Disjoint Path Covers Joining Prescribed Source and Sink Sets in Interval Graphs

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Abstract

A *disjoint path cover* of a graph is a set of internally vertex-disjoint paths that altogether cover every vertex of the graph. Given two disjoint source and sink sets, S and T, in a graph G, a k*-disjoint path cover* of G joining S and T is a disjoint path cover composed of k paths, each of which runs from a source in S to a sink in T. In this paper, we give two short proofs for the characterization of interval graphs that possess a k-disjoint path cover (of one-to-one type) joining S and T for any S and T with $|S| = |T| = 1$ and for the characterization of interval graphs that possess a k-disjoint path cover (of one-to-many type) joining S and T for any S and T with $|S| = 1$ and $|T| = k$. Also, some partial results on the many-to-many disjoint path coverability of an interval graph are addressed. In addition, we discuss interval graphs in which the disjoint-path-cover property is preserved after removal of arbitrary f or less vertices.

Keywords: Disjoint path, path cover, path partition, scattering number, fault tolerance.

1. Introduction

Let G be a finite, simple undirected graph whose vertex and edge sets are denoted by $V(G)$ and $E(G)$, respectively. A *path* from $u \in V(G)$ to $v \in V(G)$ is a sequence $(w_1, \ldots, w_l)$ of distinct vertices of G, such that $w_1 = u$, $w_l = v$, and $(w_i, w_{i+1}) \in E(G)$ for all $i \in \{1, \ldots, l-1\}$. If $l \geq 3$ and $(w_l, w_1) \in E(G)$, the sequence is called a *cycle*. A *connected component* of G is a maximal connected subgraph of G. A *vertex cut* of G is a set $X \subseteq V(G)$ such that $G \setminus X$ has two or more connected components, where $G \setminus X$ is the subgraph obtained from G by deleting all the vertices of X (or equivalently, $G \setminus X$ is the subgraph of G induced by $V(G) \setminus X$). The *connectivity* of G, denoted $\kappa(G)$, is the minimum number of vertices whose removal results in a disconnected graph or a trivial one-vertex graph. So, $\kappa(G)$ is equal to the size of a minimum vertex cut if G is a noncomplete graph; $\kappa(G) = n - 1$ if G is a complete graph K_n of order n. A graph G is k*-connected* if $\kappa(G) \geq k$.

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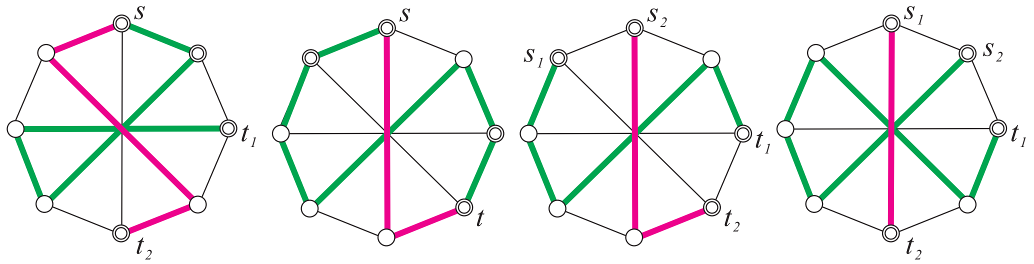


(a) One-to-many 2-DPC (b) One-to-one 2-DPC (c) Paired 2-DPC (d) Unpaired 2-DPC

Fig. 1: Examples of disjoint path covers.

It is well known that in terms of the minimum number of (vertex-)disjoint paths, the connectivity of a graph is characterized: Menger's theorem states the connectivity of a graph in terms of the number of internally disjoint paths (of one-to-one type) joining two distinct vertices, whereas the Fan Lemma states the connectivity of a graph in terms of the number of internally disjoint paths (of one-to-many type) joining a vertex to a set of vertices [3]. Moreover, it can be shown that a graph is k-connected if and only if it has k disjoint paths (of many-to-many type) joining two arbitrary vertex sets of size k each, in which a vertex that belongs to both sets is considered as a valid one-vertex path.

A *path cover* of a graph G is a set of paths in G such that every vertex of G is contained in at least one path. A *disjoint path cover* (DPC for short) of G is a set of (internally) vertex-disjoint paths that altogether cover every vertex of G. This paper is concerned with a DPC in which each path runs from a prescribed source to a prescribed sink. Given disjoint subsets $S = \{s\}$ and $T = \{t_1, \ldots, t_k\}$ of $V(G)$ for a positive integer k, a *one-to-many k-disjoint path cover* is a DPC composed of k paths, each of which joins a pair of source s and sink t_i, $i \in \{1, \ldots, k\}$. When $S = \{s\}$ and $T = \{t\}$, a DPC composed of k paths, each of which joins s and t, is named *one-to-one k-disjoint path cover*. (Refer to Fig. 1 for examples of DPCs.) The other DPC type is a *many-to-many k-disjoint path cover*, for which the k disjoint paths collectively join the disjoint sets $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$; if each source $s_i \in S$ must be joined to a specific sink $t_i \in T$, the DPC is called *paired*, and it is *unpaired* if no such constraint is imposed.

Definition 1 (Park et al. [28])**.** A graph G of order at least $k + 1$ is *one-to-many k-disjoint path coverable* if for every pair of disjoint subsets S and T of $V(G)$ with $|S| = 1$ and $|T| = k$, there exists a one-to-many k-DPC joining S and T.

Definition 2 (Park et al. [28])**.** A graph G of order at least $k + 1$ is *one-to-one k-disjoint path coverable* if for every pair of disjoint subsets S and T of $V(G)$ with $|S| = |T| = 1$, there exists a one-to-one k-DPC joining S and T.

Analogously, a graph G of order at least $2k$ is said to be *paired k-disjoint path coverable* if G has a paired (many-to-many) k-DPC joining S and T for any disjoint source and sink sets, S and T, of size k each; the graph G is said to be *unpaired k-disjoint path coverable* if G has an unpaired (many-to-many) k-DPC joining S and T for any disjoint source and sink sets, S and T, of size k each.

A cycle that visits each vertex exactly once is a *Hamiltonian cycle*; a path that visits each vertex exactly once is a *Hamiltonian path*. A graph is *Hamiltonian* if a Hamiltonian cycle exists;

a graph is *traceable* if a Hamiltonian path exists; a graph is *Hamiltonian-connected* if every two distinct vertices are joined by a Hamiltonian path. The existence of a disjoint path cover in a graph is closely related to the Hamiltonian properties (as well as the concept of vertex connectivity). A graph of order $n \geq 3$, for instance, is one-to-many 2-disjoint path coverable if and only if it is Hamiltonian-connected (because a Hamiltonian path between t_1 and t_2 forms a one-to-many 2-DPC joining $\{s\}$ and $\{t_1, t_2\}$ for every vertex $s \neq t_1, t_2$). Furthermore, a graph of order $n \geq 3$ is one-to-one 2-disjoint path coverable if and only if it is Hamiltonian (because a Hamiltonian cycle forms a one-to-one 2-DPC joining $\{s\}$ and $\{t\}$ for every pair of vertices s and t with $s \neq t$). In addition, a Hamiltonian path between two distinct vertices is in fact a 1-DPC, irrespective of its type, that joins the vertices.

An *interval graph* is the intersection graph of a family $\mathcal{I}$ of intervals on the real line, where two vertices are connected with an edge if and only if their corresponding intervals intersect. The family $\mathcal{I}$ is usually called an *interval representation* for the graph. A *proper interval graph* is an interval graph with an interval representation in which none of the intervals properly contains another. The interval graphs that are Hamiltonian/traceable were characterized by Deogun et al. [9] (Theorem 1 below); also, those that are Hamiltonian-connected were characterized by Broersma et al. [4] (Theorem 2 below). The characterizations are all in terms of the scattering number. For a noncomplete graph G, the *scattering number* $\mathrm{sc}(G)$ of G is defined as the following:

$$\mathrm{sc}(G) = \max\{c(G \setminus X) - |X| : X \subset V(G),\ c(G \setminus X) \geq 2\},$$

where $c(G \setminus X)$ denotes the number of the connected components of $G \setminus X$. A vertex cut X of G that fulfills $c(G \setminus X) - |X| = \mathrm{sc}(G)$ is called a *scattering set*. For a complete graph K_n of order n, we set $\mathrm{sc}(K_n) = 3 - n$ in this paper, as was done in [21]. (The definition of $\mathrm{sc}(K_n)$ lacks consistency in the literature; for example, $\mathrm{sc}(K_n)$ is originally undefined in [16], $\mathrm{sc}(K_n) = -n$ in [9], and $\mathrm{sc}(K_n) = -\infty$ in [4].)

Theorem 1 (Deogun et al. [9])**.** *(a) An interval graph G of order $n \geq 3$ is Hamiltonian if and only if* $\mathrm{sc}(G) \leq 0$*. (b) An interval graph G of order $n \geq 2$ is traceable if and only if* $\mathrm{sc}(G) \leq 1$.

Remark. In fact, it was proven in [9] that a cocomparability graph G is Hamiltonian (resp. traceable) if and only if $\mathrm{sc}(G) \leq 0$ (resp. $\mathrm{sc}(G) \leq 1$). Note that an interval graph is a cocomparability graph, which is defined as a graph whose edge set is exactly the set of pairs of elements that are incomparable to each other in a partial order.

Theorem 2 (Broersma et al. [4])**.** *An interval graph G of order $n \geq 3$ is Hamiltonian-connected if and only if* $\mathrm{sc}(G) \leq -1$ *or G is isomorphic to a complete graph K_3.*

Remark. The stronger result shown in Corollary 1(c) was actually proven in [4].

An ordering, $(v_1, \ldots, v_n)$, of the vertices of a graph G is a *Hamiltonian vertex ordering* if the ordering $(v_1, \ldots, v_n)$ forms a Hamiltonian path of G. The *thickness* of the Hamiltonian vertex ordering $(v_1, \ldots, v_n)$ is the maximum k such that the number of neighbors of v_i among $\{v_{i+1}, \ldots, v_n\}$ is at least $\min\{n - i, k\}$. The *Hamiltonian thickness* $\mathcal{H}(G)$ of G is defined by Li and Wu [21] as follows:

$$\mathcal{H}(G) = \begin{cases} \text{the maximum thickness over all Hamiltonian vertex orderings of } G & \text{if } G \text{ is traceable;} \\ 2 - \pi(G) & \text{otherwise,} \end{cases}$$

where $\pi(G)$ denotes the path cover number of G. In terms of the Hamiltonian thickness, interval graphs that are one-to-one k-disjoint path coverable and those that are one-to-many k-disjoint path coverable were characterized in [21], as shown below. A graph that is one-to-one k-disjoint path coverable is also said to be *spanning k-rail-connected*; a graph that is one-to-many k-disjoint path coverable is also called *spanning k-fan-connected*.

Theorem 3 (Li and Wu [21])**.** *(a) For every interval graph G, it holds* $\mathrm{sc}(G) + \mathcal{H}(G) = 2$. *(b) An interval graph G of order $n \geq k + 1$ is spanning k-rail-connected if and only if $\mathcal{H}(G) \geq k$. (c) An interval graph G of order $n \geq k + 2$ is spanning k-fan-connected if and only if $\mathcal{H}(G) \geq k + 1$.*

In this paper, we give a short proof for each of the two characterizations in Theorem 3 (b) and (c), in which the disjoint path coverability of an interval graph is directly related to its scattering number, without going through the concept of Hamiltonian thickness. In other words, we will prove that for an interval graph G of order $n \geq k + 1$ with $k \geq 2$,

- G is one-to-one k-disjoint path coverable if and only if $\mathrm{sc}(G) \leq 2 - k$;
- G is one-to-many k-disjoint path coverable if and only if $\mathrm{sc}(G) \leq 1 - k$ or G is isomorphic to a complete graph, K_{k+1}, of order $k + 1$.

The former and latter results can be seen as generalizations of Theorems 1(a) and 2, respectively, for $k \geq 2$ due to the previously mentioned relationship between the existence of a disjoint path cover and the Hamiltonian properties. On the other hand, some partial results on the many-to-many disjoint path coverability of an interval graph will be addressed. The results suggest that the many-to-many disjoint path coverability, whether paired or unpaired type, cannot be characterized only by the scattering number (and the Hamiltonian thickness); further, they allow us to predict that characterization may be possible if we extend the concept of scattering number.

In addition, the vertex fault tolerance of interval graphs that are one-to-one/one-to-many k-disjoint path coverable are discussed in this paper. That is, we consider interval graphs which remain one-to-one/one-to-many k-disjoint path coverable after removal of arbitrary f or less vertices for $f \geq 0$. The notion of vertex fault tolerance arises from reliability considerations in network design. The underlying theory is applicable to any graph property, that is characterized in terms of the scattering number, of a *hereditary* graph class, wherein every induced subgraph of a graph is contained in the same class.

2. Definitions and previous works

The disjoint path cover problems include the k-Disjoint Path Coverability and k-Disjoint Path Cover problems for a positive integer k. For each type of disjoint path covers, the k-Disjoint Path Coverability of a graph refers to a problem of deciding if the graph contains a k-DPC of the given type for any possible configurations of source and sink sets as follows:

One-to-One k-Disjoint Path Coverability
instance: A graph G with $|V(G)| \geq k + 1$.
question: Is G one-to-one k-disjoint path coverable?

One-to-Many k-Disjoint Path Coverability
instance: A graph G with $|V(G)| \geq k + 1$.
question: Is G one-to-many k-disjoint path coverable?

Paired and Unpaired k-Disjoint Path Coverability problems are defined similarly. For $k = 1$, the 1-Disjoint Path Coverability is equivalent to the Hamiltonian Connectivity problem of deciding whether a given graph is Hamiltonian-connected. The Hamiltonian Connectivity on a general graph is NP-complete [8], whereas the problem on an interval graph is linear-time solvable [4]. Meanwhile, for each type of disjoint path covers, the k-Disjoint Path Cover problem refers to that of determining whether a given graph G contains a k-DPC of the type joining given subsets S and T of $V(G)$ as follows:

One-to-One k-Disjoint Path Cover
instance: A graph G and disjoint subsets S and T of $V(G)$ with $|S| = |T| = 1$.
question: Does G contain a one-to-one k-disjoint path cover joining S and T?

One-to-Many, Paired, and Unpaired k-Disjoint Path Cover problems are defined similarly. For $k = 1$, the 1-Disjoint Path Cover problem is equivalent to the 2-Hamiltonian Path problem of deciding whether a given graph has a Hamilton path that joins two given vertices. The 2-Hamiltonian Path problem on a general graph is NP-complete [11]; it remains still open if the problem on an interval graph is polynomially solvable [2, 4, 7], but partial results can be found in [22].

The disjoint path cover problems also include the Unconstrained Disjoint Path Cover problem for which no source and sink sets are prescribed, so each path in a disjoint path cover freely joins a pair of vertices (alleviating the constraint that each path should run from a prescribed source to a prescribed sink). The problem is to determine a disjoint path cover of a graph that uses the minimum number of paths. The minimum number, known as the *path cover number*, is equal to 1 if and only if the graph is traceable. For more details, we refer to the related literature including [1, 13, 24].

The disjoint path cover problems are applicable in many areas such as software testing, database design, and code optimization [2, 24]. The problems have been studied for various classes of graphs, such as interval graphs [1, 2, 13, 22, 26], hypercubes [6, 10, 15], recursive circulants [18, 19], dense graphs [23], cubes of connected graphs [14], and grid graphs [27]. It was shown that deciding the existence of a one-to-one, one-to-many, or paired/unpaired many-to-many k-DPC in a general graph, joining given source and sink sets, is NP-complete for any fixed $k \geq 1$ [28, 29].

The results on the DPC problems of interval graphs are now briefly discussed. Regarding the Unconstrained Disjoint Path Cover problem, Arikati and Rangan [1] presented a linear-time algorithm; for the same problem, a linear-time certifying algorithm was suggested by Hung and Chang [13]. An algorithm for building a DPC that joins prescribed source and sink sets has not been reported in the literature. Only a few results on proper interval graphs, a subclass of interval graphs, can be found. A linear-time algorithm for the Unpaired k-Disjoint Path Cover problem that actually builds an unpaired k-DPC between prescribed source and sink sets was recently developed in [26], to which the One-to-One and One-to-Many k-Disjoint Path Cover problems are reduced in linear time. For a proper interval graph G and $k \geq 2$, it was proven in [25] that G is one-to-one k-disjoint path coverable if and only if G is k-connected; furthermore, G is one-to-many k-disjoint path coverable if and only if G is $(k + 1)$-connected or isomorphic to a complete graph K_{k+1}. The conditions regarding Paired and Unpaired k-Disjoint Path Coverability can be found in [20, 25].

Interval graphs are a well-studied class of graphs. One of the early characterizations of interval graphs is the following:

Theorem 4 (Gilmore and Hoffman [12]). *A graph G is an interval graph if and only if the maximal cliques of G can be linearly ordered such that, for every vertex v of G, the maximal cliques containing v occur consecutively.*

Let $C_1, \ldots, C_q$ be a linear ordering of the maximal cliques of a noncomplete interval graph G such that each vertex of G appears in consecutive cliques only. Since C_1 and C_q are maximal cliques, C_1 and C_q each contains at least one vertex that does not occur in any other clique. Let u_1 be such a vertex in C_1 and let u_n be such a vertex in C_q. The existence of a one-to-one k-DPC between u_1 and u_n was studied by Broersma et al. [4], as is shown in Theorem 5. A one-to-one k-DPC between vertices s and t is also known as a *spanning k-stave* or *k^*-container* between s and t.

Theorem 5 (Broersma et al. [4]). *A noncomplete interval graph G contains a spanning k-stave between u_1 and u_n if and only if* $\mathrm{sc}(G) \leq 2 - k$.

Alternatively, necessary conditions for a general graph G to be one-to-one k-disjoint path coverable and to be one-to-many k-disjoint path coverable have been derived in terms of the connectivity $\kappa(G)$ of G, as shown below.

Lemma 1 (Kim et al. [18]). *If a graph G of order $n \geq k + 1$ is one-to-one k-disjoint path coverable, then $\kappa(G) \geq k$.*

Lemma 2 (Lee et al. [20]). *If a graph G of order $n \geq k + 1$ is one-to-many k-disjoint path coverable, then $\kappa(G) \geq k + 1$ or G is isomorphic to a complete graph K_{k+1}.*

The graph-theoretic terms that are not defined here can be found in [3]. Our definition of $\mathrm{sc}(K_n) = 3 - n$ leads to that the scattering numbers of the n-vertex graphs form a consecutive set $\{3 - n, 4 - n, \ldots, n\}$, because the scattering number of a noncomplete graph of order n is at least $4 - n$ by Lemma 3 (below) and for every $r \in \{4 - n, \ldots, n\}$, there is a noncomplete graph whose scattering number is r. Furthermore, the definition enables us to establish a connection between the scattering number and the connectivity of a graph that is not necessarily a noncomplete graph, as shown in Lemmas 4 and 5.

Lemma 3. *Let G be a noncomplete graph of order $n \geq 2$. Then,* $\mathrm{sc}(G) \geq 4 - n$.

Proof. For a pair of nonadjacent vertices u and v of G, let $X = V(G) \setminus \{u, v\}$. It follows that $\mathrm{sc}(G) \geq c(G \setminus X) - |X| = 2 - (n - 2) = 4 - n$. ■

Lemma 4 (Zhang and Wang [30]). *Let G be a noncomplete graph of order $n \geq 2$. Then,* $\mathrm{sc}(G) \geq 2 - \kappa(G)$.

Lemma 5. *A graph G of order $n \geq 2$ is k-connected if* $\mathrm{sc}(G) \leq 2 - k$.

Proof. From Lemma 4 and the definition $\mathrm{sc}(K_n) = 3 - n$, we have $\mathrm{sc}(G) \geq 2 - \kappa(G)$ whether G is complete or not. It follows that $2 - \kappa(G) \leq \mathrm{sc}(G) \leq 2 - k$, implying $\kappa(G) \geq k$, as required. ■

Remark. The converse of Lemma 5 is not necessarily true for interval graphs; a complete bipartite graph $K_{1,3}$, which is an interval graph, is 1-connected but $\mathrm{sc}(K_{1,3}) = 2 \nleq 2 - 1 = 1$. On the other hand, the converse is known to be true for proper interval graphs, i.e., a proper interval graph G of order $n \geq 2$ is k-connected if and only if $\mathrm{sc}(G) \leq 2 - k$ [5].

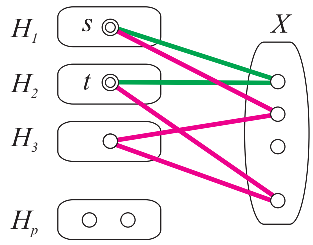


Fig. 2: Illustration of the proof of Lemma 6, in case where G is a noncomplete graph of order $n \geq k+1$.

3. Necessary conditions on general graphs

In this section, the conclusions of Lemmas 1 and 2 are strengthened in terms of the scattering number (instead of the connectivity). The condition "$\kappa(G) \geq k$" will be replaced with "$\mathrm{sc}(G) \leq 2-k$." Note that the latter is stronger than the former by Lemma 5. Of course, "$\kappa(G) \geq k+1$" will be replaced with "$\mathrm{sc}(G) \leq 2-(k+1) = 1-k$."

Lemma 6. *If a graph G of order $n \geq k+1$ is one-to-one k-disjoint path coverable, then* $\mathrm{sc}(G) \leq 2-k$.

Proof. Assume that G is a one-to-one k-disjoint path coverable graph of order $n \geq k+1$. If G is a complete graph, then $\mathrm{sc}(G) = 3-n \leq 3-(k+1) = 2-k$ as required. Let G be a noncomplete graph hereafter. Suppose $\mathrm{sc}(G) > 2-k$ for a contradiction. Then, there exists a scattering set X such that $c(G \setminus X) - |X| = \mathrm{sc}(G) > 2-k$; also, $G \setminus X$ has two or more connected components (because X is a vertex cut of G by definition). Let $H_1, \ldots, H_p$ be the connected components of $G \setminus X$, where $p = c(G \setminus X) \geq 2$. Consider a one-to-one k-DPC $\{P_1, \ldots, P_k\}$ of G joining a pair of vertices $s \in V(H_1)$ and $t \in V(H_2)$ in particular. Let a path P_i in the DPC pass through r_i, $r_i \geq 0$, connected components other than H_1 and H_2. Then, the path P_i must pass through at least r_i+1 vertices of X, as is shown in Fig. 2. (This is because, representing P_i as $(s = v_1, v_2, \ldots, v_m = t)$ for some m, every subpath $(v_j, \ldots, v_{j'})$, $j < j'$, of P_i such that $v_j \in V(H_q)$ and $v_{j'} \in V(H_{q'})$ for some q, q' with $q \neq q'$ should contain a vertex of X.) It follows that (i) $\sum_{i=1}^{k} r_i \geq p-2$ because the k paths in the DPC collectively pass through all the connected components; (ii) $\sum_{i=1}^{k}(r_i+1) \leq |X|$ because every vertex of X is contained in exactly one DPC path. Combining the two inequalities leads to $p-2+k \leq |X|$, which contradicts the fact $p-|X| = \mathrm{sc}(G) > 2-k$. This completes the proof. ■

Lemma 7. *If a graph G of order $n \geq k+1$ is one-to-many k-disjoint path coverable, then* $\mathrm{sc}(G) \leq 1-k$ *or G is isomorphic to a complete graph K_{k+1}.*

Proof. Assume that G is a one-to-many k-disjoint path coverable graph of order $n \geq k+1$. If $n = k+1$, then G is a complete graph as required. (Supposing $(u, v) \notin E(G)$ for some $u, v \in V(G)$ leads to the fact that G does not contain a one-to-many k-DPC joining $S = \{u\}$ and $T = V(G) \setminus S$, which is a contradiction.) Let $n \geq k+2$ hereafter. If G is a complete graph, then $\mathrm{sc}(G) = 3-n \leq 3-(k+2) = 1-k$, completing the proof. Now, let G be a noncomplete graph of order $n \geq k+2$. Suppose to the contrary $\mathrm{sc}(G) > 1-k$. Then, there exists a scattering set X such that $c(G \setminus X) - |X| = \mathrm{sc}(G) > 1-k$; also, we have $|X| \geq k+1$ by Lemma 2 (because X itself is a vertex cut of G). Let $H_1, \ldots, H_p$ be the connected components of $G \setminus X$, where $p = c(G \setminus X) \geq 2$.

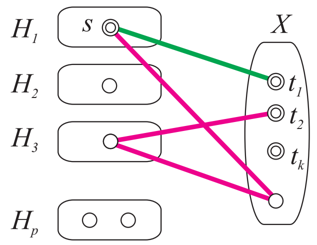


Fig. 3: Illustration of the proof of Lemma 7, in case where G is a noncomplete graph of order $n \geq k + 2$.

Consider a one-to-many k-DPC $\{P_1, \ldots, P_k\}$ of G joining a vertex $s \in V(H_1)$ and a set $T \subseteq X$ with $|T| = k$. Let a path P_i in the DPC pass through r_i, $r_i \geq 0$, connected components other than H_1. Then, the path P_i must pass through at least $r_i + 1$ vertices of X, as shown in Fig. 3. It follows that (i) $\sum_{i=1}^{k} r_i \geq p - 1$ because the k paths in the DPC collectively pass through all the connected components; (ii) $\sum_{i=1}^{k}(r_i + 1) \leq |X|$ because every vertex of X is contained in exactly one DPC path. Combining the two inequalities leads to $p - 1 + k \leq |X|$, which contradicts the fact $p - |X| = \mathrm{sc}(G) > 1 - k$. The proof is completed. ■

4. Sufficiency of the necessary conditions on interval graphs

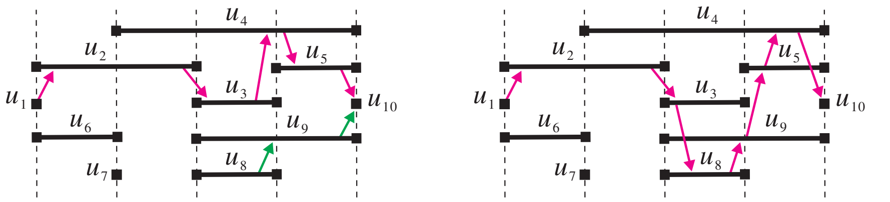
In this section, we prove that the necessary conditions of Lemmas 6 and 7, respectively, are sufficient ones for an interval graph to be one-to-one and one-to-many k-disjoint path coverable. Theorem 5 will be utilized in both proofs, as was done by Broersma et al. [4] in proving Theorem 2. We begin with a linear ordering, $C_1, \ldots, C_q$, of the maximal cliques of an interval graph G such that every vertex of G appears in consecutive cliques only. The existence of such an ordering is due to Theorem 4. Obviously, the graph G is noncomplete if and only if $q \geq 2$. Recall that u_1 denotes a vertex in $V(C_1) \setminus V(C_2)$ and u_n denotes a vertex in $V(C_q) \setminus V(C_{q-1})$ if G is noncomplete; otherwise u_1 and u_n are arbitrary distinct vertices of G.

It was proven by Keil [17] that a traceable interval graph (that possesses a Hamiltonian path) always contains a Hamiltonian path that runs from u_1 to u_n. A path in an interval graph is *monotone* if every edge (u, v) can be assigned a point from $I_u \cap I_v$, the intersection of the intervals corresponding to u and v, such that the points that are ordered in the appearance of the edges in the path form a nondecreasing sequence.

Lemma 8 (Keil [17]). *If an interval graph contains a Hamiltonian path, then it contains a monotone Hamiltonian path from u_1 to u_n.*

For distinct vertices u_i and u_j of G, there exists a maximal clique C_p, $p \in \{1, \ldots, q\}$, such that $u_i, u_j \in V(C_p)$ if and only if $(u_i, u_j) \in E(G)$. (This is because there exists a clique that contains u_i and u_j if and only if $(u_i, u_j) \in E(G)$.) Thus, a vertex u_i of G can be represented by a closed interval $I_{u_i} = [l_i, r_i]$, where $l_i = \min\{j : u_i \in V(C_j)\}$ and $r_i = \max\{j : u_i \in V(C_j)\}$. So, we have $I_{u_1} = [1, 1]$ and $I_{u_n} = [q, q]$; moreover, $(u_1, u_n) \notin E(G)$ if G is noncomplete.

Lemma 9. *Let P be a path joining u_1 and u_n of a noncomplete interval graph G, and let Q be a path of G that runs from u_1 or to u_n. If the two paths P and Q are internally disjoint, there exists a path P' joining u_1 and u_n such that $V(P') = V(P) \cup V(Q)$.*



(a) (b)

Fig. 4: An interval representation of an interval graph with 10 vertices, in which there are five maximal cliques, $C_1 = \{u_1, u_2, u_6\}$, $C_2 = \{u_2, u_4, u_6, u_7\}$, ..., $C_5 = \{u_4, u_5, u_9, u_{10}\}$: (a) there are paths, $P = (u_1, u_2, u_3, u_4, u_5, u_{10})$ and $Q = (u_8, u_9, u_{10})$, which are internally disjoint; (b) the two paths can be merged into a new path $P' = (u_1, u_2, u_3, u_8, u_9, u_5, u_4, u_{10})$.

Proof. Refer to Fig. 4. Let G' be the subgraph of G induced by $V(P) \cup V(Q)$. Then, G' is a noncomplete interval graph (because $(u_1, u_n) \notin E(G')$) that contains a Hamiltonian path (obtained from combining P and Q). Also, u_1 and u_n, respectively, are included only in the leftmost and rightmost maximal cliques of G'. Thus, G' contains a Hamiltonian path joining u_1 and u_n from Lemma 8, as required. ■

Lemma 10. *Let P be a path joining u_1 and u_n of a noncomplete interval graph G. Then, every vertex in $V(G) \setminus (V(P) \setminus \{u_1, u_n\})$ has a neighbor in $V(P) \setminus \{u_1, u_n\}$.*

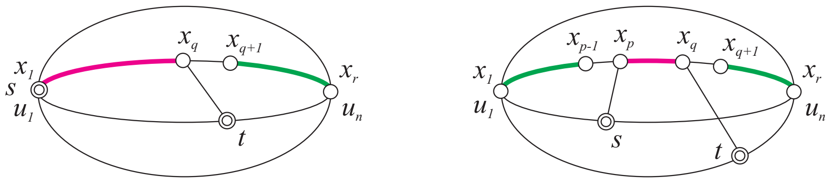
Proof. Let v be a vertex in $V(G) \setminus (V(P) \setminus \{u_1, u_n\})$. There exists a maximal clique C_i of G that includes v. The lemma follows from the fact that $V(P) \cap V(C_j) \neq \emptyset$ for all $j \in \{1, \ldots, q\}$ and moreover, $|V(P) \cap V(C_1)|, |V(P) \cap V(C_q)| \geq 2$ because I_{u_1} and I_{u_n} are single-point intervals. ■

Now we are ready to prove sufficiency of the necessary conditions of Lemmas 6 and 7.

Theorem 6. *For $k \geq 2$, an interval graph G of order $n \geq k + 1$ is one-to-one k-disjoint path coverable if and only if* $\mathrm{sc}(G) \leq 2 - k$.

Proof. The necessity is due to Lemma 6. The sufficiency proof proceeds by induction on k. Assume $\mathrm{sc}(G) \leq 2-k$. For the base step of $k = 2$, a Hamiltonian cycle exists in G by Theorem 1(a), implying that G is one-to-one 2-disjoint path coverable. Let $k \geq 3$ for the inductive step. A complete graph is obviously one-to-one k-disjoint path coverable, so it is assumed that G is noncomplete. Then, a one-to-one k-DPC $\{P_1, \ldots, P_k\}$ exists between u_1 and u_n in G by Theorem 5. Between an arbitrary pair of vertices, s and t, such that $\{s, t\} \neq \{u_1, u_n\}$, it will be shown that a one-to-one k-DPC exists in G.

Case 1: $|\{s, t\} \cap \{u_1, u_n\}| = 1$. Assume w.l.o.g. $s = u_1$ and let $t \in V(P_j)$ for some j. On a path P_i with $i \neq j$, where P_i is represented as $(x_1 = u_1, x_2, \ldots, x_r = u_n)$ for some $r \geq 3$, a neighbor x_q of the sink t with $q \notin \{1, r\}$ exists by Lemma 10. (See Fig. 5(a).) If we divide P_i into two subpaths, $P'_i = (x_1, \ldots, x_q)$ and $P''_i = (x_{q+1}, \ldots, x_r)$, and merge P_j and P''_i into a new path from u_1 to u_n through Lemma 9, then we obtain a one-to-one $(k-1)$-DPC between u_1 and u_n of the induced subgraph G', defined as $G \setminus \{x_2, \ldots, x_q\}$. It follows that $\mathrm{sc}(G') \leq 2 - (k-1)$ by Theorem 5, implying that G' is one-to-one $(k-1)$-disjoint path coverable by the induction hypothesis. Adding the path (P'_i, t) to a one-to-one $(k-1)$-DPC of G' that joins s and t results in a required one-to-one k-DPC of G.



(a) $s = u_1$ and $t \neq u_n$ (b) $\{s, t\} \cap \{u_1, u_n\} = \emptyset$

Fig. 5: Illustrations of the proof of Theorem 6.

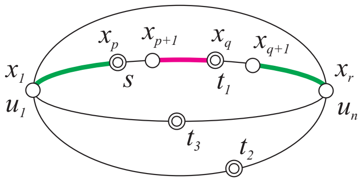
Case 2: $\{s, t\} \cap \{u_1, u_n\} = \emptyset$. The proof is similar to that for the previous case. Let $s \in V(P_j)$ and $t \in V(P_{j'})$, possibly $j' = j$. On a path P_i with $i \notin \{j, j'\}$, represented as $(x_1 = u_1, x_2, \ldots, x_r = u_n)$, there exist neighbors x_p and x_q, respectively, of s and t such that $\{p, q\} \cap \{1, r\} = \emptyset$ by Lemma 10. (See Fig. 5(b).) Assuming w.l.o.g. $p \leq q$, the path P_i is divided into three subpaths, $P_i' = (x_1, \ldots, x_{p-1})$, $P_i'' = (x_p, \ldots, x_q)$, and $P_i''' = (x_{q+1}, \ldots, x_r)$. Merging P_j and P_i' into a path between u_1 and u_n through Lemma 9, and merging again the resultant path and P_i''' into a path from u_1 to u_n, leads to a one-to-one $(k-1)$-DPC between u_1 and u_n of the induced subgraph G', defined as $G \setminus V(P_i'')$. It follows that $\mathrm{sc}(G') \leq 2 - (k-1)$ by Theorem 5, and thus G' has a one-to-one $(k-1)$-DPC between s and t by the induction hypothesis. If the path (s, P_i'', t) is added to the $(k-1)$-DPC of G', a one-to-one k-DPC between s and t of G is obtained. This completes the proof. ■

Theorem 7. *For $k \geq 2$, an interval graph G of order $n \geq k+1$ is one-to-many k-disjoint path coverable if and only if* $\mathrm{sc}(G) \leq 1 - k$ *or G is isomorphic to a complete graph K_{k+1}.*

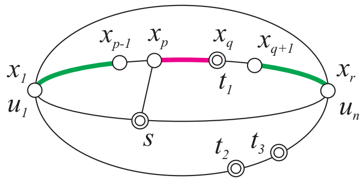
Proof. The necessity is due to Lemma 7. The sufficiency part will be proved by induction on k. Assume that $\mathrm{sc}(G) \leq 1 - k$ or G is isomorphic to K_{k+1}. For the base step of $k = 2$, G is Hamiltonian-connected by Theorem 2, meaning that G is one-to-many 2-disjoint path coverable. Let $k \geq 3$ for the inductive step. A complete graph K_n of order $n \geq k+1$ is obviously one-to-many k-disjoint path coverable, so it is further assumed that G is noncomplete, i.e., G is a noncomplete interval graph with $\mathrm{sc}(G) \leq 1 - k = 2 - (k+1)$. By Theorem 5, there exists a one-to-one $(k+1)$-DPC $\{P_1, \ldots, P_{k+1}\}$ between u_1 and u_n in G. It will be shown that for any disjoint terminal sets $S = \{s\}$ and $T = \{t_1, \ldots, t_k\}$, there exists a one-to-many k-DPC that joins them.

Claim. There exists a subpath $Q = (z_1, \ldots, z_l)$ of some path in the one-to-one $(k+1)$-DPC such that (i) $s, u_1, u_n \notin V(Q)$, (ii) $(s, z_1) \in E(G)$, and (iii) $V(Q) \cap T = \{z_l\}$.

Proof of claim. Let P_j, represented as $(x_1 = u_1, x_2, \ldots, x_r = u_n)$ for some $r \geq 3$, be a DPC path that passes through a sink, say t_1, as an intermediate vertex; such a path exists because $|T| = k \geq 3$. There are two cases to consider. First, suppose $s \in V(P_j)$, i.e., $s = x_p$ for some $p \in \{1, \ldots, r\}$. (See Fig. 6(a).) If there is a sink x_q, $p < q < r$, such that $\{x_{p+1}, \ldots, x_q\} \cap T = \{x_q\}$, it suffices to let $Q = (x_{p+1}, \ldots, x_q)$; otherwise, for a sink $x_{q'}$, $1 < q' < p$, such that $\{x_{q'}, \ldots, x_{p-1}\} \cap T = \{x_{q'}\}$, it suffices to let $Q = (x_{p-1}, x_{p-2}, \ldots, x_{q'})$. Second, suppose $s \notin V(P_j)$. (See Fig. 6(b)). Let $x_p \in V(P_j)$ denote a neighbor of s other than u_1 and u_n; such a neighbor exists by Lemma 10. If there is a sink x_q, $p \leq q < r$, such that $\{x_p, \ldots, x_q\} \cap T = \{x_q\}$, it suffices to let $Q = (x_p, \ldots, x_q)$; otherwise, for a sink $x_{q'}$, $1 < q' < p$, such that $\{x_{q'}, \ldots, x_p\} \cap T = \{x_{q'}\}$, it suffices to let $Q = (x_p, x_{p-1}, \ldots, x_{q'})$. The claim is therefore proven. □



(a) $s \in V(P_j)$



(b) $s \notin V(P_j)$

Fig. 6: Illustrations of the proof of Theorem 7, where $t_1 \in V(P_j)$ and $t_1 \neq u_1, u_n$.

Let $G' = G \setminus V(Q)$. The DPC path, P_j, that contains Q as a subpath can be divided into the three subpaths, P'_j, Q, and P''_j. Merging a DPC path P_i other than P_j and the subpath P'_j into a new path from u_1 to u_n through Lemma 9, and merging again the resultant path and P''_j leads to a one-to-one k-DPC between u_1 and u_n of G'. It follows that $\mathrm{sc}(G') \leq 2 - k = 1 - (k-1)$ by Theorem 5, meaning that G' is one-to-many $(k-1)$-disjoint path coverable by the induction hypothesis. Adding the path (s, Q) to a one-to-many $(k-1)$-DPC between s and $T \cap V(G')$ of G' results in a one-to-many k-DPC between s and T of G, as required. This completes the proof. ■

Combining Theorems 6 and 7 with the linear-time algorithm devised by Broersma et al. [4] for finding the scattering number (or, with the linear-time algorithm devised by Li and Wu [21] for finding the Hamiltonian thickness) of an interval graph yields that the One-to-One and One-to-Many k-Disjoint Path Coverability problems on an interval graph can be decided in linear time, while the two problems on a general graph are NP-complete [28].

5. Vertex Fault Tolerance

A graph G is called *f-vertex-fault Hamiltonian* if $G \setminus F$ is Hamiltonian for any fault set $F \subseteq V(G)$ with $|F| \leq f$; the graph G is *f-vertex-fault Hamiltonian-connected* (resp. *traceable*) if $G \setminus F$ is Hamiltonian-connected (resp. traceable) for any fault set $F \subseteq V(G)$ with $|F| \leq f$. (An f-vertex-fault Hamiltonian graph is also known as an f-Hamiltonian graph in the literature.) The notion of vertex fault tolerance can naturally be applied to the disjoint-path-coverability property as follows: A graph G of order $n \geq f + k + 1$ is *f-vertex-fault one-to-one* (resp. *one-to-many*) *k-disjoint path coverable* if $G \setminus F$ is one-to-one (resp. one-to-many) k-disjoint path coverable for any fault set $F \subseteq V(G)$ with $|F| \leq f$. Whenever a graph property is characterized in terms of the scattering number, the following theorem can be used to derive the vertex fault tolerance of the graphs (not necessarily interval graphs) that possess the property, as is done in Corollary 1.

Theorem 8. *Let G be a general graph of order n, and let α and f be integers with $\alpha \leq 1$ and $f \geq 0$. Then, $\mathrm{sc}(G \setminus F) \leq \alpha$ for every vertex subset F with $|F| \leq f$ if and only if $\mathrm{sc}(G) \leq \alpha - f$.*

Proof. For the sufficiency part, assume $\mathrm{sc}(G) \leq \alpha - f$. If G is a complete graph, then $\mathrm{sc}(G) = 3 - n \leq \alpha - f$, hence $\mathrm{sc}(G \setminus F) = 3 - (n - |F|) = 3 - n + |F| \leq \alpha - f + |F| \leq \alpha$ for all $F \subset V(G)$ with $|F| \leq f$, as required. Now, let G be a noncomplete graph. Toward a contradiction, suppose $\mathrm{sc}(G \setminus F) > \alpha$ for some $F \subset V(G)$ with $|F| \leq f$. Let G' denote the induced subgraph $G \setminus F$. If G' is a complete graph, then $\mathrm{sc}(G') = 3 - (n - |F|) = 3 - n + |F| > \alpha$, implying $3 - n + f > \alpha$, or equivalently $\alpha - f < 3 - n$. It follows that $\mathrm{sc}(G) \leq \alpha - f < 3 - n$, which contradicts the fact that $\mathrm{sc}(G) \geq 4 - n$ shown in Lemma 3. Let G' (as well as G) be a noncomplete graph. Then, there

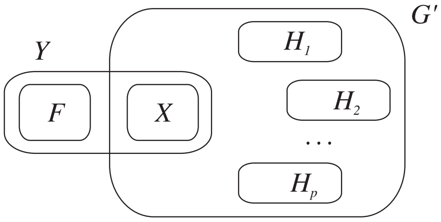


Fig. 7: Illustration of the proof of Theorem 8, where $c(G \setminus Y) = c(G' \setminus X) = p \geq 2$.

exists a scattering set X of G' such that $c(G' \setminus X) - |X| = \mathrm{sc}(G') > \alpha$. (See Fig. 7.) For $Y := X \cup F$, we have $c(G \setminus Y) = c(G' \setminus X) \geq 2$ and $|Y| = |X| + |F|$, therefore

$$\mathrm{sc}(G) \geq c(G \setminus Y) - |Y| = c(G' \setminus X) - |X| - |F| = \mathrm{sc}(G') - |F| > \alpha - f,$$

which contradicts the hypothesis $\mathrm{sc}(G) \leq \alpha - f$.

For the necessity part, assume $\mathrm{sc}(G \setminus F) \leq \alpha$ for all $F \subset V(G)$ with $|F| \leq f$. If G is a complete graph, then $\mathrm{sc}(G \setminus F) = 3 - (n - |F|) = 3 - n + |F| \leq \alpha$ for all $F \subset V(G)$ with $|F| \leq f$, implying $3 - n + f \leq \alpha$ and resulting in $\mathrm{sc}(G) = 3 - n \leq \alpha - f$, as required. It is further assumed that G is a noncomplete graph hereafter. For an arbitrary vertex cut Y of G, it suffices to show that $c(G \setminus Y) - |Y| \leq \alpha - f$.

Claim. $|Y| > f$.

Proof of claim. Suppose $|Y| \leq f$ for a contradiction. Then, $\mathrm{sc}(G \setminus Y) \leq \alpha \leq 1$ by the hypothesis. It follows that $G \setminus Y$ is 1-connected by Lemma 5, which contradicts the fact that Y is a vertex cut of G, thereby proving the claim. □

Let F be a subset of Y with $|F| = f$, and let $X = Y \setminus F$, so $X \neq \emptyset$ and $|Y| = |X| + |F|$. (See Fig. 7 again.) For the induced subgraph $G' := G \setminus F$, we have $\mathrm{sc}(G') \leq \alpha$ and $c(G' \setminus X) = c(G \setminus Y) \geq 2$, therefore

$$c(G \setminus Y) - |Y| = c(G' \setminus X) - |X| - |F| \leq \mathrm{sc}(G') - |F| \leq \alpha - |F| = \alpha - f,$$

completing the entire proof. ■

Corollary 1. *(a) An interval graph G of order $n \geq f + 3$ is f-vertex-fault Hamiltonian if and only if* $\mathrm{sc}(G) \leq -f$ *[21].*
(b) An interval graph G of order $n \geq f + 2$ is f-vertex-fault traceable if and only if $\mathrm{sc}(G) \leq 1 - f$ *[21].*
(c) An interval graph G of order $n \geq f + 3$ is f-vertex-fault Hamiltonian-connected if and only if $\mathrm{sc}(G) \leq -1 - f$ *or G is isomorphic to K_{f+3} [4, 21].*
(d) For $k \geq 2$, an interval graph G of order $n \geq f + k + 1$ is f-vertex-fault one-to-one k-disjoint path coverable if and only if $\mathrm{sc}(G) \leq 2 - k - f$ *[21].*
(e) For $k \geq 2$, an interval graph G of order $n \geq f + k + 1$ is f-vertex-fault one-to-many k-disjoint path coverable if and only if $\mathrm{sc}(G) \leq 1 - k - f$ *or G is isomorphic to a complete graph K_{f+k+1}.*

Proof. Proof for (a). Let G be an interval graph of order $n \geq f + 3$. Then,
G is f-vertex-fault Hamiltonian
$\Leftrightarrow$ $G \setminus F$ is Hamiltonian for any fault set $F \subseteq V(G)$ with $|F| \leq f$ (by definition)

$\Leftrightarrow$ $\mathrm{sc}(G \setminus F) \leq 0$ for any fault set $F \subseteq V(G)$ with $|F| \leq f$ (by Theorem 1(a))
$\Leftrightarrow$ $\mathrm{sc}(G) \leq -f$ (by Theorem 8).

Analogously, proofs for (b), (c), (d), and (e) are completed by combining Theorem 8 with Theorems 1(b), 2, 6, and 7, respectively. We give a proof for (c) in which the condition forms a clause with an exclusive disjunction; proofs for the others are omitted.

Proof for (c). Let G be an interval graph of order $n \geq f+3$. Then,
G is f-vertex-fault Hamiltonian-connected
$\Leftrightarrow$ $G \setminus F$ is Hamiltonian-connected for all $F \subseteq V(G)$ with $|F| \leq f$ (by definition)
$\Leftrightarrow$ $\mathrm{sc}(G \setminus F) \leq -1$ or $G \setminus F$ is isomorphic to K_3 for all $F \subseteq V(G)$ with $|F| \leq f$ (by Theorem 2)

$$\Leftrightarrow \begin{cases} \mathrm{sc}(G \setminus F) \leq -1 \text{ for all } F \subseteq V(G) \text{ with } |F| \leq f & \text{if } n \geq f+4 \\ G \text{ is isomorphic to } K_{f+3} & \text{if } n = f+3 \end{cases}$$

$$\Leftrightarrow \begin{cases} \mathrm{sc}(G) \leq -1-f \text{ (by Theorem 8)} & \text{if } n \geq f+4 \\ G \text{ is isomorphic to } K_{f+3} & \text{if } n = f+3 \end{cases}$$

$\Leftrightarrow$ $\mathrm{sc}(G) \leq -1-f$ or G is isomorphic to K_{f+3}. ■

Corollary 2 (Li and Wu [21]). *Let G be an interval graph of order $n \geq f+k+2$ with $f \geq 0$ and $k \geq 2$. Then, the following statements are equivalent:*
(a) G is f-vertex-fault one-to-many k-disjoint path coverable.
(b) G is $(f+1)$-vertex-fault one-to-one k-disjoint path coverable.
(c) G is $(f+k-2)$-vertex-fault Hamiltonian-connected.
(d) G is $(f+k-1)$-vertex-fault Hamiltonian.
(e) G is $(f+k)$-vertex-fault traceable.
(f) $\mathrm{sc}(G) \leq 1-k-f$.

Proof. Equivalence between (a) and (f) is due to Corollary 1(e). Also, the statements (b) through (e) are equivalent to (f) by Corollary 1 (a) through (d). ■

6. Many-to-many disjoint path covers

Also in terms of the connectivity, necessary conditions for a general graph to be paired k-disjoint path coverable and to be unpaired k-disjoint path coverable were established as follows:

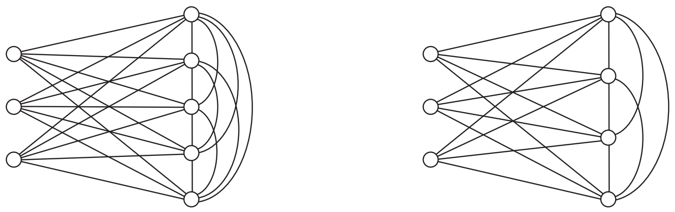
Lemma 11 (Park et al. [28]). *If a graph G of order $n \geq 2k$ is paired k-disjoint path coverable, then $\kappa(G) \geq 2k-1$.*

Lemma 12 (Park et al. [29]). *If a graph G of order $n \geq 2k$ is unpaired k-disjoint path coverable, then $\kappa(G) \geq k$.*

Continuing the approach taken in Sections 3 and 4, one may wonder if the connectivity conditions of the above lemmas can be strengthened to scattering-number conditions, and if the new conditions become sufficient ones on interval graphs. In this section, some partial results on many-to-many disjoint path coverability of an interval graph are given.

6.1. Paired many-to-many disjoint path covers

It has been known that a paired k-disjoint path coverable graph is paired $(k-1)$-disjoint path coverable for $k \geq 2$ [28], implying that a paired 2-disjoint path coverable graph is Hamiltonian-connected (or equivalently, 1-disjoint path coverable). A *complete split graph* is a graph whose vertex set can be partitioned into an independent set and a clique such that every vertex in the



(a) $G_{3,5}$ (b) $G_{3,4}$

Fig. 8: Examples of complete split graphs, where $G_{3,5}$ is paired 3-disjoint path coverable but $G_{3,4}$ is not even paired 2-disjoint path coverable.

independent set is adjacent to every vertex in the clique. We denote by $G_{p,q}$ a complete split graph with an independent set of size p and a clique of size q. Refer to Fig. 8 for examples. Note that $G_{p,q}$ is an interval graph and its scattering number is $p - q$.

Lemma 13. *A complete split graph $G_{3,2k-1}$ is paired k-disjoint path coverable for $k \geq 3$.*

Proof. Let G denote a complete split graph $G_{3,2k-1}$, whose vertex set is partitioned into an independent set I of size 3 and a clique C of size $2k - 1$. It will be shown that any disjoint terminal sets $S = \{s_1, \dots, s_k\}$ and $T = \{t_1, \dots, t_k\}$ are joined by a paired k-DPC. (See Fig. 8(a).) If some source-sink pair, s_i and t_i, are contained in I, then for a nonterminal vertex v in C (i.e., $v \notin S \cup T$), it suffices to let $P_i = (s_i, v, t_i)$ and build a paired $(k - 1)$-DPC joining $S \setminus \{s_i\}$ and $T \setminus \{t_i\}$ in the subgraph $G \setminus V(P_i)$. (A paired $(k - 1)$-DPC exists because the subgraph is isomorphic to a complete graph K_{2k-1}.) Hereafter, let no such source-sink pair exist. If $|I \cap (S \cup T)| \geq 2$, then for some i, j with $i \neq j$ such that $\{s_i, t_i\} \cap I \neq \emptyset$ and $\{s_j, t_j\} \cap I \neq \emptyset$, it suffices to let $P_i = (s_i, t_i)$, $P_j = (s_j, t_j)$ and build a paired $(k-2)$-DPC in the subgraph $G \setminus (V(P_i) \cup V(P_j))$. Finally, suppose $|I \cap (S \cup T)| = 1$. For two nonterminal vertices $v, w \in I$ and two source-sink pairs $s_i, t_i, s_j, t_j \in C$, it suffices to let $P_i = (s_i, v, t_i)$, $P_j = (s_j, w, t_j)$ and build a paired $(k - 2)$-DPC in the subgraph $G \setminus (V(P_i) \cup V(P_j))$. Thus, the lemma is proven. ■

Lemma 13 implies that it is not possible to strengthen the conclusion of Lemma 11 by replacing "$\kappa(G) \geq 2k - 1$" with "$\mathrm{sc}(G) \leq 2 - (2k - 1) = 3 - 2k$" for any $k \geq 3$. Nonetheless, the strengthening is possible for $k = 2$ as shown below.

Lemma 14. *If a graph G of order $n \geq 4$ is paired 2-disjoint path coverable, then $\mathrm{sc}(G) \leq -1$.*

Proof. Assume that G is paired 2-disjoint path coverable. If G is a complete graph, then $\mathrm{sc}(G) = 3 - n \leq -1$ as required, so we further assume that G is a noncomplete graph. Suppose $\mathrm{sc}(G) \geq 0$ for a contradiction. Then, there exists a scattering set X such that $c(G \setminus X) - |X| = \mathrm{sc}(G) \geq 0$; moreover, $|X| \geq 3$ by Lemma 11. Let $H_1, \dots, H_p$ be the connected components of $G \setminus X$, where $p = c(G \setminus X)$. Consider a paired 2-DPC $\{P_1, P_2\}$ that joins disjoint terminal sets $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$ such that $s_1, t_1, s_2 \in X$ and $t_2 \in V(H_1)$, where P_1 runs from s_1 to t_1 and P_2 runs from s_2 to t_2. The path P_1 passes through at most $|V(P_1) \cap X| - 1$ connected components; the path P_2 passes through at most $|V(P_2) \cap X|$ connected components. It follows that the two paths collectively pass through at most $|V(P_1) \cap X| + |V(P_2) \cap X| - 1 = |X| - 1$ connected components, leading to $|X| - 1 \geq p$. This contradicts the hypothesis $p - |X| = \mathrm{sc}(G) \geq 0$, completing the proof. ■

The condition $\mathrm{sc}(G) \leq -1$ is not sufficient, even for interval graphs G to be paired 2-disjoint path coverable, because the graph $G_{3,4}$ shown in Fig. 8(b) is not paired 2-disjoint path coverable. (No paired 2-DPC joining S and T exists in $G_{3,4}$ if $S \cup T$ forms a clique and $V(G) \setminus (S \cup T)$ forms an independent set.)

6.2. Unpaired many-to-many disjoint path covers

It is worth mentioning that for $k \geq 2$, not every unpaired k-disjoint path coverable graph is unpaired $(k-1)$-disjoint path coverable [29]. In fact, it was proven in [29] that a complete bipartite graph $K_{m,m}$, $m \geq 2$, is unpaired m-disjoint path coverable but not unpaired k-disjoint path coverable for any $1 \leq k < m$. A complete bipartite graph $K_{m,m}$ is a spanning subgraph of a complete split graph $G_{m,m}$, so we have that:

Lemma 15. *A complete split graph $G_{k,k}$, $k \geq 1$, is unpaired k-disjoint path coverable.*

Lemma 15 leads to the fact that replacing the conclusion "$\kappa(G) \geq k$" of Lemma 12 with "$\mathrm{sc}(G) \leq 2-k$" is not possible for any $k \geq 3$, even if G is restricted to an interval graph. Nevertheless, we can derive a necessary condition "$\mathrm{sc}(G) \leq 1-k$" stronger than "$\mathrm{sc}(G) \leq 2-k$" for large $n \geq \lfloor \frac{k}{2} \rfloor \lceil \frac{k}{2} \rceil + 2k$ as follows:

Theorem 9. *Let G be a graph of order $n \geq \lfloor \frac{k}{2} \rfloor \lceil \frac{k}{2} \rceil + 2k$ for $k \geq 2$. If G is unpaired k-disjoint path coverable, then* $\mathrm{sc}(G) \leq 1-k$.

Proof. Assume that G is unpaired k-disjoint path coverable. If G is a complete graph, then $\mathrm{sc}(G) = 3-n \leq 3 - (\lfloor \frac{k}{2} \rfloor \lceil \frac{k}{2} \rceil + 2k) \leq 3-(1+2k) \leq 1-k$, as required. It is further assumed that G is a noncomplete graph. Suppose $\mathrm{sc}(G) \geq 2-k$ for a contradiction. Then, there exists a scattering set X such that $c(G \setminus X) - |X| = \mathrm{sc}(G) \geq 2-k$; moreover $|X| \geq k$ because G is k-connected by Lemma 12. Let $H_1, \ldots, H_p$ be the connected components of $G \setminus X$ such that $|V(H_1)| \geq \cdots \geq |V(H_p)|$, where $p = c(G \setminus X) \geq |X| + 2 - k$.

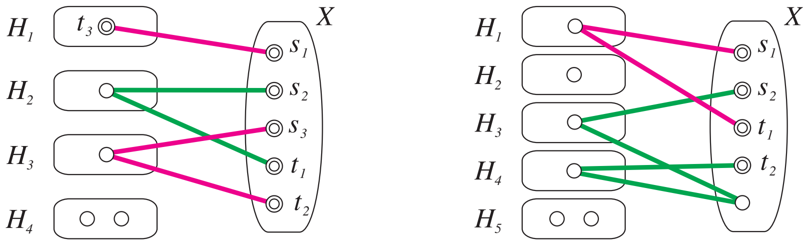
Case 1: $k \leq |X| \leq 2k$. Let $|X| = k+r$ for some $0 \leq r \leq k$. We first show in the following claim that there exist source and sink sets, S and T, that admit $r+1$ connected components each of which is disjoint to $S \cup T$.

Claim. The $p-r-1$ big components $H_1, \ldots, H_{p-r-1}$ contain at least $k-r$ vertices in total.

Proof of claim. The total number of vertices contained in the $p-r-1$ big components $H_1, \ldots, H_{p-r-1}$ is

$$\sum_{i=1}^{p-r-1} |V(H_i)| \geq \left\lceil (n-k-r) \cdot \frac{p-r-1}{p} \right\rceil \geq \left\lceil (n-k-r) \cdot \frac{1}{r+2} \right\rceil$$

because $\frac{p-r-1}{p}$ over all $p \geq r+2$ has the minimum value when $p = r+2$. It suffices to show that $\left\lceil (n-k-r) \cdot \frac{1}{r+2} \right\rceil \geq k-r$. Toward a contradiction, suppose $\left\lceil (n-k-r) \cdot \frac{1}{r+2} \right\rceil < k-r$, or



(a) Case 1 (b) Case 2

Fig. 9: Illustrations of the proof of Theorem 9.

equivalently, $(n-k-r)\cdot\frac{1}{r+2} \le k-r-1$. It follows that

$$
\begin{aligned}
n &\le (r+2)(k-r-1)+k+r \\
&= -r^2+(k-2)r+3k-2 \\
&= -\left(r-\frac{k-2}{2}\right)^2+\frac{k^2}{4}+2k-1 \\
&\le \begin{cases} \frac{k^2}{4}+2k-1 & = \lfloor\frac{k}{2}\rfloor\lceil\frac{k}{2}\rceil+2k-1 \quad \text{for even} \quad k, \\ -\frac{1}{4}+\frac{k^2}{4}+2k-1 & = \lfloor\frac{k}{2}\rfloor\lceil\frac{k}{2}\rceil+2k-1 \quad \text{for odd} \quad k, \end{cases}
\end{aligned}
$$

which is a contradiction to the hypothesis $n \ge \lfloor\frac{k}{2}\rfloor\lceil\frac{k}{2}\rceil+2k$. Thus, the claim is proven. □

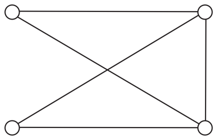
Consider an unpaired k-DPC $\{P_1,\dots,P_k\}$ that joins disjoint terminal sets $S=\{s_1,\dots,s_k\}$ and $T=\{t_1,\dots,t_k\}$ such that $S\subseteq X$, $X\subseteq S\cup T$, and $T\subseteq X\cup\bigcup_{i=1}^{p-r-1}V(H_i)$, as illustrated in Fig. 9(a). Note that $|S\cap X|=k$, $|T\cap X|=r$, and $V(H_i)\cap(S\cup T)=\emptyset$ for all $i\in\{p-r,\dots,p\}$. A DPC path P_i that runs to a sink in X passes through at most one among the $r+1$ components $H_{p-r},\dots,H_p$, whereas a DPC path P_j that runs to a sink in $\bigcup_{i=1}^{p-r-1}V(H_i)$ passes through no component among $H_{p-r},\dots,H_p$. It follows that the k paths in the DPC altogether pass through at most r components among the $r+1$ components $H_{p-r},\dots,H_p$. This is a contradiction to the fact that $\{P_1,\dots,P_k\}$ is an unpaired k-DPC joining S and T, completing the proof for Case 1.

Case 2: $|X|>2k$. Let S and T be disjoint terminal sets of size k each such that $S\cup T\subset X$, as illustrated in Fig. 9(b). Consider an unpaired k-DPC joining S and T, in which each path P_i passes through at most $|V(P_i)\cap X|-1$ connected components. The DPC paths pass through at most $|X|-k$ connected components in total, leading to $|X|-k\ge p$. This contradicts the hypothesis $p-|X|=\mathrm{sc}(G)\ge 2-k$, thereby completing the entire proof. ■

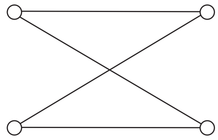
Similar to the paired DPC case, we can strengthen the conclusion of Lemma 12 for $k=2$ as shown in Corollary 3. The converse of the corollary does not hold true by Theorem 9 (even if G is restricted to an interval graph).

Corollary 3. *If a graph G of order $n\ge 4$ is unpaired 2-disjoint path coverable, then* $\mathrm{sc}(G)\le 0$.

Proof. The corollary follows from Theorem 9 for $n\ge 5$, so let $n=4$. Assume that G is unpaired 2-disjoint path coverable. Then, $\kappa(G)\ge 2$ by Lemma 12. It follows that G is isomorphic to K_4, $G_{2,2}$, or $K_{2,2}$, leading to $\mathrm{sc}(G)\le 0$. ■



(a) $G_{2,2}$



(b) $K_{2,2}$

Fig. 10: Two graphs that are unpaired 2-disjoint path coverable but not Hamiltonian-connected.

Almost every unpaired 2-disjoint path coverable graph (except the two shown in Fig. 10) is Hamiltonian-connected as shown below.

Theorem 10. *If a graph G of order $n \geq 4$ is unpaired 2-disjoint path coverable, then G is Hamiltonian-connected or isomorphic to one of the two graphs, $G_{2,2}$ and $K_{2,2}$.*

Proof. Assume that G is an unpaired 2-disjoint path coverable graph. A complete graph is obviously Hamiltonian-connected, so we further assume that G is noncomplete. Let s and t be distinct vertices of G. If $n \geq 5$, then there exist distinct vertices $u, v \in V(G) \setminus \{s, t\}$ such that $(u, v) \in E(G)$. (Suppose otherwise, $\{s, t\}$ would be a vertex cut of G, leading to $\mathrm{sc}(G) \geq (n-2) - 2 = n - 4 \geq 1$, which contradicts the conclusion of Corollary 3.) Then, a Hamiltonian path between s and t can be built by combining the two paths in an unpaired 2-DPC joining $\{s, t\}$ and $\{u, v\}$ through the edge (u, v). Finally, if $n = 4$, then G must be 2-connected by Lemma 12. Among the graphs of order 4 other than K_4, there are only two graphs that are 2-connected, shown in Fig. 10. The two are unpaired 2-disjoint path coverable and are not Hamiltonian-connected. Thus, the theorem is proven. ■

It is open to characterize interval graphs that are paired k-disjoint path coverable and those that are unpaired k-disjoint path coverable.

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References

[1] S. R. Arikati, C. P. Rangan, Linear algorithm for optimal path cover problem on interval graphs, Information Processing Letters 35 (3) (1990) 149–153.

[2] K. Asdre, S. D. Nikolopoulos, The 1-fixed-endpoint path cover problem is polynomial on interval graphs, Algorithmica 58 (3) (2010) 679–710.

[3] J. A. Bondy, U. S. R. Murty, Graph Theory, 2nd printing, 2008.

[4] H. Broersma, J. Fiala, P. A. Golovach, T. Kaiser, D. Paulusma, A. Proskurowski, Linear-time algorithms for scattering number and Hamilton-connectivity of interval graphs, Journal of Graph Theory 79 (4) (2015) 282–299.

[5] C. Chen, C.-C. Chang, Connected proper interval graphs and the guard problem in spiral polygons, in: Combinatorics and Computer Science, Springer, 39–47, 1996.

[6] X.-B. Chen, Paired many-to-many disjoint path covers of the hypercubes, Information Sciences 236 (2013) 218–223.

[7] P. Damaschke, Paths in interval graphs and circular arc graphs, Discrete Mathematics 112 (1-3) (1993) 49–64.

[8] A. M. Dean, The computational complexity of deciding Hamiltonian-connectedness, Congressus Numerantium 93 (1993) 209–214.
[9] J. S. Deogun, D. Kratsch, G. Steiner, 1-Tough cocomparability graphs are Hamiltonian, Discrete Mathematics 170 (1) (1997) 99–106.
[10] T. Dvořák, P. Gregor, Partitions of faulty hypercubes into paths with prescribed endvertices, SIAM Journal on Discrete Mathematics 22 (4) (2008) 1448–1461.
[11] M. R. Garey, D. S. Johnson, Computers and Intractability: A Guide to the Theory of NP-completeness, 1979.
[12] P. C. Gilmore, A. J. Hoffman, A characterization of comparability graphs and of interval graphs, Canadian Journal of Mathematics 16 (1964) 539–548.
[13] R.-W. Hung, M.-S. Chang, Linear-time certifying algorithms for the path cover and Hamiltonian cycle problems on interval graphs, Applied Mathematics Letters 24 (5) (2011) 648–652.
[14] I. Ihm, J.-H. Park, A linear-time algorithm for finding a paired 2-disjoint path cover in the cube of a connected graph, Discrete Applied Mathematics 218 (2017) 98–112.
[15] S. Jo, J.-H. Park, K.-Y. Chwa, Paired many-to-many disjoint path covers in faulty hypercubes, Theoretical Computer Science 513 (2013) 1–24.
[16] H. A. Jung, On a class of posets and the corresponding comparability graphs, Journal of Combinatorial Theory, Series B 24 (2) (1978) 125–133.
[17] J. M. Keil, Finding Hamiltonian circuits in interval graphs, Information Processing Letters 20 (4) (1985) 201–206.
[18] S.-Y. Kim, J.-H. Lee, J.-H. Park, Disjoint path covers in recursive circulants $G(2^m, 4)$ with faulty elements, Theoretical Computer Science 412 (35) (2011) 4636–4649.
[19] S.-Y. Kim, J.-H. Park, Paired many-to-many disjoint path covers in recursive circulants $G(2^m, 4)$, IEEE Transactions on Computers 62 (12) (2013) 2468–2475.
[20] J.-H. Lee, J.-H. Park, General-demand disjoint path covers in a graph with faulty elements, International Journal of Computer Mathematics 89 (5) (2012) 606–617.
[21] P. Li, Y. Wu, Spanning connectedness and Hamiltonian thickness of graphs and interval graphs, Discrete Mathematics and Theoretical Computer Science 16 (2) (2015) 125–210.
[22] P. Li, Y. Wu, A linear time algorithm for the 1-fixed-endpoint path cover problem on interval graphs, SIAM Journal on Discrete Mathematics 31 (1) (2017) 210–239.
[23] H.-S. Lim, H.-C. Kim, J.-H. Park, Ore-type degree conditions for disjoint path covers in simple graphs, Discrete Mathematics 339 (2) (2016) 770–779.
[24] S. C. Ntafos, S. L. Hakimi, On path cover problems in digraphs and applications to program testing, IEEE Transactions on Software Engineering 5 (5) (1979) 520–529.
[25] J.-H. Park, Conditions for disjoint path coverability in proper interval graphs, Journal of KIISE: Computer Systems and Theory 34 (10) (2007) 539–554.
[26] J.-H. Park, J. Choi, H.-S. Lim, Algorithms for finding disjoint path covers in unit interval graphs, Discrete Applied Mathematics 205 (2016) 132–149.
[27] J.-H. Park, I. Ihm, Many-to-many two-disjoint path covers in cylindrical and toroidal grids, Discrete Applied Mathematics 185 (2015) 168–191.
[28] J.-H. Park, H.-C. Kim, H.-S. Lim, Many-to-many disjoint path covers in hypercube-like interconnection networks with faulty elements, IEEE Transactions on Parallel and Distributed Systems 17 (3) (2006) 227–240.
[29] J.-H. Park, H.-C. Kim, H.-S. Lim, Many-to-many disjoint path covers in the presence of faulty elements, IEEE Transactions on Computers 58 (4) (2009) 528–540.
[30] S. Zhang, Z. Wang, Scattering number in graphs, Networks 37 (2) (2001) 102–106.