2-DPC with edges  $(x, \bar{x})$  and  $(y, \bar{y})$ . Of course, we discard the edges  $(s_1, x)$  and  $(t_1, y)$ . If  $|\{\bar{x}, \bar{y}\} \cap \{s_2, t_2\}| = 1$ , say  $\bar{x} = s_2$ , it suffices to find a  $\bar{y}$ - $t_2$  hamiltonian path  $P_h$  in  $H_1 \setminus s_2$  and then merge  $C_h$  and  $P_h$  with  $(x, \bar{x})$  and  $(y, \bar{y})$ . Finally in case  $\{\bar{x}, \bar{y}\} = \{s_2, t_2\}$ , let subpath  $(t_1, P_b) = (P'_b, z)$ . It suffices to find a  $\bar{s}_1$ - $\bar{z}$  hamiltonian path  $P'_h$  in  $H_1 \setminus \{s_2, t_2\}$  and merge  $C_h$  and  $P'_h$  with edges  $(s_1, \bar{s}_1)$  and  $(z, \bar{z})$ . The existence of  $P'_h$  is due to Lemma 3. The proof is completed.  $\square$

**Lemma 23.** *When  $k_0 = 1$ ,  $k_2 = 1$ , and  $f_1 = m - 3$ , there exists an  $f$ -fault unpaired  $k$ -DPC in  $H_0 \oplus H_1$  for any  $m \geq 6$ .*

PROOF. There exists a hamiltonian cycle  $C_h$  in  $H_1 \setminus F_1$ , and let  $C_h = (t_2, x, P_a, y)$  for some subpath  $P_a$ . Then,  $\bar{x} \neq t_1$  or  $\bar{y} \neq t_1$ . Assume  $\bar{y} \neq t_1$ . If  $\bar{y} \notin \{s_1, s_2\}$ , it suffices to find 2-DPC $[\{s_1, s_2\}, \{t_1, \bar{y}\} | H_0, \emptyset]$  and merge the 2-DPC and  $C_h$  with edge  $(\bar{y}, y)$ . If  $\bar{y} \in \{s_1, s_2\}$ , say  $\bar{y} = s_2$ , it suffices to find an  $s_1$ - $t_1$  hamiltonian path  $P_h$  in  $H_0 \setminus s_2$  and merge  $P_h$  and  $C_h$  with  $(\bar{y}, y)$ . Thus, we have the lemma.  $\square$

**Lemma 24.** *When  $k_0 = 2$  and  $f_1 = m - 3$ , there exists an  $f$ -fault unpaired  $k$ -DPC in  $H_0 \oplus H_1$  for any  $m \geq 6$ .*

PROOF. We consider the first case that for some terminal, say  $s_1, \bar{s}_1$  is fault-free. There exists a hamiltonian cycle  $C_h$  in  $H_1 \setminus F_1$ , and let  $C_h = (\bar{s}_1, x, P_a, y)$  for some subpath  $P_a$ . Assume w.l.o.g.  $\bar{y} \neq s_2$ . If  $\bar{y} \notin \{t_1, t_2\}$ , it suffices find 2-DPC $[\{\bar{y}, s_2\}, \{t_1, t_2\} | H_0, \{s_1\}]$  and merge the 2-DPC and  $C_h$  with edge  $(s_1, \bar{s}_1)$  and  $(\bar{y}, y)$ . If  $\bar{y} \in \{t_1, t_2\}$ , say  $\bar{y} = t_1$ , we find an  $s_2$ - $t_2$  hamiltonian path in  $H_0 \setminus \{s_1, t_1\}$  and let  $s_1$ - $t_1$  path be  $(s_1, C_h \setminus (\bar{s}_1, y), t_1)$ . For the second case, we assume that  $\bar{s}_1, \bar{t}_1, \bar{s}_2$ , and  $\bar{t}_2$  are all faulty. This implies  $f_1 \geq 4$  and thus  $m \geq 7$ . We claim that there exists a free edge  $(x, \bar{x})$  with  $x \in V(H_0)$  such that  $x$  is adjacent to  $s_1$ . There are  $m - 1$  candidate edges. The number of blocking elements is at most  $m - 3$  since  $\bar{s}_2, \bar{t}_1$ , and

$\bar{t}_2$  are all faulty. Thus, the claim is proved. Let a hamiltonian cycle  $C_h$  in  $H_1 \setminus F_1$  be  $(\bar{x}, w, P_b, z)$ . Since  $\bar{z} \notin \{s_1, s_2, t_1, t_2\}$ , it suffices to find 2-DPC $[\{\bar{z}, s_2\}, \{t_1, t_2\} | H_0, \{s_1, x\}]$  and merge the 2-DPC and  $C_h$  with edges  $(s_1, x)$ ,  $(x, \bar{x})$ , and  $(\bar{z}, z)$ . This completes the proof.  $\square$

**Case 2:**  $k_1 = 0$ ,  $f_0 = f$ , and  $k_0 \geq 1$  or  $f_0 \geq 1$ .

We present two basic Procedures DPC-E and DPC-F depending on whether  $k_0 = k$  or not.

**Procedure DPC-E**( $H_0 \oplus H_1, S, T, F$ )

*/\*  $k_1 = 0$ ,  $f_0 = f$ , and  $k_0 = k$ . \*/*

1. Regarding  $s_1$  and  $t_1$  as *virtual* free vertices, find a  $(k_0 - 1)$ -DPC $[S_0 \setminus s_1, T_0 \setminus t_1 | H_0, F_0]$ .
2. If there exists a path  $P_i$  in the DPC which passes through both  $s_1$  and  $t_1$ , let  $P_i = (s_i, P_x, x, P_1, y, P_y, t_{\sigma_i})$ , where  $P_1$  is an  $s_1$ - $t_1$  path. If  $P_i$  and  $P_j$  pass through  $s_1$  and  $t_1$ , respectively, let  $P_i = (s_i, P_x, x, s_1, P_a, t_{\sigma_i})$  and  $P_j = (s_j, P_b, t_1, y, P_y, t_{\sigma_j})$ .
3. Find an  $\bar{x}$ - $\bar{y}$  hamiltonian path in  $H_1$ .
4. Merge the DPC and the hamiltonian path with edges  $(x, \bar{x})$  and  $(y, \bar{y})$ .

**Lemma 25.** *When  $k_1 = 0$ ,  $f_0 = f$ , and  $k_0 = k$ , Procedure DPC-E constructs an  $f$ -fault unpaired  $k$ -DPC for any  $m \geq 6$  unless  $k_0 = 2$  and  $f_0 = m - 3$ .*

PROOF. It holds that  $f_0 + (k_0 - 1) = f + k - 1 = m - 2$ . If  $k_0 - 1 \geq 2$ , the  $(k_0 - 1)$ -DPC in Step 1 exists. If  $k_0 - 1 = 1$ , the  $(k_0 - 1)$ -DPC exists when  $f_0 \leq m - 4$ . Thus, the  $(k_0 - 1)$ -DPC exists unless  $k_0 = 2$  and  $f_0 = m - 3$ . The existence of the  $\bar{x}$ - $\bar{y}$  hamiltonian path in Step 3 is straightforward.  $\square$

**Lemma 26.** *When  $k_0 = 2$  and  $f_0 = m - 3$ , there exists an  $f$ -fault unpaired  $k$ -DPC in  $H_0 \oplus H_1$  for any  $m \geq 6$ .*

PROOF. There exists a hamiltonian cycle  $C_h$  in  $H_0 \setminus F_0$ . From  $C_h$ , we can construct four disjoint paths starting from the four terminals. If  $C_h = (s_1, P_x, x, s_2, P_y, y, t_1, P_z, z, t_2, P_w, w)$ , then it suffices to remove edges  $(x, s_2)$ ,  $(y, t_1)$ ,  $(z, t_2)$ , and  $(w, s_1)$ . The order of terminals in  $C_h$  does not matter. The four disjoint paths and 2-DPC $[\{\bar{x}, \bar{y}\}, \{\bar{z}, \bar{w}\} | H_1, \emptyset]$  are merged to obtain a desired 2-DPC.  $\square$

In the remaining part of Case 2, we assume  $k_0 < k$ . It implies  $k_2 \geq 1$ .

**Procedure DPC-F**( $H_0 \oplus H_1, S, T, F$ )

*/\*  $k_1 = 0, f_0 = f, k_0 < k$ , and  $k_0 \geq 1$  or  $f_0 \geq 1$ . \*/*

1. Pick up  $k_2 - 1$  free edges joining vertices in  $H_0$  and vertices in  $H_1$ . Let  $X_0$  be the set of endvertices of the free edges in  $H_0$  and  $X_1$  be in  $H_1$ .
2. Regarding  $s_1$  as a virtual free vertex, find a  $(k_0 + k_2 - 1)$ -DPC[ $S_0 \setminus s_1, T_0 \cup X_0 | H_0, F_0$ ]. Assume path  $P_i$  in the DPC passes through  $s_1$ , and let  $P_i = (s_i, P_a, x, s_1, P_b, t_{\sigma_i})$ .
3. Case  $\bar{x}$  is not a sink:
  - (a) Find a  $k_2$ -DPC[ $X_1 \cup \{\bar{x}\}, T_1 | H_1, \emptyset$ ].
  - (b) Merge the two DPC's with the free edges and edge  $(x, \bar{x})$ .
4. Case  $\bar{x}$  is a sink and  $k_2 \geq 2$ :
  - (a) Find a  $(k_2 - 1)$ -DPC[ $X_1, T_1 \setminus \bar{x} | H_1, \{\bar{x}\}$ ].
  - (b) Merge the two DPC's with the free edges and edge  $(x, \bar{x})$ .
5. Case  $\bar{x}$  is a sink and  $k_2 = 1$ :
  - (a) Pick up an edge  $(y, z)$  on a path in the DPC such that  $y, z \neq x$ .
  - (b) Find a  $\bar{y}$ - $\bar{z}$  hamiltonian path in  $H_1 \setminus \bar{x}$ .
  - (c) Merge the DPC and the hamiltonian path with edges  $(x, \bar{x})$ ,  $(y, \bar{y})$ , and  $(z, \bar{z})$ .

**Lemma 27.** *When  $k_1 = 0, f_0 = f, k_0 < k$ , and  $k_0 \geq 1$  or  $f_0 \geq 1$ , Procedure DPC-F constructs an  $f$ -fault unpaired  $k$ -DPC for any  $m \geq 6$  unless (a)  $k_0 = k_2 = 1$  and  $f_0 = m - 3$ , or (b)  $k_2 = 2$  and  $f_0 = m - 3$ .*

PROOF. The existence of  $k_2 - 1$  free edges in Step 1 is straightforward. For Step 2, note that  $f_0 + (k_0 + k_2 - 1) = f + k - 1 = m - 2$ . If  $k_0 + k_2 - 1 \geq 2$ , the  $(k_0 + k_2 - 1)$ -DPC exists. Otherwise, it exists when  $f_0 \leq m - 4$ . Thus, the  $(k_0 + k_2 - 1)$ -DPC in Step 2 exists unless the exceptional cases (a) or (b). It holds that  $k_2 < f_0 + k_0 + k_2 = f + k = m - 1$ . Thus, the  $k_2$ -DPC in Step 3 exists if  $k_2 \geq 2$ . It also exists if  $k_2 = 1$ , due to Lemma 3. Similarly, we can see that the 1-fault  $(k_2 - 1)$ -DPC in Step 4 exists whether  $k_2 - 1 \geq 2$  or not. The existence of the  $\bar{y}$ - $\bar{z}$  hamiltonian path in Step 5 is straightforward. Thus, we have the lemma.  $\square$

**Lemma 28.** *When  $k_0 = k_2 = 1$  and  $f_0 = m - 3$ , there exists an  $f$ -fault unpaired  $k$ -DPC in  $H_0 \oplus H_1$  for any  $m \geq 6$ .*

PROOF. There exists a hamiltonian cycle  $C_h = (s_1, P_x, x, s_2, P_y, y, t_1, P_z, z)$  in  $H_0 \setminus F_0$ . We decompose  $C_h$  into three disjoint paths starting from the three terminals in  $H_0$ :  $(s_1, P_x, x)$ ,  $(s_2, P_y, y)$ , and  $(t_1, P_z, z)$ . Let  $\bar{z} \neq t_2$  first. If  $\bar{x}, \bar{y} \neq t_2$ , it suffices to find 2-DPC $[\{\bar{x}, \bar{y}\}, \{\bar{z}, t_2\} | H_1, \emptyset]$  and merge  $C_h$  and the 2-DPC. Otherwise, say  $\bar{x} = t_2$ , it suffices to find a  $\bar{y}$ - $\bar{z}$  hamiltonian path in  $H_1 \setminus t_2$  and merge  $C_h$  and the hamiltonian path. Suppose  $\bar{z} = t_2$ . We use another representation of  $C_h$ , which is obtained by traversing  $C_h$  in reverse order. Let  $C_h = (s_1, P_u, u, t_1, P_v, v, s_2, P_w, w)$ . If  $\bar{v} \neq t_2$ , we can construct a desired DPC in the same way as before. Now, let  $\bar{z} = \bar{v} = t_2$ , which means  $z = v = t_1$  and both  $(P_y, y)$  and  $(P_z, z)$  are empty. Then,  $C_h = (s_1, P_x, x, s_2, t_1)$ . It suffices to find an  $\bar{x}$ - $t_2$  hamiltonian path in  $H_1$  and merge  $C_h$  and the hamiltonian path with edge  $(x, \bar{x})$ . The proof is completed.  $\square$

**Lemma 29.** *When  $k_2 = 2$  and  $f_0 = m - 3$ , there exists an  $f$ -fault unpaired  $k$ -DPC in  $H_0 \oplus H_1$  for any  $m \geq 6$ .*

PROOF. There exists a hamiltonian cycle  $C_h = (s_1, P_x, x, s_2, P_y, y)$  in  $H_0 \setminus F_0$ . We assume w.l.o.g.  $\{\bar{x}, \bar{y}\} \neq \{t_1, t_2\}$ . (Suppose otherwise, then we use another representation of  $C_h$  obtained by traversing  $C_h$  in reverse order.) If  $\{\bar{x}, \bar{y}\} \cap \{t_1, t_2\} = \emptyset$ , it suffices to find 2-DPC $[\{\bar{x}, \bar{y}\}, \{t_1, t_2\} | H_1, \emptyset]$  and merge  $C_h$  and the 2-DPC. If  $|\{\bar{x}, \bar{y}\} \cap \{t_1, t_2\}| = 1$ , say  $\bar{x} = t_1$ , it suffices to find a  $\bar{y}$ - $t_2$  hamiltonian path in  $H_1 \setminus t_1$  and merge  $C_h$  and the hamiltonian path. This completes the proof.  $\square$

**Case 3:**  $k_2 = k$  and  $f = 0$ .

In this case, all the sources are contained in  $H_0$  and all the sinks are contained in  $H_1$ . There are no faults. By the assumption of  $f + k = m - 1$ , we have  $k_2 = m - 1$ . In the recursive structure of  $G^m$ , there are four components  $G_0, G_1, G_2$ , and  $G_3$ , which are  $(m - 2)$ -dimensional RC-like graphs. Unless all the  $m - 1$  sources are contained in  $G_i$  and all the sinks are contained in  $G_{(i+2) \bmod 4}$  for some  $i$ , letting  $H'_0$  (resp.  $H'_1$ ) be the subgraph induced by the vertices in  $G_1$  and  $G_2$  (resp. in  $G_3$  and  $G_0$ ), our problem is reduced to one of the two cases considered before. Thus, we assume w.l.o.g. that all the sources are contained in  $G_0$  and all the sinks are contained in  $G_2$ .

The following procedure will construct an unpaired  $(m - 1)$ -DPC in which  $m - 3$  paths pass through  $G_1$  and do not pass through  $G_3$ . The remaining two paths in the DPC will pass through  $G_3$ . They may or may not pass through  $G_1$ .

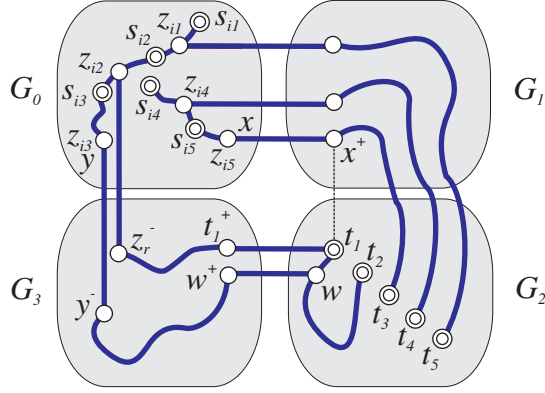


Figure 4: Illustration of Procedure DPC-G.

**Procedure DPC-G**( $[G_0 \oplus G_1] \oplus [G_2 \oplus G_3]$ ,  $S$ ,  $T$ ,  $F$ )

*/\*  $k = m - 1$ ,  $f = 0$ ,  $S \subset V(G_0)$ , and  $T \subset V(G_2)$ . See Fig. 4. \*/*

1. Let  $x$  be the vertex in  $G_0$  such that  $x^+ = t_1^-$ .
2. Let  $Z = \{t_1^+\} \cup \{u^+ : u \in V(G_2), (u, t_1) \in E(G_2)\}$ . Pick up a vertex  $y$  in  $G_0$  such that  $y^- \notin Z$ .
3. Pick up two sources, say  $s_{i_1}$  and  $s_{i_q}$  such that  $\{s_{i_1}, s_{i_q}\} \cap \{x, y\} = \emptyset$ . Regarding sources other than  $s_{i_1}$  and  $s_{i_q}$  as virtual free vertices, find a 2-DPC $[\{s_{i_1}, s_{i_q}\}, \{x, y\} | G_0, \emptyset]$ .
4. Let  $s_{i_1}$ -path in the DPC be  $(s_{i_1}, P_{i_1}, z_{i_1}, s_{i_2}, P_{i_2}, z_{i_2}, \dots, s_{i_{q-1}}, P_{i_{q-1}}, z_{i_{q-1}})$ , and let  $s_{i_q}$ -path be  $(s_{i_q}, P_{i_q}, z_{i_q}, \dots, s_{i_k}, P_{i_k}, z_{i_k})$ , where  $\{z_{i_{q-1}}, z_{i_k}\} = \{x, y\}$ . Then, we have  $k$  disjoint  $s_j$ - $z_j$  paths,  $1 \leq j \leq k$ , that cover  $V(G_0)$ .
5. Let  $r$  be an arbitrary index such that  $r \neq i_{q-1}, i_k$ . Let  $Y = \{y^-, z_r^-\}$  and  $X = \{z_j^+ : z_j \neq y, z_r\}$ . Then,  $Y \subset V(G_3)$  and  $X \subset V(G_1)$ .
6. Regarding  $t_1$  as a virtual source, find an  $(m - 2)$ -DPC $[X \cup \{t_1\}, T \setminus t_1 | G_1 \oplus G_2, \emptyset]$ . Let the  $t_1$ -path in the DPC be  $(t_1, w, P_w, t_p)$  for some  $p$ .
7. Let  $W = \{t_1^+, w^+\}$ . If  $z_r^- \notin W$ , find a 2-DPC $[Y, W | G_3, \emptyset]$ . If  $z_r^- \in W$ , find a 1-DPC $[\{y^-\}, W \setminus z_r^- | G_3, \{z_r^-\}]$ .
8. Merge the three DPC's, the 2-DPC in  $G_0$ , the  $(m - 2)$ -DPC in  $G_1 \oplus G_2$ , and the 2-DPC or 1-fault 1-DPC in  $G_3$ , with edges  $\{(u^-, u) : u \in X\}$ ,  $\{(u^+, u) : u \in Y\}$ , and  $\{(w, w^+), (t_1, t_1^+)\}$ .



**Lemma 30.** *When  $k = m - 1$ ,  $f = 0$ ,  $S \subset V(G_0)$ , and  $T \subset V(G_2)$ , Procedure DPC-G constructs an  $f$ -fault unpaired  $k$ -DPC for any  $m \geq 6$ .*

PROOF. The vertex  $y$  of Step 2 exists since  $|Z| = m - 1 < 2^{m-2}$  for any  $m \geq 6$ . The degree  $m - 2$  of  $G_0$  is at least 4, thus the 2-DPC in  $G_0$  exists by Lemma 19 and induction hypothesis. The  $s_j$ - $z_j$  path in  $G_0$  will be extended to pass through vertices in  $G_1$  if  $z_j^+ \in X$ ; otherwise,  $z_j^- \in Y$  and the path will be extended to pass through vertices in  $G_3$ . Note that  $|X| = m - 3$  and  $|Y| = 2$ . The existence of  $(m - 2)$ -DPC in Step 6 is due to induction hypothesis. Recall that  $G_1 \oplus G_2$  is an  $(m - 1)$ -dimensional RC-like graph. By the choice of  $x$  in Step 1,  $t_1^-$  is a source of the  $(m - 2)$ -DPC in Step 6. Thus,  $w$  is certainly a vertex in  $G_2$ . Now, we have constructed  $m - 3$  disjoint paths terminating at  $T \setminus \{t_1, t_p\}$ . To construct two paths terminating at  $\{t_1, t_p\}$ , Step 7 of the procedure works. Observe that  $W \subset Z$  and  $y^- \notin W$  by the choice of  $y$ . The 2-DPC in  $G_3$  exists by Lemma 19 and induction hypothesis. The 1-fault 1-DPC in  $G_3$  also exists by Lemma 3. This completes the proof.  $\square$

## 6. Concluding Remarks

In this paper, it was shown that recursive circulant  $G(2^m, 4)$  is  $f$ -fault one-to-one  $k$ -disjoint path coverable for any  $f$  and  $k \geq 2$  with  $f + k \leq m$  when  $m \geq 3$ , and is  $f$ -fault unpaired many-to-many  $k$ -disjoint path coverable for any  $f$  and  $k \geq 2$  with  $f + k \leq m - 1$  when  $m \geq 5$ . The constructions presented in this paper are recursive and not so complicated. According to them, we can design efficient algorithms for finding the two types of disjoint path covers. Furthermore, the bound  $f + k \leq m$  for a one-to-one DPC problem and the bound  $f + k \leq m - 1$  for an unpaired DPC problem are both optimal.

It has been proven in [26] that  $G(2^m, 4)$ ,  $m \geq 4$ , is  $f$ -fault *paired* many-to-many  $k$ -disjoint path coverable for any  $f$  and  $k \geq 2$  with  $f + 2k \leq m$ . For a graph  $G$  to be  $f$ -fault paired many-to-many  $k$ -disjoint path coverable, it is necessary that  $\kappa(G) \geq f + 2k - 1$  [25]. The gap between the bound  $f + 2k \leq m$  for a paired DPC problem addressed in [26] and the bound  $f + 2k \leq m + 1$  of necessity is one. Recently, it was found that  $G(32, 4)$  is 0-fault paired many-to-many 3-disjoint path coverable. It sheds light on the optimal construction of paired many-to-many disjoint path covers in  $G(2^m, 4)$ .

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## Appendix

**Lemma 31.**  $G(8, 4) \times K_2$  has a one-to-one 3-DPC for any  $s, t \in V(H_0)$  when  $f_1 = 1$  ( $f_0 = f_2 = 0$ ).

PROOF. There exists a one-to-one 3-DPC in  $H_0$  joining  $s$  and  $t$ , and there exists a hamiltonian cycle  $C_h$  in  $H_1 \setminus F_1$ . If there exists an edge  $(x, y)$  on some path in the DPC such that  $(\bar{x}, \bar{y})$  is an edge of  $C_h$ , a desired one-to-one 3-DPC can be obtained by replacing  $(x, y)$  with  $(x, \bar{x}, P_h, \bar{y}, y)$ , where  $P_h = C_h \setminus (\bar{x}, \bar{y})$ . The number of edges in  $G(8, 4)$  is 12. The 3-DPC passes through 9 edges and the hamiltonian cycle passes through at least 7 edges. Thus, at least four satisfy the required condition.  $\square$

**Lemma 32.**  $G(16, 4) \times K_2$  has a one-to-one 3-DPC for any  $s, t \in V(H_0)$  when  $F = \{\bar{s}, \bar{t}\}$ .

PROOF. The proof is similar to that of Lemma 31. Recall the assumption of  $(s, t) \notin E(G^m) \setminus F$ .  $H_0$  has a one-to-one 3-DPC, and  $H_1 \setminus F_1$  has a hamiltonian cycle  $C_h$ . It suffices to show that there exists an edge  $(x, y)$  on some path in the DPC such that  $(\bar{x}, \bar{y})$  is an edge of  $C_h$ . The number of edges in  $G(16, 4)$  incident to neither  $s$  nor  $t$  is  $24 (= 32 - 4 \cdot 2)$ . The 3-DPC passes through 17 edges, among them 11 edges are incident to neither  $s$  nor  $t$ . The hamiltonian cycle passes through 14 edges, which are incident to neither  $\bar{s}$  nor  $\bar{t}$ . Thus, there exists at least one edge satisfying the required condition.  $\square$

**Lemma 33.**  $G(8, 4) \times K_2$  has a one-to-one 3-DPC for any  $s \in V(H_0)$  and  $t \in V(H_1)$  when  $F = \{\bar{t}\}$ .

PROOF. By the assumption of  $F \neq \{v_f\}$  for any  $v_f$  with  $(s, v_f), (t, v_f) \in E(G^m)$ , we have  $(s, \bar{t}) \notin E(G^m)$ . Let  $V(H_0) = \{v_0, v_1, \dots, v_7\}$  and  $(v_i, v_j) \in E(H_0)$  if and only if  $j \equiv i + 1$  or  $i + 4 \pmod{8}$ . Assume w.l.o.g.  $\bar{t} = v_0$  and  $s \in \{v_2, v_3\}$ . Since  $H_0 \setminus F_0$  has a hamiltonian cycle  $(v_1, v_2, v_6, v_7, v_3, v_4, v_5)$ , we have a one-to-many 2-DPC in  $H_0 \setminus F_0$  joining  $s$  and  $\{v_1, v_5\}$ . Furthermore,  $H_1$  has a one-to-many 3-DPC  $\mathcal{P}$  joining  $\{\bar{s}, \bar{v}_1, \bar{v}_5\}$  and  $t$  as follows: for  $s = v_2$ ,  $\mathcal{P} = \{(\bar{s}, \bar{v}_3, \bar{v}_4, t), (\bar{v}_1, t), (\bar{v}_5, \bar{v}_6, \bar{v}_7, t)\}$ ; for  $s = v_3$ ,  $\mathcal{P} = \{(\bar{s}, \bar{v}_2, \bar{v}_6, \bar{v}_7, t), (\bar{v}_1, t), (\bar{v}_5, \bar{v}_4, t)\}$ . A one-to-one 3-DPC in  $H_0 \oplus H_1 \setminus F$  is obtained from the one-to-many 2-DPC in  $H_0 \setminus F_0$  and the one-to-many 3-DPC in  $H_1$ .  $\square$

**Lemma 34.** For any  $s \in V(H_0)$  and  $t \in V(H_1)$ ,  $G(16, 4) \times K_2$  has a one-to-one 4-DPC when  $F = \{\bar{t}\}$  and has a one-to-one 3-DPC when  $F = \{\bar{t}, \bar{s}\}$ .

PROOF. The proof is similar to that of Lemma 33. When  $F = \{\bar{t}\}$ , by the assumption of  $F \neq \{v_f\}$  for any  $v_f$  with  $(s, v_f), (t, v_f) \in E(G^m)$ , we let  $(s, \bar{t}) \notin E(G^m)$ . When  $F = \{\bar{t}, \bar{s}\}$ , we also assume  $(s, \bar{t}) \notin E(G^m)$ ; suppose otherwise, we can obtain a one-to-one 3-DPC from a 0-fault one-to-one 5-DPC in  $G(16, 4) \times K_2$  without faulty elements by removing the two paths  $(s, \bar{s}, t)$  and  $(s, \bar{t}, t)$  from the 5-DPC. Notice that  $(s, \bar{t})$  is an edge of  $G(16, 4) \times K_2$  iff  $(t, \bar{s})$  is an edge. Let  $V(H_0) = \{v_0, v_1, \dots, v_{15}\}$  and  $(v_i, v_j) \in E(H_0)$  if and only if  $j \equiv i + 1$  or  $i + 4 \pmod{16}$ . Assume w.l.o.g.  $\bar{t} = v_0$  and  $s \in \{v_2, v_3, v_5, v_6, v_7, v_8\}$ .  $H_0 \setminus F_0$  has a one-to-one 3-DPC between  $s$  and  $v_{15}$ . The vertices precede  $v_{15}$  on the three paths in the DPC are  $v_3, v_{11}$ , and  $v_{14}$ , which are the fault-free vertices adjacent to  $v_{15}$ . Therefore, there exists a one-to-many 3-DPC in  $H_0 \setminus F_0$  joining  $s$  and  $\{v_{11}, v_{14}, v_{15}\}$ .

When  $F = \{\bar{t}\}$ , it suffices to construct a one-to-many 4-DPC  $\mathcal{P}$  in  $H_1$  joining  $\{\bar{s}, v_{11}, v_{14}, v_{15}\}$  and  $t$  as follows. Let  $P_3 = (v_{14}, v_{13}, v_{12}, t)$  and  $P_4 = (v_{15}, t)$ .

For  $s = v_2$ ,  $\mathcal{P} = \{(\bar{s}, \bar{v}_3, \bar{v}_4, t), (v_{11}, v_{10}, \bar{v}_9, \bar{v}_8, \bar{v}_7, \bar{v}_6, \bar{v}_5, \bar{v}_1, t), P_3, P_4\}$ ;  
for  $s = v_3$ ,  $\mathcal{P} = \{(\bar{s}, \bar{v}_2, \bar{v}_1, t), (v_{11}, v_{10}, \bar{v}_9, \bar{v}_8, \bar{v}_7, \bar{v}_6, \bar{v}_5, \bar{v}_4, t), P_3, P_4\}$ ;  
for  $s = v_5$ ,  $\mathcal{P} = \{(\bar{s}, \bar{v}_1, t), (v_{11}, v_{10}, \bar{v}_9, \bar{v}_8, \bar{v}_7, \bar{v}_6, \bar{v}_2, \bar{v}_3, \bar{v}_4, t), P_3, P_4\}$ ;  
for  $s = v_6$ ,  $\mathcal{P} = \{(\bar{s}, \bar{v}_2, \bar{v}_3, \bar{v}_7, \bar{v}_8, \bar{v}_4, t), (v_{11}, v_{10}, \bar{v}_9, \bar{v}_5, \bar{v}_1, t), P_3, P_4\}$ ;  
for  $s = v_7$ ,  $\mathcal{P} = \{(\bar{s}, \bar{v}_3, \bar{v}_2, \bar{v}_6, \bar{v}_5, \bar{v}_1, t), (v_{11}, v_{10}, \bar{v}_9, \bar{v}_8, \bar{v}_4, t), P_3, P_4\}$ ;  
for  $s = v_8$ ,  $\mathcal{P} = \{(\bar{s}, \bar{v}_7, \bar{v}_3, \bar{v}_4, t), (v_{11}, v_{10}, \bar{v}_9, \bar{v}_5, \bar{v}_6, \bar{v}_2, \bar{v}_1, t), P_3, P_4\}$ .

When  $F = \{\bar{t}, \bar{s}\}$ , it suffices to construct a one-to-many 3-DPC  $\mathcal{P}'$  in  $H_1 \setminus \bar{s}$  joining  $\{v_{11}, v_{14}, v_{15}\}$  and  $t$  as follows.

For  $s = v_2$ ,  $\mathcal{P}' = \{(v_{11}, v_{10}, \bar{v}_9, \bar{v}_8, \bar{v}_4, \bar{v}_3, \bar{v}_7, \bar{v}_6, \bar{v}_5, \bar{v}_1, t), P_3, P_4\}$ ;  
for  $s = v_3$ ,  $\mathcal{P}' = \{(v_{11}, v_{10}, \bar{v}_9, \bar{v}_8, \bar{v}_7, \bar{v}_6, \bar{v}_2, \bar{v}_1, \bar{v}_5, \bar{v}_4, t), P_3, P_4\}$ ;  
for  $s = v_5$ ,  $\mathcal{P}' = \{(v_{11}, v_{10}, \bar{v}_9, \bar{v}_8, \bar{v}_4, \bar{v}_3, \bar{v}_7, \bar{v}_6, \bar{v}_2, \bar{v}_1, t), P_3, P_4\}$ ;  
for  $s = v_6$ ,  $\mathcal{P}' = \{(v_{11}, v_{10}, \bar{v}_9, \bar{v}_5, \bar{v}_4, \bar{v}_8, \bar{v}_7, \bar{v}_3, \bar{v}_2, \bar{v}_1, t), P_3, P_4\}$ ;  
for  $s = v_7$ ,  $\mathcal{P}' = \{(v_{11}, v_{10}, \bar{v}_6, \bar{v}_5, \bar{v}_9, \bar{v}_8, \bar{v}_4, \bar{v}_3, \bar{v}_2, \bar{v}_1, t), P_3, P_4\}$ ;  
for  $s = v_8$ ,  $\mathcal{P}' = \{(v_{11}, \bar{v}_7, \bar{v}_6, v_{10}, \bar{v}_9, \bar{v}_5, \bar{v}_4, \bar{v}_3, \bar{v}_2, \bar{v}_1, t), P_3, P_4\}$ . □