Disjoint Path Covers with Path Length Constraints in Restricted Hypercube-Like Graphs

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Abstract

A *disjoint path cover* of a graph is a set of pairwise vertex-disjoint paths that altogether cover every vertex of the graph. In this paper, we prove that given k sources, $s_1, \ldots, s_k$, in an m-dimensional restricted hypercube-like graph with a set F of faults (vertices and/or edges), associated with k positive integers, $l_1, \ldots, l_k$, whose sum is equal to the number of fault-free vertices, there exists a disjoint path cover composed of k fault-free paths, each of whose paths starts at s_i and contains l_i vertices for $i \in \{1, \ldots, k\}$, provided $|F| + k \leq m - 1$. The bound, $m - 1$, on $|F| + k$ is the best possible.

Keywords: Hypercube-like graph, disjoint path cover, path partition, prescribed sources, prescribed path lengths, fault tolerance, interconnection network.

1. Introduction

One of the central issues in the study of interconnection networks is the detection of parallel paths, which is naturally related to routing among nodes and fault tolerance of the network [18, 28]. An interconnection network is frequently modeled as a graph, where vertices and edges represent the nodes and communication links of the network, respectively. Parallel paths correspond to pairwise disjoint paths of the graph. Disjoint path is, moreover, a fundamental notion from which many graph properties can be deduced [3, 28].

A *Disjoint Path Cover* (*DPC* for short) of a graph is a set of pairwise disjoint paths that altogether cover every vertex of the graph. The disjoint path cover problem has applications in many areas such as software testing, database design, and code optimization [2, 31]. In addition, the problem is concerned with applications where the full utilization of network nodes is important [37]. The disjoint path covers, with or without additional constraints on the paths, have been the subject of research for several decades. They can be categorized as several types, as discussed in the following, according to whether the terminals (sources and sinks) or the lengths of paths are prescribed or not. A path runs from its *source* to its *sink*.

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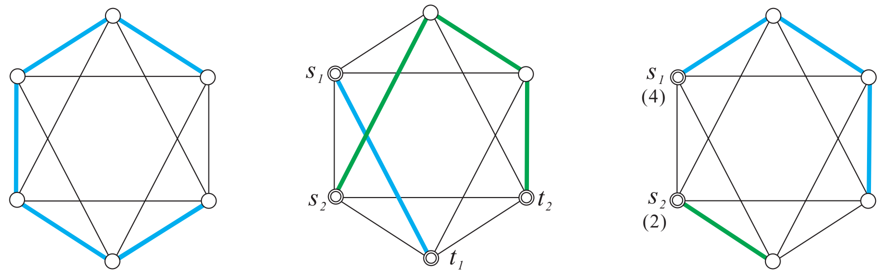


(a) An unconstrained DPC

(b) A DPC with prescribed sources and sinks

(c) A DPC with prescribed sources and path lengths

Fig. 1: Three types of disjoint path covers.

Unconstrained DPCs. No prescribed terminals or lengths of paths are given in this type. The problem is to determine a disjoint path cover of a graph that uses the minimum number of paths. Fig. 1(a) shows an example of unconstrained DPCs. The minimum number is called the *path cover number* of the graph. Obviously, the path cover number of a graph is equal to one if and only if the graph contains a Hamiltonian path. Every hypercube-like graph [43] has a Hamiltonian path, and thus its path cover number is one. For more discussion on the disjoint path covers of this type, refer to, for example, [1, 19, 27, 31, 42].

DPCs with prescribed sources and sinks. Given pairwise disjoint terminal sets $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ of a graph, each representing k *sources* and *sinks*, the *many-to-many k-DPC* is a disjoint path cover each of whose paths joins a pair of source and sink. An example is shown in Fig. 1(b). There are two simpler variants: The *one-to-many k-DPC* for $S = \{s\}$ and $T = \{t_1, \ldots, t_k\}$ is a disjoint path cover made of k paths, each joining a pair of source s and sink t_i, $i \in \{1, \ldots, k\}$. The *one-to-one k-DPC* for $S = \{s\}$ and $T = \{t\}$ is a disjoint path cover in which each path joins an identical pair of source s and sink t. The disjoint path covers of this type have been studied for graphs such as hypercubes [5, 6, 9, 10, 16, 21], recursive circulants [22, 23], hypercube-like graphs [20, 24, 25, 37, 38], cubes of connected graphs [34], k-ary n-cubes [40, 46], grid graphs [35], unit interval graphs [32], simple graphs [26], and DCell networks [44].

DPCs with prescribed sources and path lengths. Consider a given set of k sources, $S = \{s_1, \ldots, s_k\}$, in a graph G, associated with k positive integers, $l_1, \ldots, l_k$, such that $\sum_{i=1}^{k} l_i = |V(G)|$, where $V(G)$ denotes the vertex set of G. A *prescribed-source-and-length k-DPC* of G is a disjoint path cover composed of k paths, each of which is an s_i-path of length l_i for $i \in \{1, \ldots, k\}$, where s_i-path is a path starting at source s_i. See Fig. 1(c) for an example. Here, the *length* of a path refers to the number of vertices in the path. This type of disjoint path covers will be mainly discussed in this paper. From now on, a disjoint path cover whose type is not specified is assumed to be a prescribed-source-and-length type.

The problem of determining whether there exists a k-DPC in a general graph is NP-complete for any fixed $k \geq 1$, which can be reduced from the well-known HAMILTONIAN PATH problem in [15]. Research on prescribed-source-and-length type is relatively scarce, although some results for hypercubes can be found in [7, 30]. Nebeský [30] proved that there exists a k-DPC in an m-dimensional hypercube, $m \geq 4$, for any set of k sources, $S = \{s_1, \ldots, s_k\}$, associated with k

positive *even* integers, $l_1, \ldots, l_k$, whose sum is equal to 2^m, subject to $k \leq m-1$. More general results established by Choudum et al. can be found in [7].

In this paper, we study the disjoint path cover problem for the class of Restricted Hypercube-Like graphs (RHL graphs for short) [36], which are a subset of nonbipartite hypercube-like graphs that have received considerable attention over recent decades. The class includes most nonbipartite hypercube-like networks found in the literature; for example, twisted cubes [17], crossed cubes [12], Möbius cubes [8], recursive circulants $G(2^m, 4)$ with odd m [33], multiply twisted cubes [11], Mcubes [41], and generalized twisted cubes [4]. The definition of RHL graphs is deferred to the next section (Definition 4). An m-dimensional RHL graph has 2^m vertices of degree m.

As node and/or link failure is inevitable in a large network, fault tolerance is essential to the network performance. Throughout this paper, let F denote a set of faults (vertices and/or edges) corresponding to the set of node and/or link failures. When a graph contains faulty elements, its k-disjoint path cover naturally means a k-disjoint path cover of the graph, $G \setminus F$, with the faults deleted.

Definition 1. A graph G is called *k-path partitionable* if G has a k-DPC for any set of k sources, $s_1, \ldots, s_k$, associated with any k positive integers, $l_1, \ldots, l_k$, such that $\sum_{i=1}^{k} l_i = |V(G)|$. A graph G is said to be *f_b-fault k-path partitionable* if $G \setminus F$ is k-path partitionable for any fault set F with $|F| \leq f_b$.

Our main result on the construction of disjoint path covers in RHL graphs can be stated as follows:

Theorem 1. *Every m-dimensional RHL graph is f_b-fault k-path partitionable for any $f_b \geq 0$ and $k \geq 1$ subject to $f_b + k \leq m-1$, where $m \geq 3$.*

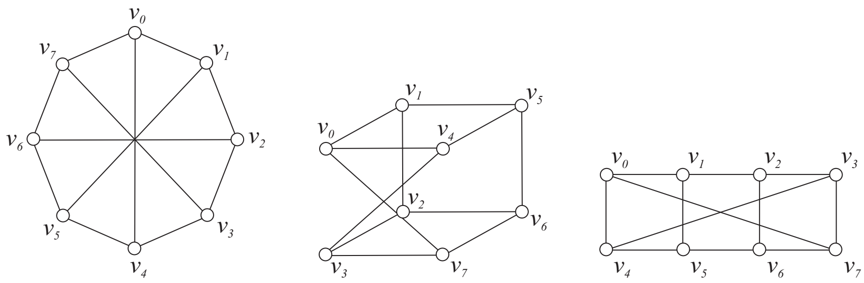
Note that the bound, $m-1$, on $f_b + k$ in Theorem 1 is the maximum possible. (Suppose otherwise, no m-dimensional RHL graph would have a k-DPC when, for some vertex $v \in V(G) \setminus (S \cup F)$, its $m-k$ neighbors are vertex faults whereas its remaining k neighbors are sources, s_1 through s_k, with $l_i \neq 2$ for all i.)

In fact, we will give a proof for a stronger result, stated in Theorem 2 below for $m \geq 4$, than Theorem 1 asserts, which states that each s_i-path in a k-DPC is not allowed to have a sink contained in a given vertex subset W_i of size $|W_i| \leq \delta(G) - f_b - k$, where $\delta(G)$ denotes the minimum degree of a graph G. Note that the bound on $|W_i|$ is set to the maximum possible value. (Suppose otherwise, there would exist no k-DPC for the case when the degree of s_1 is equal to $\delta(G)$, $l_1 = 2$, and every neighbor of s_1 is contained in $(S \setminus \{s_1\}) \cup F \cup W_1$.)

Definition 2. A graph G is called *f_b-fault k-path partitionable in a strong sense* if G is f_b-fault k-path partitionable and, moreover, for any given k subsets W_i of $V(G)$, $i \in \{1, \ldots, k\}$, such that (i) $|W_i| \leq \delta(G) - f_b - k$ and (ii) $s_i \notin W_i$ whenever $l_i = 1$, there exists a k-DPC in $G \setminus F$ each of whose paths, P_i, has length l_i and runs from s_i to a sink not contained in W_i for $i \in \{1, \ldots, k\}$.

Theorem 2. *Every m-dimensional RHL graph is f_b-fault k-path partitionable in a strong sense for any $f_b \geq 0$ and $k \geq 1$ subject to $f_b + k \leq m-1$, where $m \geq 4$.*

The remainder of this paper is organized as follows: Section 2 presents some preliminaries. For the proof of Theorem 2, four basic procedures for constructing k-DPCs are developed in Section 3. A few exceptional cases not covered by the basic procedures are dealt with in Section 4. Finally, the paper is concluded in Section 5.



(a) $G(8,4)$ (b) TQ_3 (c) Möbius ladder

Fig. 2: The 3-dimensional RHL graph.

2. Preliminaries

We denote by k-DPC$[\{(s_1, l_1, W_1), \ldots, (s_k, l_k, W_k)\} \,|\, G, F]$ a k-DPC for given s_i, l_i, and W_i, $i \in \{1, \ldots, k\}$, in a graph G with fault set F. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively. Throughout this paper, a path in a graph is represented as a sequence of vertices, $(v_1, \ldots, v_n)$, where $(v_i, v_{i+1}) \in E(G)$ for every $i \in \{1, \ldots, n-1\}$. An *s-path* refers to a path starting at a vertex s; an *s–t path* refers to a path from s to t.

Many interconnection networks can be expanded into higher dimensional networks by connecting two lower dimensional networks. Given a bijection ϕ from $V(G_0)$ to $V(G_1)$ for two graphs G_0 and G_1 containing the same number of vertices, we denote by $G_0 \oplus_\phi G_1$ a new graph with vertex set $V(G_0) \cup V(G_1)$ and edge set $E(G_0) \cup E(G_1) \cup \{(v, \phi(v)) : v \in V(G_0)\}$. Here, G_0 and G_1 are called the *components* of $G_0 \oplus_\phi G_1$, where every vertex v in one component has a unique neighbor, denoted by $\bar{v}$, in the other one. To simplify the notation, we often omit the bijection ϕ from $\oplus_\phi$.

In 1993, Vaidya et al. [43] introduced the hypercube-like graphs, which are defined recursively with a graph operator $\oplus_\phi$ as follows:

Definition 3 (Vaidya et al. [43])**.** A graph that belongs to HL_m is called an *m-dimensional hypercube-like graph*, where

- $HL_0 = \{K_1\}$, and
- $HL_m = \{G_0 \oplus_\phi G_1 : G_0, G_1 \in HL_{m-1}, \phi$ is a bijection from $V(G_0)$ to $V(G_1)\}$ for $m \geq 1$.

Then, $HL_1 = \{K_2\}$; $HL_2 = \{Q_2\}$; $HL_3 = \{Q_3, G(8,4)\}$. Here, K_n is a complete graph with n vertices, Q_m is an m-dimensional hypercube, and $G(8,4)$ is a recursive circulant, shown in Fig. 2(a), that has a vertex set $\{v_0, \ldots, v_7\}$ and an edge set $\{(v_i, v_j) : i + 1 \text{ or } i + 4 \equiv j \pmod 8\}$. The graph $G(8,4)$ is isomorphic to a 3-dimensional twisted cube TQ_3, and also isomorphic to a Möbius ladder with four spokes [29], as shown in Fig. 2. The hypercube-like graphs are sometimes referred to as bijective connection networks [13, 14, 45].

RHL graphs, which are a subset of nonbipartite hypercube-like graphs, are defined as follows:

Definition 4. (See [36].) A graph that belongs to RHL_m is called an *m-dimensional RHL graph*, where

- $RHL_3 = \{G(8,4)\}$, and

- $RHL_m = \{G_0 \oplus_\phi G_1 : G_0, G_1 \in RHL_{m-1}, \phi$ is a bijection from $V(G_0)$ to $V(G_1)\}$ for $m \geq 4$.

Lemma 1. *(See [24].) Let G be an m-dimensional RHL graph, where* $m \geq 3$.
(a) G has 2^m *vertices of degree m. Moreover, it is nonbipartite.*
(b) G has no triangle (cycle of length three).
(c) There are at most two common neighbors for any pair of vertices in G.

For the proof of our theorems, we will utilize the fault-Hamiltonicity of RHL graphs. A graph G is said to be f_b-*fault Hamiltonian* (resp. f_b-*fault Hamiltonian-connected*) if there exists a Hamiltonian cycle (resp. if each pair of vertices are joined by a Hamiltonian path) in $G \setminus F$ for any set F of faults (vertices and/or edges) with $|F| \leq f_b$. The fault-Hamiltonicity of RHL graphs was studied by the authors in [36].

Lemma 2. *(See [36].) (a) Every m-dimensional RHL graph,* $m \geq 3$, *is* $(m-3)$-*fault Hamiltonian-connected.*
(b) Every m-dimensional RHL graph, $m \geq 3$, *is* $(m-2)$-*fault Hamiltonian.*

Theorem 2, our stronger result, is concerned with four- or higher-dimensional RHL graphs; however, some interesting properties on disjoint path covers of the 3-dimensional RHL graph, $G(8, 4)$, can be discovered. They will be employed in proving Theorem 2. Firstly, Theorem 1 holds true for $m = 3$, as shown below.

Proof of Theorem 1 for $m = 3$. A 1-DPC can be constructed from a Hamiltonian cycle of $G(8, 4)\setminus F$, where $|F| \leq 1$. A 2-DPC can be extracted from a Hamiltonian s_1–s_2 path of $G(8, 4)\setminus F$, where $F = \emptyset$. The existence of a Hamiltonian path/cycle is due to Lemma 2. □

On the contrary, Theorem 2 for $m = 3$ does not hold true, as shown in Lemma 3. Nevertheless, $G(8, 4)$ still has good DPC properties for $k = 2$, as discussed in Lemmas 3 and 4. The proofs for the two lemmas are deferred to the appendix.

Lemma 3. *There exists a* 2-$DPC[\{(s_1, l_1, W_1), (s_2, l_2, W_2)\} \,|\, G(8, 4), \emptyset]$ *for every pair of triplets* (s_i, l_i, W_i), $i \in \{1, 2\}$, *such that* $|W_i| \leq 1$ *and* $l_1 + l_2 = 8$, *with the unique exception up to symmetry that* $s_1 = v_0$, $s_2 = v_4$, $l_1 = l_2 = 4$, $W_1 = \{v_3\}$, *and* $W_2 = \{v_1\}$.

Lemma 4. *(a) If* $(l_1, l_2) = (6, 2)$ *or* $(5, 3)$, *there exists a* 2-$DPC[\{(s_1, l_1, W_1), (s_2, l_2, \emptyset)\} \,|\, G(8, 4), \emptyset]$ *for any* s_1, s_2, *and* W_1 *with* $|W_1| \leq 3$.
(b) There exists a 2-$DPC[\{(s_1, 3, W_1), (s_2, 5, \emptyset)\} \,|\, G(8, 4), \emptyset]$ *for any* s_1, s_2, *and* W_1 *with* $|W_1| \leq 2$.

Consider a fault set F, set of k sources $S = \{s_1, \ldots, s_k\}$, and k vertex subsets $W_1, \ldots, W_k$ in an m-dimensional RHL graph $G_0 \oplus G_1$, where G_0 and G_1 are $(m-1)$-dimensional RHL graphs. We denote by F_i the fault set in G_i, $i \in \{0, 1\}$, and by F_2 the set of edge faults between G_0 and G_1. Then, $F = F_0 \cup F_1 \cup F_2$. Let $f = |F|$, $f_0 = |F_0|$, $f_1 = |F_1|$, and $f_2 = |F_2|$, so that $f = f_0 + f_1 + f_2$. Similarly, S_0 and S_1 denote the sets of sources contained in G_0 and G_1, respectively, so that $S = S_0 \cup S_1$. Let $k_0 = |S_0|$ and $k_1 = |S_1|$, so that $k_0 + k_1 = k$. Let I_0 and I_1 be the index sets such that $I_0 = \{j : s_j \in S_0\}$ and $I_1 = \{j : s_j \in S_1\}$, and let $I = I_0 \cup I_1$. For $j \in I$, we let $W_j^0 = W_j \cap V(G_0)$ and $W_j^1 = W_j \cap V(G_1)$.

Remark. Theorem 1 (and Theorem 2) cannot be extended directly for the hypercube-like graphs. This is because there exists a bipartite hypercube-like graph, which is equitable, that is, the two parts of the bipartition have the same number of vertices. If the sources $s_1, \ldots, s_k$ are all contained in one part with some l_i being odd, no k-DPC could exist in the graph. The same is true of the "near" bipartite hypercube-like graphs G, because $G \setminus F$ would be a (not necessarily equitable) bipartite graph for some fault set F of small size.

3. Proof of Theorem 2

In this section, we prove Theorem 2 by induction on m; however, some special cases of the proof will be discussed in the following section. Let $G_0 \oplus G_1$ be an m-dimensional RHL graph, $m \geq 4$, where G_0 and G_1 are $(m-1)$-dimensional RHL graphs. Given a fault set F and k triplets (s_i, l_i, W_i) for $i \in I$ subject to

1. $f + k \leq m - 1$ and
2. $f + k + |W_i| \leq m$ for every $i \in I$,

a k-DPC$[\{(s_i, l_i, W_i) : i \in I\} \,|\, G_0 \oplus G_1, F]$ will be constructed. It is implicit in our discussion that $k \geq 1$, $\sum_{i \in I} l_i$ is equal to the number of fault-free vertices of $G_0 \oplus G_1$, And, moreover, $s_i \notin W_i$ whenever $l_i = 1$.

First, we assert in the following lemma that, for the proof of Theorem 2, it suffices to consider the case where F contains edge faults only, i.e., $F \subseteq E(G_0 \oplus G_1)$.

Lemma 5. *If there exists a k-DPC$[\{(s_i, l_i, W_i) : i \in I\} \,|\, G_0 \oplus G_1, F']$ for any fault set F' with $F' \subseteq E(G_0 \oplus G_1)$ and k triplets (s_i, l_i, W_i) subject to the above two conditions, then there exists a k-DPC$[\{(s_i, l_i, W_i) : i \in I\} \,|\, G_0 \oplus G_1, F]$ for any fault set F and k triplets (s_i, l_i, W_i) subject to the above two conditions.*

Proof. Let F contain f_v vertex faults, $\{v_1, \ldots, v_{f_v}\}$. We will show that the desired k-DPC can be obtained from some $(k + f_v)$-DPC as follows: If we regard each vertex fault v_j as a *virtual source*, say s_{k+j}, with $l_{k+j} = 1$ and $W_{k+j} = \emptyset$, then the above two conditions are also satisfied. Precisely speaking, for $S' := S \cup \{s_{k+1}, \ldots, s_{k+f_v}\}$ and $F' := F \setminus \{v_1, \ldots, v_{f_v}\}$ with $s_{k+j} = v_j$ for $j \in \{1, \ldots, f_v\}$, it holds that (i) $|F'| + |S'| = |F| + |S| \leq m - 1$ and (ii) $|F'| + |S'| + |W_i| \leq m$ for every $i \in I \cup \{k+1, \ldots, k+f_v\}$. Then, there exists a $(k + f_v)$-DPC for F' and the $k + f_v$ triplets by the hypothesis of this lemma. Note that F' contains edge faults only. Removing the s_{k+j}-paths for all $j \in \{1, \ldots, f_v\}$ from the $(k + f_v)$-DPC results in the desired k-DPC, completing the proof. □

The following lemma dealing with the base case of $m = 4$ is verified using computer programs that exhaustively search for DPCs, mostly on the basis of a variation of depth-first search. The source codes can be downloaded from `http://tcs.catholic.ac.kr/~jhpark/papers/verifying_Lemma_6.zip`.

Lemma 6. *Every 4-dimensional RHL graph is f_b-fault k-path partitionable in a strong sense for any $f_b \geq 0$ and $k \geq 1$ subject to $f_b + k \leq 3$.*

In the remainder of this section, let $m \geq 5$. The induction hypothesis states that G_0 and G_1, which are $(m-1)$-dimensional RHL graphs, are f'_b-fault k'-path partitionable in a strong sense for any $f'_b \geq 0$ and $k' \geq 1$ subject to $f'_b + k' \leq (m-1) - 1 = m - 2$. Without loss of generality, we assume that:

- $I_0 = \{1, \ldots, k_0\}$, $I_1 = \{k_0 + 1, \ldots, k_0 + k_1\}$;
- $l_1 \geq \cdots \geq l_{k_0}$ and $l_{k_0+1} \geq \cdots \geq l_{k_0+k_1}$;
- $W_i = \emptyset$ if $l_i = 1$;
- $W_i \cap S = \emptyset$ for every $i \in I$;
- F contains no vertex faults;
- $L_0 \geq 2^{m-1}$ and $L_1 \leq 2^{m-1}$, where $L_0 = \sum_{i \in I_0} l_i$ and $L_1 = \sum_{j \in I_1} l_j$.

For our proof, two easy cases are considered first: the case of a unique source, i.e., $k = 1$ in Lemma 7 and the case of L_0 being equal to the number of vertices of G_0, 2^{m-1}, in Lemma 8.

Lemma 7. *If $k = 1$, there exists a k-DPC$[\{(s_i, l_i, W_i) : i \in I\} \,|\, G_0 \oplus G_1, F]$ in $G_0 \oplus G_1$.*

Proof. The proof is an immediate consequence of Lemma 2(a) if $f \leq m-3$, and of Lemma 2(b) if $f = m-2$ ($|W_1| \leq 1$). Notice that from a Hamiltonian cycle, we can extract two Hamiltonian s_1-paths, one of which has a sink that is not contained in W_1. □

From Lemma 7, we further assume

$$k \geq 2, \text{ and thus } f \leq m-3. \tag{1}$$

Lemma 8. *If $L_0 = 2^{m-1}$, then $G_0 \oplus G_1$ has a k-DPC$[\{(s_i, l_i, W_i) : i \in I\} \,|\, G_0 \oplus G_1, F]$.*

Proof. It follows that $L_1 = 2^{m-1}$ and $1 \leq k_0, k_1 < k$. There exists a k_0-DPC$[\{(s_i, l_i, W_i^0) : i \in I_0\} \,|\, G_0, F_0]$ by the induction hypothesis, because (i) $f_0 + k_0 \leq f + k - 1 \leq (m-1) - 1$ and (ii) for each $i \in I_0$, it holds that $f_0 + k_0 + |W_i^0| \leq f + k - 1 + |W_i| \leq m - 1$. Similarly, there exists a k_1-DPC$[\{(s_j, l_j, W_j^1) : j \in I_1\} \,|\, G_1, F_1]$. Then, the union of the k_0-DPC of G_0 and the k_1-DPC of G_1 results in the required k-DPC of $G_0 \oplus G_1$. □

We further assume $L_0 > 2^{m-1}$ or, equivalently,

$$\Delta := L_0 - 2^{m-1} > 0. \tag{2}$$

Now, it is not possible to embed all the s_i-paths for $s_i \in S_0$ wholly within G_0. One natural approach would be as follows: (a) Decompose $(l_1, \ldots, l_{k_0})$ into $(l'_1, \ldots, l'_{k_0}) + (l''_1, \ldots, l''_{k_0})$ subject to $\sum_{i \in I_0} l'_i = 2^{m-1}$ and $1 \leq l'_i \leq l_i$ for every $i \in I_0$. (b) For each $i \in I_0$, embed an s_i-subpath of length l'_i in G_0 and, if $l''_i \geq 1$, embed a subpath of length l''_i in G_1, and then connect the two subpaths into a single s_i-path of length l_i. (c) Embed the s_j-paths for $j \in I_1$ wholly within G_1.

Here, the sink of an s_i-subpath, say x_i, embedded in G_0, if $l''_i \geq 1$, should be a vertex such that $\overline{x_i} \notin S_1$ and $(x_i, \overline{x_i}) \notin F_2$. Moreover, if $l''_i = 1$ (i.e., $l'_i = l_i - 1$), then $\overline{x_i}$ should not be contained in W_i^1. We will try a decomposition of $(l_1, \ldots, l_{k_0})$ such that $l''_i = 0$ or $l''_i \geq 2$ for every $i \in I_0$, which may be possible only if $\Delta \geq 2$, as the condition attached to x_i should be stronger if $l''_i = 1$. In addition, we prefer a decomposition with some $l''_j = 0$, plausible only if $k_0 \geq 2$, which will make it easier to embed the remaining (sub)paths in G_1.

In the remainder of this section, we present four basic procedures for constructing k-DPCs for three cases: (i) $k_0 \geq 2$ and $\Delta \geq 2$; (ii) $k_0 \geq 2$ and $\Delta = 1$; (iii) $k_0 = 1$. For most situations, one of the four procedures will be applicable, leaving a few exceptional cases deferred to Section 4.

3.1. Case when $k_0 \geq 2$ and $\Delta \geq 2$

The aforementioned approach will be taken for this case. There are two basic procedures depending on whether $f_0 + k_0 < f + k$ or not. The first procedure, shown below, insists on the most preferable decomposition so that (i) $l''_i \neq 1$ for every $i \in I_0$ and (ii) $l''_i = 0$ for some $i \in I_0$. This procedure works well unless $k_0 = 2$ and $l_2 \geq 2^{m-1} - 1$, as shown in Lemmas 9 and 10.

Procedure DPC-A$(\{(s_i, l_i, W_i) : i \in I\}$**,** F**,** $G_0 \oplus G_1)$
/* $k_0 \geq 2$, $\Delta \geq 2$, and $f_0 + k_0 < f + k$. See Fig. 3. */

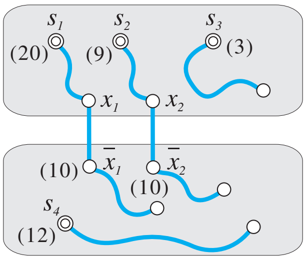


Fig. 3: Illustration of Procedure DPC-A, where $m = 6$, $k = 4$, $(l'_1, l''_1) = (20, 10)$, $(l'_2, l''_2) = (9, 10)$, $l_3 = 3$, and $l_4 = 12$.

1: Decompose $(l_1, \ldots, l_{k_0})$ into $(l'_1, \ldots, l'_{k_0}) + (l''_1, \ldots, l''_{k_0})$ subject to the five conditions:
  A1: $1 \le l'_i \le l_i$ for every $i \in I_0$;
  A2: $\sum_{i \in I_0} l'_i = 2^{m-1}$;
  A3: $l'_{k_0} = l_{k_0}$ $(l''_{k_0} = 0)$;
  A4: if $l_i \le 3$, then $l'_i = l_i$;
  A5: if $l_i \ge 4$, then either $l'_i = l_i$ or $l'_i, l''_i \ge 2$.
  Let $I'_0 = \{i \in I_0 : l'_i < l_i\}$ and $k'_0 = |I'_0|$.
2: Find a k_0-DPC$[\{(s_i, l'_i, W'_i) : i \in I_0\} \,|\, G_0, F_0]$, where

$$W'_i = \begin{cases} \{w \in V(G_0) : \overline{w} \in S_1 \text{ or } (w, \overline{w}) \in F_2\} & \text{if } i \in I'_0, \\ W^0_i & \text{if } i \notin I'_0. \end{cases}$$

  Let x_i denote the sink of an s_i-path in the k_0-DPC.
3: Find a $(k'_0 + k_1)$-DPC$[\{(\overline{x_i}, l''_i, W^1_i) : i \in I'_0\} \cup \{(s_j, l_j, W^1_j) : j \in I_1\} \,|\, G_1, F_1]$.
4: Merge the k_0-DPC of G_0 and the $(k'_0 + k_1)$-DPC of G_1 with edges $(x_i, \overline{x_i})$ for $i \in I'_0$.

Lemma 9. *The decomposition of Step 1 in Procedure DPC-A is possible unless $k_0 = 2$ and $l_2 \ge 2^{m-1} - 1$.*

Proof. We first claim that $l_1 \ge 5$. Suppose otherwise, $L_0 \le 4k_0$ by the assumption that $l_1 \ge l_i$ for every $i \in I_0$. Then, from $L_0 = 2^{m-1} + \Delta$, it follows that

$$2^{m-1} = L_0 - \Delta \le 4k_0 - \Delta \le 4(f + k) - \Delta \le 4(m - 1) - 2,$$

which is invalid for every $m \ge 5$. Thus, the claim is proved.

If $k_0 = 2$, then by the hypothesis of our lemma, $l_2 \le 2^{m-1} - 2$. From the definition of Δ, it follows that

$$l_1 - \Delta = 2^{m-1} - l_2 \ge 2.$$

Thus, the decomposition of

$$(l_1, l_2) = (l_1 - \Delta, l_2) + (\Delta, 0)$$

satisfies the five conditions A1 through A5 of Step 1.

For the case $k_0 \ge 3$, we prove by induction on Δ that the decomposition of Step 1 is possible under the following constraints: **X1**: $\Delta = \sum_{i \in I_0} l_i - 2^{m-1} \ge 2$; **X2**: for some $1 \le r \le k_0 - 1$, it holds that $l_1 \ge \cdots \ge l_r \ge 4$ and $l_{r+1}, \ldots, l_{k_0 - 1} \le 3$; **X3**: $l_1 \ge 5$; **X4**: $l_{k_0} \le \lfloor 2^m / k_0 \rfloor$. Initially, all the four constraints X1 through X4 are satisfied.

If $2 \le \Delta \le l_r - 2$, it suffices to let

$$l'_r = l_r - \Delta \text{ and } l''_r = \Delta;\ l'_i = l_i \text{ and } l''_i = 0 \text{ for every } i \neq r.$$

Clearly, the decomposition satisfies all the five conditions of Step 1, and we are done.

For the remaining case of $\Delta \geq l_r - 1$, we first claim $r \geq 2$. Suppose $r = 1$ for a contradiction. Then,

$$\textstyle\sum_{i\in I_0} l_i \leq l_1 + 3(k_0 - 2) + l_{k_0} \leq (\Delta + 1) + 3(k_0 - 2) + l_{k_0}.$$

From $\sum_{i\in I_0} l_i = 2^{m-1} + \Delta$, it follows that

$$2^{m-1} + \Delta \leq (\Delta + 1) + 3(k_0 - 2) + l_{k_0},$$

or equivalently

$$2^{m-1} \leq l_{k_0} + 3k_0 - 5 \leq \lfloor 2^m/k_0 \rfloor + 2k_0 + k_0 - 5 \leq \lfloor 2^m/k_0 \rfloor + 2k_0 + (m-1) - 5.$$

If $k_0 \geq 4$, then

$$2^{m-1} \leq \lfloor 2^m/4 \rfloor + 2(m-1) + (m-1) - 5 \leq 2^{m-2} + 3m - 8,$$

which is invalid for every $m \geq 5$, leading to a contradiction. If $k_0 = 3$, then

$$2^{m-1} \leq \lfloor 2^m/3 \rfloor + 2 \cdot 3 + (m-1) - 5 \leq 2^m/3 + m,$$

which is also a contradiction, proving the claim.

Now, first suppose that $\Delta = l_r - 1$ ($r \geq 2$). If $l_r = 4$ ($\Delta = 3$), it suffices to let

$$l'_1 = l_1 - \Delta \text{ and } l''_1 = \Delta;\ l'_i = l_i \text{ and } l''_i = 0 \text{ for every } i \neq 1.$$

If $l_r \geq 5$, it suffices to let

$$l'_r = 3 \text{ and } l''_r = l_r - 3;\ l'_{r-1} = l_{r-1} - 2 \text{ and } l''_{r-1} = 2;\ l'_i = l_i \text{ and } l''_i = 0 \text{ for every } i \neq r, r-1.$$

The decompositions obviously satisfy all the five conditions of Step 1, and we are done.

Finally, suppose $\Delta \geq l_r$ ($r \geq 2$). If we let

$$p_r = 2 \text{ and } p_i = l_i \text{ for every } i \neq r,$$

then we have

$$\Delta' := \textstyle\sum_{i\in I_0} p_i - 2^{m-1} = \sum_{i\in I_0} l_i - (l_r - 2) - 2^{m-1} = \Delta - (l_r - 2) \geq 2.$$

It is straightforward to check that the new $(p_1, \ldots, p_{k_0})$ still obeys the constraints X2, X3, and X4, as well as X1. Therefore, by the induction hypothesis, there exists a decomposition

$$(p_1, \ldots, p_{k_0}) = (p'_1, \ldots, p'_{k_0}) + (p''_1, \ldots, p''_{k_0})$$

that satisfies the five conditions A1 through A5. Then,

$$(l_1, \ldots, l_{k_0}) = (p'_1, \ldots, p'_{k_0}) + (l''_1, \ldots, l''_{k_0}),$$

where $l''_r = p''_r + (l_r - 2)$ and $l''_i = p''_i$ for every $i \neq r$. This decomposition obviously satisfies all the five conditions A1 through A5 of Step 1, completing the proof. □

Lemma 10. *When $k_0 \geq 2$, $\Delta \geq 2$, and $f_0 + k_0 < f + k$, Procedure DPC-A constructs a k-DPC unless $k_0 = 2$ and $l_2 \geq 2^{m-1} - 1$.*

Proof. The decomposition of Step 1 in Procedure DPC-A is possible by Lemma 9. The k_0-DPC of Step 2 exists by the induction hypothesis, because (i) $f_0 + k_0 \leq f + k - 1 \leq (m-1) - 1$, (ii) for $i \in I'_0$, it holds that $f_0 + k_0 + |W'_i| \leq f_0 + k_0 + (k_1 + f_2) \leq f + k \leq m - 1$, and (iii) for $i \in I_0 \setminus I'_0$, it also holds that $f_0 + k_0 + |W'_i| \leq f + k - 1 + |W_i| \leq m - 1$. The existence of the $(k'_0 + k_1)$-DPC of Step 3 is due to the fact $k'_0 + k_1 < k$; precisely speaking, because (i) $f_1 + (k'_0 + k_1) \leq f + k - 1 \leq (m-1) - 1$ and (ii) for each $i \in I'_0 \cup I_1$, it holds that $f_1 + (k'_0 + k_1) + |W^1_i| \leq f + k - 1 + |W_i| \leq m - 1$. Therefore, the lemma is proved. □

The exceptional case of Lemma 10, where $k_0 = 2$ and $l_1 \geq l_2 \geq 2^{m-1} - 1$, will be discussed later in Section 4.2.

The second basic procedure shown below considers the case $f_0 + k_0 = f + k$, or equivalently, $f_0 = f$ and $k_0 = k$. Notice that $f_1 = f_2 = 0$, $k_1 = 0$, and $\Delta = 2^{m-1}$. The procedure assumes $f + k \leq m - 2$; the remaining case of $f + k = m - 1$ will be deferred to Section 4.4. Observe that we can apply the induction hypothesis to obtain a k_0-DPC in $G_0 \setminus F_0$ for the k_0 triplets (s_i, l'_i, W'_i) for $i \in I_0$, only if $|W'_i| < m - f - k$ for all i. Let $J = \{i \in I_0 : |W^0_i| < m - f - k\}$. Then, $|W^0_i| = m - f - k$ ($W^1_i = \emptyset$) for every $i \in I_0 \setminus J$.

Procedure DPC-B($\{(s_i, l_i, W_i) : i \in I\}$, F, $G_0 \oplus G_1$)
/* $k_0 \geq 2$, $\Delta \geq 2$, and $f_0 + k_0 = f + k \leq m - 2$. See Fig. 4. */

1: Decompose $(l_1, \ldots, l_{k_0})$ into $(l'_1, \ldots, l'_{k_0}) + (l''_1, \ldots, l''_{k_0})$ subject to the four conditions:
   B1: $1 \leq l'_i \leq l_i$ for every $i \in I_0$;
   B2: $\sum_{i \in I_0} l'_i = 2^{m-1}$;
   B3: $l'_i < l_i$ for every $i \in I_0 \setminus J$;
   B4: $l'_j = l_j$ for some $j \in J$. Moreover, $l'_j \neq l_j - 1$ ($l''_j \neq 1$) for every $j \in J$.
   Let $I'_0 = \{i \in I_0 : l'_i < l_i\}$ and $k'_0 = |I'_0|$.
2: Find a k_0-DPC$[\{(s_i, l'_i, W'_i) : i \in I_0\} \,|\, G_0, F_0]$, where

$$W'_i = \begin{cases} \emptyset & \text{if } i \in I'_0, \\ W^0_i & \text{if } i \notin I'_0. \end{cases}$$

   Let x_i denote the sink of an s_i-path in the k_0-DPC.
3: Find a k'_0-DPC$[\{(\overline{x}_i, l''_i, W^1_i) : i \in I'_0\} \,|\, G_1, F_1]$.
4: Merge the k_0-DPC of G_0 and the k'_0-DPC of G_1 with edges $(x_i, \overline{x}_i)$ for $i \in I'_0$.

Lemma 11. *When $k_0 \geq 2$, $\Delta \geq 2$, and $f_0 + k_0 = f + k \leq m - 2$, Procedure DPC-B constructs a k-DPC if $J \neq \emptyset$ and $l_p \leq 2^{m-1} - (k-1)$, where $p = \arg\min_{j \in J} l_j$.*

Proof. Necessarily in our decomposition, $l'_j = l_j$ for every $j \in J$ such that $l_j = 2$. It is straightforward to see that the decomposition of Step 1 is possible if $J \neq \emptyset$ and

$$\begin{cases} l_p + 2|J_2| + (k - 1 - |J_2|) \leq 2^{m-1} & \text{if } l_p = 1, \\ l_p + 2(|J_2| - 1) + (k - |J_2|) \leq 2^{m-1} & \text{if } l_p = 2, \\ l_p + (k-1) \leq 2^{m-1} & \text{if } l_p \geq 3, \end{cases}$$

where $J_2 = \{j \in J : l_j = 2\}$. Note that $J_2 = \emptyset$ if $l_p \geq 3$, and $l_p = \min_{j \in J} l_j$. The left-hand sides of the inequalities for $l_p \in \{1, 2\}$ are both equal to $k + |J_2|$, where $k + |J_2| \leq 2k \leq 2(m-1) \leq 2^{m-1}$

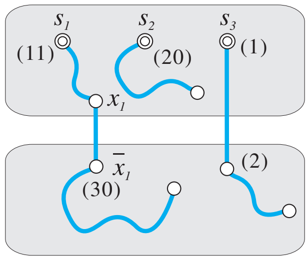


Fig. 4: Illustration of Procedure DPC-B, where $m = 6$, $k = 3$, $J = \{1, 2\}$, $(l'_1, l''_1) = (11, 30)$, $l_2 = 20$, and $(l'_3, l''_3) = (1, 2)$.

for every $m \geq 5$. The inequality for $l_p \geq 3$ is the same as the second precondition of the lemma. Therefore, the three inequalities hold under the hypothesis of our lemma, and the decomposition of Step 1 is possible.

The k_0-DPC of Step 2 exists by the induction hypothesis, because (i) $f_0 + k_0 = f + k \leq m-2 = (m-1)-1$ and (ii) for every $i \in I_0$, it holds that $f_0 + k_0 + |W'_i| \leq f + k + (m - f - k - 1) \leq m - 1$. Note that $l''_i = 1$ only if $i \in I_0 \setminus J$. Thus, $\overline{x_i} \notin W^1_i$ $(= \emptyset)$ whenever $l''_i = 1$. Similarly, the k'_0-DPC of Step 3 also exists, because (i) $f_1 + k'_0 \leq k - 1 \leq (m-1) - 1$ and (ii) for every $i \in I'_0$, it holds that $f_1 + k'_0 + |W^1_i| \leq (k-1) + (m - f - k) \leq m - 1$. Thus, the lemma is proved. □

The exceptional case of Lemma 11, where either $J = \emptyset$ or $J \neq \emptyset$ and $l_p \geq 2^{m-1} - (k-2)$, will be covered in Section 4.3.

3.2. *Case when $k_0 \geq 2$ and $\Delta = 1$*

In this case, we have $k_1 \geq 1$. Again, following the aforementioned approach, we pursue a decomposition of $(l_1, \ldots, l_{k_0})$ into $(l'_1, \ldots, l'_{k_0}) + (l''_1, \ldots, l''_{k_0})$ in which $l'_q = l_q - 1$ for a unique $q \in I_0$ and $l'_i = l_i$ for every $i \in I_0$ other than q. Care should be taken when selecting q among the k_0 indices in I_0, because the sink, denoted by x_q, of the s_q-subpath of length $l_q - 1$ should satisfy the condition $\overline{x_q} \notin W^1_q$ as well as $\overline{x_q} \notin S_1$ and $(x_q, \overline{x_q}) \notin F_2$. Fortunately, there always exists $i \in I_0$ with $l''_i = 0$, which is one of the preferable features of our decomposition. Excluding a very special case, the basic procedure shown below works well, as proved in Lemma 12. Let $Z = \{z \in V(G_1) : (z, \bar{z}) \in F_2\}$. For a vertex subset X of $G_0 \oplus G_1$, $\overline{X}$ denotes $\{\bar{x} : x \in X\}$.

Procedure DPC-C($\{(s_i, l_i, W_i) : i \in I\}$, F, $G_0 \oplus G_1$)
/* $k_0 \geq 2$ and $\Delta = 1$. See Fig. 5. */

1: Pick up $s_q \in S_0$ such that
   C1: $l_q \geq 3$ and $|(\overline{S_1} \cup \bar{Z} \cup \overline{W^1_q}) \setminus S_0| < m - f_0 - k_0$, or
   C2: $l_q = 2$ and $s_q \notin \overline{S_1} \cup \bar{Z} \cup \overline{W^1_q}$.
2: Decompose $(l_1, \ldots, l_{k_0})$ into $(l'_1, \ldots, l'_{k_0}) + (l''_1, \ldots, l''_{k_0})$ such that $l'_q = l_q - 1$ and $l'_i = l_i$ for every $i \in I_0 \setminus \{q\}$.
3: Find a k_0-DPC$[\{(s_i, l'_i, W'_i) : i \in I_0\} \,|\, G_0, F_0]$, where

$$W'_i = \begin{cases} \emptyset & \text{if } i = q \text{ and } l_q = 2, \\ (\overline{S_1} \cup \bar{Z} \cup \overline{W^1_q}) \setminus S_0 & \text{if } i = q \text{ and } l_q \geq 3, \\ W^0_i & \text{if } i \in I_0 \setminus \{q\}. \end{cases}$$

   Let x_q denote the sink of an s_q-path in the k_0-DPC.

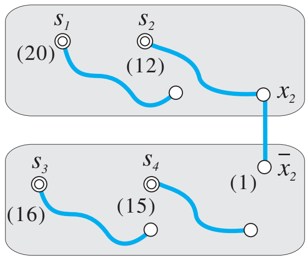


Fig. 5: Illustration of Procedure DPC-C, where $m = 6$, $k = 4$, $q = 2$, $l_1 = 20$, $(l'_2, l''_2) = (12, 1)$, $l_3 = 16$, and $l_4 = 15$.

4: Find $(1 + k_1)$-DPC$[\{(\overline{x_q}, 1, \emptyset)\} \cup \{(s_j, l_j, W_j^1) : j \in I_1\} \,|\, G_1, F_1]$.
5: Merge the k_0-DPC and the $(1 + k_1)$-DPC with edge $(x_q, \overline{x_q})$.

Lemma 12. *When $k_0 \geq 2$ and $\Delta = 1$, Procedure DPC-C constructs a k-DPC unless the following four conditions are satisfied simultaneously: (a) $f_1 = 0$; (b) $S_0 \cap \overline{S_1} = \emptyset$, $S_0 \cap \bar{Z} = \emptyset$, and $S_1 \cap Z = \emptyset$; (c) $|W_i^1| = m - f - k$ and $W_i^1 \cap (\overline{S_0} \cup S_1 \cup Z) = \emptyset$ for every $i \in I_0$ with $l_i \geq 3$; (d) $\overline{s_i} \in W_i^1$ for every $i \in I_0$ with $l_i = 2$.*

Proof. First, we show that picking up s_q of Step 1 is possible. There exists $i \in I_0$ such that $l_i \geq 3$; suppose otherwise, $2^{m-1} + 1 = L_0 \leq 2k_0 \leq 2(m - 1)$, which is impossible for every $m \geq 5$. For $i \in I_0$ with $l_i \geq 3$,

$$\begin{aligned} |(\overline{S_1} \cup \bar{Z} \cup \overline{W_i^1}) \setminus S_0| &\leq |\overline{S_1} \cup \bar{Z} \cup \overline{W_i^1}| = |S_1 \cup Z \cup W_i^1| \\ &\leq |S_1| + |Z| + |W_i^1| \\ &\leq k_1 + f_2 + (m - f - k) = m - f_0 - f_1 - k_0 \\ &\leq m - f_0 - k_0. \end{aligned}$$

The equality $|(\overline{S_1} \cup \bar{Z} \cup \overline{W_i^1}) \setminus S_0| = m - f_0 - k_0$ holds true for every $i \in I_0$ with $l_i \geq 3$ only if the three conditions (a), (b), and (c) are satisfied simultaneously. If at least one of the three is violated, then $|(\overline{S_1} \cup \bar{Z} \cup \overline{W_i^1}) \setminus S_0| < m - f_0 - k_0$ for some i, which implies that it suffices to pick up s_i. Now, assume that all the three conditions (a) through (c) are satisfied, whereas condition (d) is violated by the hypothesis of the lemma. Then, for every $j \in I_0$, it holds that $s_j \notin \overline{S_1} \cup \bar{Z}$. It suffices to pick up an arbitrary s_j such that $l_j = 2$ and $\overline{s_j} \notin W_j^1$, concluding that the picking up process of Step 1 is possible.

It is straightforward to see that the procedure successfully constructs the desired k-DPC if the k_0-DPC of Step 3 and the $(1 + k_1)$-DPC of Step 4 exist. The existence of k_0-DPC of Step 3 is due to the fact that (i) $f_0 + k_0 \leq f + k - 1 \leq (m - 1) - 1$, (ii) for $i = q$ and $l_q = 2$, it holds that $f_0 + k_0 + |W'_i| = f_0 + k_0 \leq f + k - 1 < m - 1$, (iii) for $i = q$ and $l_q \geq 3$, it also holds that $f_0 + k_0 + |W'_i| \leq f_0 + k_0 + (m - f_0 - k_0 - 1) = m - 1$, and (iv) for $i \in I_0 \setminus \{q\}$, it holds that $f_0 + k_0 + |W'_i| \leq f + k - 1 + |W_i| \leq m - 1$. Note that $k_0 \geq 2$ and $k_1 \geq 1$. The $(1 + k_1)$-DPC of Step 4 also exists because (i) $f_1 + (1 + k_1) = f_1 + (1 + k - k_0) = (f_1 + k) - (k_0 - 1) \leq (f + k) - (k_0 - 1) \leq (m - 1) - 1$, (ii) for $i = q$, it holds that $f_1 + (1 + k_1) + |\emptyset| \leq f + k \leq m - 1$, and (iii) for $i \in I_1$, it also holds that $f_1 + (1 + k_1) + |W_i^1| = f_1 + k - (k_0 - 1) + |W_i^1| \leq f + k + |W_i| - (k_0 - 1) \leq m - 1$. Therefore, the proof is complete. □

The exceptional case of Lemma 12 will be discussed in Section 4.5.

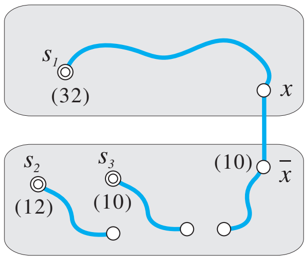


Fig. 6: Illustration of Procedure DPC-D, where $m = 6$, $k = 3$, $(l'_1, l''_1) = (32, 10)$, $l_2 = 12$, and $l_3 = 10$.

3.3. Case when $k_0 = 1$

Using the aforementioned approach again, we decompose l_1 into $l'_1 + l''_1$, where $l'_1 = 2^{m-1}$ and $l''_1 = \Delta$. We have no room of selecting a source of G_0 on the contrary to the previous cases. Instead, we carefully choose the sink position of an s_1-subpath of length l'_1. The procedure below assumes $f_1 + k \leq m - 2$; the remaining case of $f_1 + k = m - 1$ is deferred to Section 4.7. The procedure for this case also works well, except for some special cases. An edge (u, v) is called *free* if $u, v \notin S$ and $(u, v) \notin F$. Notice that the condition $|W_1^1| = m - f_1 - k$ is equivalent to $f_1 = f$ and $|W_1| = |W_1^1| = m - f - k$. This is because $m - f - k \leq m - f_1 - k = |W_1^1| \leq |W_1| \leq m - f - k$.

Procedure DPC-D($\{(s_i, l_i, W_i) : i \in I\}$, F, $G_0 \oplus G_1$)
/* $k_0 = 1$ and $f_1 + k \leq m - 2$. See Fig. 6. */

1: Pick up a free edge $(x, \bar{x})$ with $x \in V(G_0)$ such that

$$\begin{cases} \bar{x} \notin W_1^1 & \text{if } \Delta = 1 \text{ or } |W_1^1| < m - f_1 - k, \\ \bar{x} \in W_1^1 & \text{if } \Delta \geq 2 \text{ and } |W_1^1| = m - f_1 - k. \end{cases}$$

2: Find a Hamiltonian path joining s_1 and x in $G_0 \setminus F_0$.

3: Find a k-DPC$[\{(\bar{x}, \Delta, W'_1)\} \cup \{(s_j, l_j, W_j^1) : j \in I_1\} \,|\, G_1, F_1]$, where

$$W'_1 = \begin{cases} \emptyset & \text{if } \Delta = 1, \\ W_1^1 & \text{if } \Delta \geq 2 \text{ and } |W_1^1| < m - f_1 - k, \\ W_1^1 \setminus \{\bar{x}\} & \text{if } \Delta \geq 2 \text{ and } |W_1^1| = m - f_1 - k. \end{cases}$$

4: Merge the Hamiltonian path and the k-DPC of G_1 with edge $(x, \bar{x})$.

Lemma 13. *When $k_0 = 1$ and $f_1 + k \leq m - 2$, Procedure DPC-D constructs a k-DPC if $f_0 \leq m - 4$ and $|W_j^1| < m - f_1 - k$ for every $j \in I_1$.*

Proof. We first show that it is always possible to pick up a free edge $(x, \bar{x})$ of Step 1. When $\Delta = 1$ or $|W_1^1| < m - f_1 - k$, there are 2^{m-1} candidate edges whereas at most m of them could be blocked (by k sources, f_2 faults, and $m - f - k$ vertices in W_1^1). Thus, there remain at least $2^{m-1} - m \geq 11$ candidates unblocked. When $\Delta \geq 2$ and $|W_1^1| = m - f_1 - k$, there are $m - f_1 - k$ candidates, and at most one of them could be blocked by s_1 of G_0. Note that $f_1 = f$, and thus $f_2 = 0$. Moreover, $W_1^1 \cap S = \emptyset$ from our assumption that $W_i \cap S = \emptyset$ for every $i \in I$. Thus, there remain at least $(m - f_1 - k) - 1 \geq m - (m - 2) - 1 \geq 1$ candidates unblocked. The Hamiltonian path of Step 2 exists from Lemma 2(a). The k-DPC of Step 3 exists because (i) $f_1 + k \leq m - 2 = (m - 1) - 1$,

(ii) $f_1 + k + |W_1'| \leq f_1 + k + (m - f_1 - k - 1) = m - 1$, and (iii) for every $j \in I_1$, it holds that $f_1 + k + |W_j^1| \leq f_1 + k + (m - f_1 - k - 1) = m - 1$. The sink of the s_1-path is not contained in W_1 by the choice of $(x, \bar{x})$ and the definition of W_1' in Step 3. The proof is complete. □

The exceptional case of Lemma 13, where $f_0 = f = m - 3$ or $|W_j^1| \geq m - f_1 - k$ for some $j \in I_1$, will be discussed in Section 4.6.

4. The Exceptional Cases

Dealing with the special cases not covered by the four basic procedures is not an easy task. We rely on various Hamiltonian properties of faulty RHL graphs, including the fundamental one given in Lemma 2. Before we examine the exceptional cases in depth, we study the fault-Hamiltonicity of RHL graphs first in Section 4.1. The three exceptional cases of Section 3.1 will be addressed in Sections 4.2 through 4.4; the one exception of Section 3.2 and the two exceptions of Section 3.3 will be addressed in Sections 4.5, 4.6, and 4.7.

4.1. Hamiltonian properties of RHL graphs

We begin with several properties on paths of the 3-dimensional RHL graph, $G(8, 4)$, of Fig. 2 in Lemmas 14 through 17. They will be utilized to derive the Hamiltonian properties of four- or higher-dimensional RHL graphs in subsequent Lemmas 18, 19, and 20. The proofs of Lemmas 14 and 16 are deferred to the appendix.

Lemma 14. *Let $G = G(8, 4)$ and let $F \subset V(G) \cup E(G)$ be its fault set with $|F| = 1$.*
(a) If $F = \{(x, y)\}$, every pair of vertices $s \in \{x, y\}$ and $t \in V(G) \setminus \{s\}$ are joined by a Hamiltonian path of $G \setminus F$.
(b) If $F = \{v_0\}$, every pair of vertices $s \in \{v_4\}$ and $t \in V(G) \setminus (F \cup \{s\})$ are joined by a Hamiltonian path of $G \setminus F$. Moreover, there exists a Hamiltonian path joining s and t in $G \setminus F$ such that $s = v_1$ and $t \in \{v_2, v_4, v_5, v_7\}$, $s = v_2$ and $t \in \{v_1, v_4, v_6\}$, or $s = v_3$ and $t \in \{v_4, v_7\}$.

A *hypo-Hamiltonian path* in a graph is a path whose length is one shorter than that of a Hamiltonian path.

Lemma 15. *(See [39].) Let $G(8, 4)$ have one vertex fault v_0. Every pair of vertices, s and t, of $G(8, 4) \setminus \{v_0\}$ are joined by a hypo-Hamiltonian path provided that $\{s, t\} \neq \{v_2, v_6\}$, $\{v_3, v_4\}$, and $\{v_4, v_5\}$.*

The graph $G(8, 4)$ even has a 3-DPC for some (l_1, l_2, l_3), provided that $W_1 = W_2 = W_3 = \emptyset$, as follows.

Lemma 16. *If $(l_1, l_2, l_3) = (3, 3, 2)$, $(4, 2, 2)$, or $(5, 2, 1)$, then there exists a 3-DPC$[\{(s_1, l_1, \emptyset), (s_2, l_2, \emptyset), (s_3, l_3, \emptyset)\} \,|\, G(8, 4), \emptyset]$ for any three distinct sources s_1, s_2, and s_3.*

An *unpaired many-to-many* 2*-DPC* joining $\{s_1, s_2\}$ and $\{t_1, t_2\}$ in a graph is a set of two pairwise disjoint paths, $\{P_1, P_2\}$, that altogether cover every vertex of the graph, where P_1 is an s_1–t_1 path and P_2 is an s_2–t_2 path, or P_1 is an s_1–t_2 path and P_2 is an s_2–t_1 path.

Lemma 17. *(See [39].) Let $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$ be pairwise disjoint vertex subsets of $G(8, 4)$. Then, $G(8, 4)$ has an unpaired many-to-many 2-DPC joining S and T except the following cases up to symmetry:*

- $S = \{v_0, v_1\}$ *and* $T = \{v_3, v_6\}$*;*
- $S = \{v_0, v_2\}$ *and* $T = \{v_3, v_5\}$ *or* $\{v_5, v_7\}$*;*
- $S = \{v_0, v_3\}$ *and* $T = \{v_1, v_6\}$*,* $\{v_2, v_5\}$*, or* $\{v_5, v_6\}$*;*
- $S = \{v_0, v_4\}$ *and* $T = \{v_2, v_6\}$*.*

An arbitrary 4-dimensional RHL graph G is 1-fault Hamiltonian-connected and 2-fault Hamiltonian by Lemma 2; however, it is neither 2-fault Hamiltonian-connected nor 3-fault Hamiltonian. We are to hunt for some more Hamiltonian properties of G in Lemmas 18 and 19 below.

Lemma 18. *Let* $F = \{v_f\} \subset V(G)$ *be a fault set of a* 4*-dimensional RHL graph* G*. Then, for every pair of vertices* s *and* t *in* $V(G) \setminus F$*, there exists a vertex* $x \in V(G) \setminus (F \cup \{s, t\})$ *adjacent to* v_f *such that* $G \setminus F'$ *has a Hamiltonian* s*–*t *path where* $F' = F \cup \{x\}$*.*

Proof. Let $G = G_0 \oplus G_1$, where G_0 and G_1 are 3-dimensional RHL graphs, $G(8, 4)$. Assume w.l.o.g. $v_f \in V(G_0)$. Suppose $s, t \in V(G_0)$ for the first case. If we regard v_f as a *virtual* source, there exists a 3-DPC$[\{(v_f, 2, \emptyset), (s, 5, \emptyset), (t, 1, \emptyset)\} \,|\, G_0, \emptyset]$ by Lemma 16. Let x and y denote the sinks of a v_f-path and an s-path in the DPC, respectively. Then, a Hamiltonian s–t path of $G \setminus F'$ is obtained from the concatenation of an s-path in the DPC, a Hamiltonian $\bar{y}$–$\bar{t}$ path of G_1, and a one-vertex path (t). Suppose $s \in V(G_0)$ and $t \in V(G_1)$ for the second case. There exists a 2-DPC$[\{(v_f, 2, \emptyset), (s, 6, \{\bar{t}\})\} \,|\, G_0, \emptyset]$ by Lemma 3 (also by Lemma 4(a)). If we let x and y be the sinks of a v_f-path and an s-path in the DPC, then the desired Hamiltonian s–t path is the concatenation of the s-path and a Hamiltonian $\bar{y}$–t path of G_1.

Suppose $s, t \in V(G_1)$ for the last case. We claim that there exists a vertex $z \in V(G_1) \setminus \{s, t\}$ such that (i) $\bar{z} \neq v_f$ and (ii) there exists an unpaired many-to-many 2-DPC joining $\{s, t\}$ and $\{z, u\}$ for every $u \in V(G_1) \setminus \{s, t, z\}$. By Lemma 17, there exist two vertices satisfying (ii); for example, if $\{s, t\} = \{v_0, v_3\}$, then v_4 and v_7 will do. Thus, at least one of the two also satisfies (i), proving the claim. There exists a 2-DPC$[\{(v_f, 2, \emptyset), (\bar{z}, 6, \{\bar{s}, \bar{t}\})\} \,|\, G_0, \emptyset]$ by Lemma 4(a). If we let x and y be the sinks of v_f-path and $\bar{z}$-path in the DPC, respectively, the desired Hamiltonian s–t path is the concatenation of three paths: the $\bar{z}$–y path and two paths in an unpaired many-to-many 2-DPC of G_1 joining $\{s, t\}$ and $\{z, \bar{y}\}$. The proof is complete. □

Lemma 19. *Let* $F = \{v_f, w_f\} \subset V(G)$ *be a fault set of a* 4*-dimensional RHL graph* G*. Then, there exist two vertices* $x_i \in V(G) \setminus F$*,* $i \in \{1, 2\}$*, such that* $G \setminus F'_i$ *has a Hamiltonian cycle, where* $F'_i = F \cup \{x_i\}$*. Furthermore, if* v_f *is adjacent to* w_f*, there exist four such vertices* x_i*,* $i \in \{1, \ldots, 4\}$*.*

Proof. Let $G = G_0 \oplus G_1$, where G_0 and G_1 are 3-dimensional RHL graphs. We show that there exist four (resp. two) vertices, x_i, such that $G \setminus F'_i$ has a Hamiltonian cycle if $(v_f, w_f) \in E(G)$ (resp. if $(v_f, w_f) \notin E(G)$).

Case 1: $(v_f, w_f) \in E(G)$. Let $v_f, w_f \in V(G_0)$ for the first subcase. There are two cases up to symmetry: $F = \{v_0, v_1\}$ or $\{v_0, v_4\}$. We let $X = \{v_2, v_4, v_5, v_7\}$ if $F = \{v_0, v_1\}$; let $X = \{v_1, v_3, v_5, v_7\}$ if $F = \{v_0, v_4\}$. It is straightforward to see that for every $x_i \in X$, $G_0 \setminus (F \cup \{x_i\})$ has a Hamiltonian path P_h. (This is because $G_0 \setminus (F \cup \{x_i\})$ is a connected graph that has a cycle of length four or five.) For two end-vertices u and v of P_h, we can construct a Hamiltonian cycle of $G \setminus (F \cup \{x_i\})$ by merging P_h and a Hamiltonian $\bar{u}$–$\bar{v}$ path of G_1. Second, let $v_f \in V(G_0)$ and $w_f \in V(G_1)$, so that $\overline{v_f} = w_f$. Assume w.l.o.g. that $v_f = v_0$, where $V(G_0) = \{v_0, \ldots, v_7\}$. $G_1 \setminus \{w_f\}$ has a Hamiltonian cycle $C = (z_0, z_1, \ldots, z_6, z_0)$, where we assume $\overline{v_4} = z_0$. For $x_1 = z_6$, a Hamiltonian cycle of $G \setminus (F \cup \{x_1\})$ is constructed from $C \setminus \{z_6\}$ and a Hamiltonian v_4–$\overline{z_5}$ path of $G_0 \setminus \{v_f\}$.

The existence of the Hamiltonian v_4–$\overline{z_5}$ path is by Lemma 14(b). Also for $x_2 = z_1$, a Hamiltonian cycle of $G \setminus (F \cup \{x_2\})$ is obtained from $C \setminus \{z_1\}$ and a Hamiltonian v_4–$\overline{z_2}$ path of $G_0 \setminus \{v_f\}$. Notice that $x_1, x_2 \in V(G_1)$. In a symmetric way, we can also find two more desired vertices, x_3 and x_4, in G_0.

Case 2: $(v_f, w_f) \notin E(G)$. Let $v_f, w_f \in V(G_0)$ first. There are two cases up to symmetry: $F = \{v_0, v_2\}$ or $\{v_0, v_3\}$. We let $X = \{v_4, v_6\}$ if $F = \{v_0, v_2\}$; and let $X = \{v_4, v_7\}$ if $F = \{v_0, v_3\}$. Then, $G_0 \setminus (F \cup \{x_i\})$ has a Hamiltonian path for every $x_i \in X$. Similar to the first subcase of Case 1, we can conclude that $G \setminus (F \cup \{x_i\})$ has a Hamiltonian cycle for each $x_i \in X$. Second, let $v_f \in V(G_0)$ and $w_f \in V(G_1)$. We first claim that there exists a pair of vertices u and v in G_0 such that (i) $\{u, \bar{u}, v, \bar{v}\} \cap F = \emptyset$, (ii) $G_0 \setminus \{v_f\}$ has a Hamiltonian u–v path P_h, and (iii) $G_1 \setminus \{w_f\}$ has a hypo-Hamiltonian $\bar{u}$–$\bar{v}$ path P'_h. By Lemma 14(b), there are $(4 + 3 + 2 + 6 + 2 + 3 + 4)/2 = 12$ unordered vertex-pairs, each of which is joined by a Hamiltonian path of $G_0 \setminus \{v_f\}$. Among them, at least six pairs also satisfy (i), as $\overline{w_f}$ may block at most six. From Lemma 15, there remain at least three pairs satisfying (iii), proving the claim. For the vertex $x_1 \in V(G_1) \setminus \{w_f\}$ such that $x_1 \notin V(P'_h)$, a Hamiltonian cycle of $G \setminus (F \cup \{x_1\})$ is obtained from P_h and P'_h. Symmetrically, we can also find one more vertex, x_2, in G_0. This completes the proof. □

Finally, we consider Hamiltonian paths that share a common source s in an m-dimensional RHL graph with at most $m - 2$ faults, where $m \geq 4$. There exist two vertices, t_i, each admits a Hamiltonian s–t_i path because the RHL graph is $(m - 2)$-fault Hamiltonian by Lemma 2(b). (Think of the vertices next to s in a Hamiltonian cycle.)

Lemma 20. *Let G be an m-dimensional RHL graph, where $m \geq 4$, and let $F \subset V(G) \cup E(G)$ be its fault set with $|F| \leq m - 2$. For every vertex $s \in V(G) \setminus F$, there exist three vertices t_i, $i \in \{1, 2, 3\}$, depending on s, such that $G \setminus F$ has a Hamiltonian s–t_i path for every $i \in \{1, 2, 3\}$.*

Proof. Let $G = G_0 \oplus G_1$, where G_0 and G_1 are $(m-1)$-dimensional RHL graphs. Assume w.l.o.g. that $s \in V(G_0)$.

Case 1: $f_0 = m - 2$ ($f_1 = f_2 = 0$). There exists a Hamiltonian path $P_h = (u_1, \ldots, u_n)$, where $n \geq 2^{m-1} - (m - 2) \geq 6$, in $G_0 \setminus F_0$ because G_0 is $(m - 3)$-fault Hamiltonian. Let $s = u_j$ for some j, and assume w.l.o.g. that $j \leq \lceil n/2 \rceil$. If $j = 1$, then for any vertex $t_i \in V(G_1) \setminus \{\overline{u_n}\}$, there exists a Hamiltonian s–t_i path, (P_h, P_i), in $G \setminus F$, where P_i is a Hamiltonian $\overline{u_n}$–t_i path of G_1. If $j \geq 2$, we have three Hamiltonian paths of $G \setminus F$ for $\{t_1, t_2, t_3\} = \{u_n, u_{j+1}, u_{j-1}\}$ as follows: $(u_j, \ldots, u_1, P_1, u_{j+1}, \ldots, u_n)$, where P_1 is a Hamiltonian $\overline{u_1}$–$\overline{u_{j+1}}$ path of G_1; $(u_j, \ldots, u_1, P_2, u_n, \ldots, u_{j+1})$, where P_2 is a Hamiltonian $\overline{u_1}$–$\overline{u_n}$ path of G_1; $(u_j, \ldots, u_n, P_3, u_1, \ldots, u_{j-1})$, where P_3 is a Hamiltonian $\overline{u_n}$–$\overline{u_1}$ path of G_1.

Case 2: $f_0 = m - 3$ ($f_1 + f_2 \leq 1$). There exists a Hamiltonian cycle in $G_0 \setminus F_0$. From the cycle, we can extract a Hamiltonian s–x path P_h of $G_0 \setminus F_0$ for some $x \in V(G_0) \setminus F_0$ such that $\{\bar{x}, (x, \bar{x})\} \cap F = \emptyset$, because $f_1 + f_2 \leq 1$. There also exists a Hamiltonian cycle, $C_1 = (\bar{x}, u, \ldots, v, \bar{x})$, in $G_1 \setminus F_1$, as $f_1 \leq 1 \leq m - 3$. Then, we have two Hamiltonian paths of $G \setminus F$: an s–t_1 path $(P_h, C_1 \setminus \{(\bar{x}, v)\})$ for $t_1 = v$, and an s–t_2 path $(P_h, C_1 \setminus \{(\bar{x}, u)\})$ for $t_2 = u$. Here, $t_1, t_2 \in V(G_1)$. One more Hamiltonian path is obtained from the Hamiltonian cycle, $C = (s, y, \ldots, z, s)$, of $G \setminus F$, where at least one of y and z, say z, is a vertex of G_0. The Hamiltonian path, $C \setminus \{(s, z)\}$, joins s and t_3 for $t_3 = z$.

Case 3: $f_0 \leq m - 4$. In this case, $G_0 \setminus F_0$ is Hamiltonian-connected from Lemma 2(a).

Case 3.1: $f_1 = m - 2$ ($f_0 = f_2 = 0$). There exists a Hamiltonian path $P_h = (u_1, \ldots, u_n)$ in $G_1 \setminus F_1$. Suppose $\bar{s} \notin \{u_1, u_n\}$ for the first subcase. We have two Hamiltonian paths for

$\{t_1, t_2\} = \{u_1, u_n\}$ as follows: (P_1, P_h), where P_1 is a Hamiltonian s–$\overline{u_1}$ path of G_0, and (P_2, P_h^R), where P_2 is a Hamiltonian s–$\overline{u_n}$ path of G_0 and P_h^R is the reverse, $(u_n, \ldots, u_1)$, of P_h. Note that $t_1, t_2 \in V(G_1)$. For some $t_3 \in V(G_0)$, a Hamiltonian s–t_3 path can be obtained from a Hamiltonian cycle of $G \setminus F$. Suppose $\bar{s} \in \{u_1, u_n\}$ for the second subcase, say $\bar{s} = u_1$. There exist two vertices $t_i \in V(G_0)$, $i \in \{1, 2\}$, such that $G_0 \setminus \{s\}$ has a Hamiltonian $\overline{u_n}$–t_i path P_i, because G_0 is 1-fault Hamiltonian. Then, we have two desired Hamiltonian paths, (s, P_h, P_1) and (s, P_h, P_2), of $G \setminus F$. Also for $t_3 = u_1$, there is a Hamiltonian s–t_3 path, (P_3, P_h^R), where P_3 is a Hamiltonian s–$\overline{u_n}$ path of G_0.

Case 3.2: $f_1 \leq m - 3$. The subgraph $G_1 \setminus F_1$ has a Hamiltonian cycle C_1 whose length, n, is at least $2^{m-1} - (m - 3)$. We claim that there exists an edge (u, v) of C_1 such that (i) $\bar{u}, \bar{v} \neq s$ and (ii) $\{\bar{u}, (u, \bar{u}), \bar{v}, (v, \bar{v})\} \cap F = \emptyset$. There are n candidate edges in C_1 whereas each element of $\{s\} \cup F$ could block at most two of them. Thus, there remain at least $n - (2 + 2f)$ candidates, where $n - (2 + 2f) \geq (2^{m-1} - (m - 3)) - (2 + 2(m - 2)) = 2^{m-1} - (3m - 5) \geq 1$, proving the claim. If we let $P_h = C_1 \setminus \{(u, v)\}$, then P_h is a Hamiltonian u–v path of $G_1 \setminus F_1$. Similar to the first subcase of Case 3.1, we can construct Hamiltonian s–t_i paths for $\{t_1, t_2\} = \{u, v\}$ and $t_3 \in V(G_0)$. The proof is complete. □

4.2. Exceptional case of Lemma 10

We consider the exceptional case of Lemma 10: $k_0 = 2$ and $l_2 \geq 2^{m-1} - 1$. We have $l_1 + l_2 \geq 2^m - 2$, so that $k \leq 4$. A k-DPC will be constructed under the additional conditions: $k_0 \geq 2$, $\Delta \geq 2$, and $f_0 + k_0 < f + k$.

Lemma 21. *In the exceptional case of Lemma 10, where $k_0 = 2$, $l_2 \geq 2^{m-1} - 1$, and $f_0 + k_0 < f + k$, there exists a k-DPC$[\{(s_i, l_i, W_i) : i \in I\} \,|\, G_0 \oplus G_1, F]$.*

Proof. There are two cases depending on whether $f_1 < f$ or not.

Case 1: $f_1 < f$. We employ the existence of an f_1-fault k-DPC in G_1 as follows:

Procedure DPC-E1 // See Fig. 7(a).

1: Let $(l_1, l_2) = (l'_1, l'_2) + (l''_1, l''_2)$ such that $l'_1 = l'_2 = 2^{m-2}$.
2: Find a 2-DPC$[\{(s_1, l'_1, W), (s_2, l'_2, W)\} \,|\, G_0, F_0]$, where $W = \overline{S_1} \cup \{y \in V(G_0) : (y, \bar{y}) \in F_2\}$. Let x_i denote the sink of an s_i-path in the 2-DPC.
3: Find a k-DPC$[\{(\overline{x_i}, l''_i, W_i^1) : i = 1, 2\} \cup \{(s_j, l_j, W_j^1) : j \in I_1\} \,|\, G_1, F_1]$.
4: Merge the 2-DPC of G_0 and the k-DPC of G_1 with edges $(x_i, \overline{x_i})$ for $i \in \{1, 2\}$.

It is obvious that $2 \leq l'_i, l''_i < l_i$ for each $i \in \{1, 2\}$. The 2-DPC of Step 2 exists because (i) $f_0 + k_0 \leq f + k - 1 \leq (m - 1) - 1$ and (ii) $f_0 + k_0 + |W| \leq f_0 + k_0 + (k_1 + f_2) \leq f + k \leq m - 1$. The k-DPC of Step 3 also exists because (i) $f_1 + k \leq f - 1 + k \leq (m - 1) - 1$ and (ii) for every $i \in I$, it holds that $f_1 + k + |W_i^1| \leq f - 1 + k + |W_i| \leq m - 1$. Thus, the procedure successfully produces a k-DPC of $G_0 \oplus G_1$.

Case 2: $f_1 = f$ *($f_0 = f_2 = 0$).*

Case 2.1: $l_2 = 2^{m-1} - 1$. It is assumed that $\overline{s_1} \notin S_1$ in Procedure DPC-E2.

Procedure DPC-E2 // See Fig. 7(b).

1: Let $(l_1, l_2) = (l'_1, l'_2) + (l''_1, l''_2)$ where $l'_1 = 1$ and $l'_2 = l_2$.
2: Find a 2-DPC$[\{(s_1, l'_1, \emptyset), (s_2, l'_2, W_2^0)\} \,|\, G_0, \emptyset]$.
3: Find a $(k - 1)$-DPC$[\{(\overline{s_1}, l''_1, W_1^1)\} \cup \{(s_j, l_j, W_j^1) : j \in I_1\} \,|\, G_1, F_1]$.
4: Merge the 2-DPC of G_0 and the $(k - 1)$-DPC of G_1 with edge $(s_1, \overline{s_1})$.

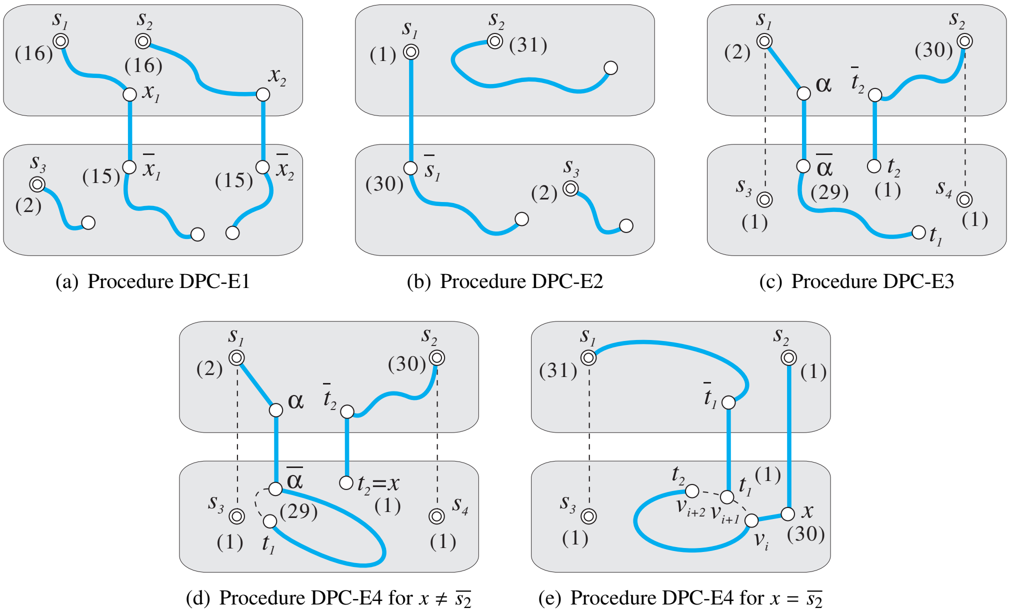


Fig. 7: Illustrations of the proof of Lemma 21, where $m = 6$.

The 2-DPC of Step 2 exists because (i) $f_0 + k_0 \leq f + k - 1 < (m-1) - 1$ and (ii) $f_0 + k_0 + |W_2^0| \leq f + k - 1 + |W_2| \leq m - 1$. The $(k-1)$-DPC of Step 3 also exists because (i) $f_1 + (k-1) = f + k - 1 \leq (m-1) - 1$ and (ii) for each $i \in I \setminus \{2\}$, it holds that $f_1 + (k-1) + |W_i^1| \leq f + k - 1 + |W_i| \leq m - 1$. Thus, the procedure works correctly.

Procedure DPC-E2 can also be used for the case $l_1 = 2^{m-1} - 1$ and $\overline{s_2} \notin S_1$, if we interchange the roles of s_1 and s_2. Thus, the following two cases remain:

- $k = 4$, $l_1 = l_2 = 2^{m-1} - 1$, and $\{\overline{s_1}, \overline{s_2}\} = \{s_3, s_4\}$;
- $k = 3$, $l_1 = 2^{m-1}$, $l_2 = 2^{m-1} - 1$, and $\overline{s_1} = s_3$.

We will devise two procedures, depending on whether $f + k \leq m - 2$ or not, which are applicable to both of the remaining cases. The procedures use the Hamiltonian properties of faulty RHL graphs developed in Section 4.1. Procedure DPC-E3 works if $f + k \leq m - 2$.

Procedure DPC-E3 // See Fig. 7(c).

1: Pick up a vertex, t_2, of G_1 such that $t_2 \notin \overline{S_0} \cup S_1 \cup W_2^1$.
2: Pick up a neighbor, α, of s_1 in G_0 such that $G_0 \setminus \{s_1, \alpha\}$ has a Hamiltonian s_2–$\overline{t_2}$ path P_2.
3: Find a Hamiltonian $\overline{\alpha}$–t_1 path P_1 in $G_1 \setminus (F_1 \cup S_1 \cup \{t_2\})$ for some $t_1 \notin W_1^1$.
4: Let the s_1-path be (s_1, α, P_1) and the s_2-path be (P_2, t_2). Every other s_i-path is a one-vertex path (s_i).

For Step 2, if $m \geq 6$, it suffices to pick up a neighbor, different from s_2 and $\overline{t_2}$, of s_1 by Lemma 2(a); if $m = 5$, the existence of α is due to Lemma 18. Observe that $|F_1 \cup S_1 \cup \{t_2\}| = f + (k-2) + 1 = f + k - 1$. Step 3 is plausible by Lemma 2(a) if $|F_1 \cup S_1 \cup \{t_2\}| \leq m - 4$, or equivalently, if $f + k \leq m - 3$; by Lemma 20 if $f + k = m - 2$, where $|W_i| \leq 2$ for every $i \in I$.

Finally, we present a procedure for the two remaining cases that works if $f + k = m - 1$. We can assume $\{u, v\} \cap S_1 = \emptyset$ for every edge fault $(u, v) \in F_1$, as $l_i = 1$ for every $i \in I_1$; suppose otherwise, the desired k-DPC can be produced by Procedure DPC-E3 under the fault set $F' = F \setminus \{(u, v)\}$. Recall that $|W_i| \le 1$ for every $i \in I$.

Procedure DPC-E4 // See Fig. 7 (d) and (e).

1: Pick up a vertex $x \in V(G_1) \setminus (S_1 \cup W_2^1)$ such that $G_1 \setminus (F_1 \cup S_1 \cup \{x\})$ has a Hamiltonian cycle $C = (v_0, v_1, \ldots, v_{n-1}, v_0)$, where $n = 2^{m-1} - (k_1 + 1)$.

2: Case when $x \ne \overline{s_2}$: Let $t_2 = x$.

- a: Pick up a neighbor, α, of s_1 in G_0 such that $G_0 \setminus \{s_1, \alpha\}$ has a Hamiltonian s_2–$\overline{t_2}$ path P_2.
- b: Extract a Hamiltonian $\bar{\alpha}$–t_1 path P_1 from C for some $t_1 \in V(C) \setminus W_1^1$.
- c: Let the s_1-path be (s_1, α, P_1) and the s_2-path be (P_2, t_2). Every other s_i-path is a one-vertex path (s_i).

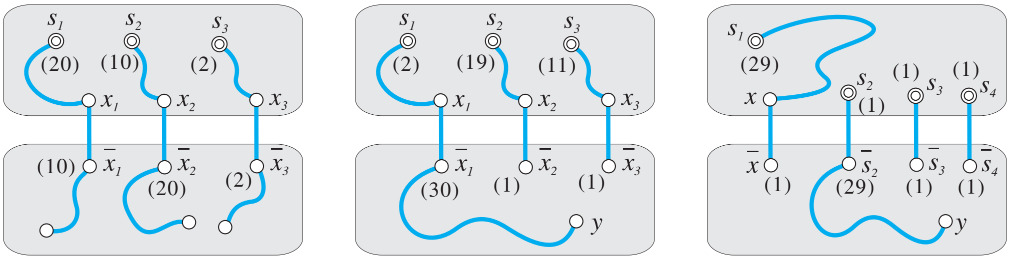
3: Case when $x = \overline{s_2}$: Here, we have $k = 3$, $l_1 = 2^{m-1}$, $l_2 = 2^{m-1} - 1$, and $\overline{s_1} = s_3$.

- a: Pick up a neighbor $v_i \in V(C)$ of x in $G_1 \setminus F_1$ such that (i) $v_{i+1} \notin W_1^1$ and $v_{i+2} \notin W_2^1$ or (ii) $v_{i-1} \notin W_1^1$ and $v_{i-2} \notin W_2^1$. Assume w.l.o.g. that $v_{i+1} \notin W_1^1$ and $v_{i+2} \notin W_2^1$, and let $t_1 = v_{i+1}$ and $t_2 = v_{i+2}$.
- b: Find a Hamiltonian s_1–$\overline{t_1}$ path P_1 in $G_0 \setminus \{s_2\}$.
- c: Let the s_1-path be (P_1, t_1) and the s_2-path be (s_2, x, P_2), where P_2 is the v_i–v_{i+2} path obtained from $C \setminus \{t_1\}$. Every other s_i-path is a one-vertex path (s_i).

We first claim that the vertex x in Step 1 exists. If $F_1 = \emptyset$ (where $f + k = m - 1$, $f_1 = f$, $k \le 4$, $m \ge 5$, and thus $k = 4$ and $m = 5$), then x exists by Lemma 19. If there exists an edge fault $(u, v) \in F_1$ (with $\{u, v\} \cap S_1 = \emptyset$), then for at least one of u and v, say u, we have $u \notin W_2^1$. It suffices to let $x = u$ by Lemma 2(b), as $G_1 \setminus (F_1 \cup S_1 \cup \{x\})$ is identical to $G_1 \setminus (F_1' \cup S_1 \cup \{x\})$, where $F_1' = F_1 \setminus \{(u, v)\}$, and $|F_1' \cup S_1 \cup \{x\}| = (f - 1) + (k - 2) + 1 = m - 3$. Thus, the claim is proved. The remainder of the proof is similar to that of Procedure DPC-E3. Note that the vertex v_i of C in Step 3a exists because x has at least $(m - 1) - (|S_1| + f) = (m - 1) - (k - 2 + f) = 2$ neighbors contained in C.

Case 2.2: $l_2 = 2^{m-1}$ ($k_0 = k = 2$, $l_1 = 2^{m-1}$). If $f + k \le m - 2$, Procedure DPC-E3 can be recycled to construct a k-DPC for this case just by redefining α as s_1 itself (instead of a neighbor of s_1). Let us now suppose that $f + k = m - 1$ ($|W_1|, |W_2| \le 1$). If there exists an edge fault $(u, v) \in F_1$ such that $u \notin \{\overline{s_1}, \overline{s_2}\} \cup W_1^1$, then we have an s_1-path that is the concatenation of a Hamiltonian s_1–$\bar{u}$ path of $G_0 \setminus \{s_2\}$ and a one-vertex path (u). As there exists a Hamiltonian cycle, $(x_1, \ldots, x_{2^{m-1}-1}, x_1)$, in $G_1 \setminus (F_1' \cup \{u\})$, where $F_1' = F_1 \setminus \{(u, v)\}$, assuming w.l.o.g. that $\overline{s_2} = x_1$ and $x_{2^{m-1}-1} \notin W_2^1$, we have an s_2-path $(s_2, x_1, \ldots, x_{2^{m-1}-1})$. Symmetrically, a 2-DPC can be constructed if there exists an edge fault $(u, v) \in F_1$ such that $u \notin \{\overline{s_1}, \overline{s_2}\} \cup W_2^1$.

There remains a case that $\{u, v\} \subseteq \{\overline{s_1}, \overline{s_2}\} \cup (W_1^1 \cap W_2^1)$ for every edge fault $(u, v) \in F_1$. It follows that $f \le 2$ (where $f + k = m - 1$, $k = 2$, and $m \ge 5$), and thus we have $f = 2$, $m = 5$, and $W_1^1 \cap W_2^1 \ne \emptyset$. Recall that no RHL graph has a triangle from Lemma 1(b). Let $W_1^1 \cap W_2^1 = \{w\}$ (where $w \ne \overline{s_1}, \overline{s_2}$) and assume w.l.o.g. that $(\overline{s_1}, w) \in F_1$. There exists a Hamiltonian cycle, $(x_1, \ldots, x_{16}, x_1)$, in $G_1 \setminus F_1$ by Lemma 2(b). Let $\overline{s_1} = x_1$. Then, we have $w \notin \{x_2, x_{16}\}$ by our construction. Furthermore, it holds that $w \notin \{x_3, x_{15}\}$; suppose otherwise, there would exist a triangle, (x_1, x_2, x_3, x_1) or $(x_1, x_{16}, x_{15}, x_1)$, in G_1, which is a contradiction. Assuming w.l.o.g. that $x_2 \ne \overline{s_2}$, we have an s_1-path $(s_1, x_1, x_{16}, \ldots, x_3)$ and an s_2-path (P_2, x_2), where P_2 is a Hamiltonian s_2–$\overline{x_2}$ path of $G_0 \setminus \{s_1\}$. Therefore, the lemma is proved. □



(a) Procedure DPC-F1 (b) Procedure DPC-F2, where $p = 1$ (c) Procedure DPC-F3

Fig. 8: Illustrations of the proof of Lemma 22, where $m = 6$.

4.3. Exceptional case of Lemma 11

We now deal with the exceptional case: either $J = \emptyset$ or $J \neq \emptyset$ and $l_p \geq 2^{m-1} - (k-2)$, where $J = \{i \in I_0 : |W_i^0| < m - f - k\}$ and $p = \arg\min_{j \in J} l_j$. Recall the conditions that $k_0 \geq 2$, $\Delta \geq 2$, $f_0 = f$, $k_0 = k$, and $f + k \leq m - 2$. In this case, $|W_i^0| = m - f - k$ ($W_i^1 = \emptyset$) for every $i \in I_0 \setminus J$, and $l_j \geq 2^{m-1} - (k-2)$ for every $j \in J$. Moreover, $l_i \geq 2$ for every $i \in I_0$; suppose otherwise, $l_p = 1 \not\geq 2^{m-1} - (k-2)$, which is a contradiction.

Lemma 22. *In the exceptional case of Lemma 11, where $f_0 = f$, $k_0 = k$, $f+k \leq m-2$, and either $J = \emptyset$ or $J \neq \emptyset$ and $l_p \geq 2^{m-1} - (k-2)$, there exists a k-DPC$[\{(s_i, l_i, W_i) : i \in I\} \,|\, G_0 \oplus G_1, F]$.*

Proof. Note that $f_1 = f_2 = 0$ and $k_1 = 0$. We claim $|J| \leq 2$. Suppose otherwise, $\sum_{j \in J} l_j \geq 3l_p \geq 3(2^{m-1} - (k-2)) \geq 2^m + 2^{m-1} - 3(m-4) > 2^m$, which is a contradiction for every $m \geq 5$, proving the claim. There are three cases.

Case 1: either $J = \emptyset$ or $J \neq \emptyset$ and $f \geq 1$. We employ the existence of a (f_1-fault) k-DPC in G_1 as follows:

Procedure DPC-F1 // See Fig. 8(a).

1: Decompose $(l_1, \ldots, l_{k_0})$ into $(l'_1, \ldots, l'_{k_0}) + (l''_1, \ldots, l''_{k_0})$ subject to the three conditions:
   F1a: $1 \leq l'_i \leq l_i - 1$ ($l''_i \geq 1$) for all $i \in I_0 \setminus J$;
   F1b: $1 \leq l'_j \leq l_j - 2$ ($l''_j \geq 2$) for all $j \in J$;
   F1c: $\sum_{i \in I_0} l'_i = 2^{m-1}$.
2: Find a k_0-DPC$[\{(s_i, l'_i, \emptyset) : i \in I_0\} \,|\, G_0, F_0]$. Let x_i denote the sink of the s_i-path in the k_0-DPC.
3: Find a k_0-DPC$[\{(\overline{x}_i, l''_i, W_i^1) : i \in I_0\} \,|\, G_1, \emptyset]$.
4: Merge the two k_0-DPCs with edges $(x_i, \overline{x}_i)$ for $i \in I_0$.

The decomposition of Step 1 is possible because (i) $l_i \geq 2$ for every $i \in I_0$, (ii) $l_j \geq l_p \geq 2^{m-1} - (k-2) \geq 2^{m-1} - (m-4) \geq 15$ for every $j \in J$, and (iii) $\sum_{i \in I_0 \setminus J} 1 + \sum_{j \in J} 2 \leq 2k \leq 2(m-2) < 2^{m-1}$ for every $m \geq 5$. The k_0-DPC of Step 2 exists because (i) $f_0 + k_0 = f + k \leq m - 2 = (m-1) - 1$ and (ii) $f_0 + k_0 + |\emptyset| < m - 1$. The k_0-DPC of Step 3 also exists as (i) $f_1 + k_0 \leq f + k \leq (m-1) - 1$, (ii) when $J = \emptyset$, it holds that $f_1 + k_0 + |W_i^1| \leq f + k + 0 < m - 1$ for every $i \in I_0$, and (iii) when $J \neq \emptyset$ and $f \geq 1$, it holds that $f_1 + k_0 + |W_i^1| \leq 0 + k + (m - f - k) = m - f \leq m - 1$ for every $i \in I_0$. Thus, the correctness of our procedure is proved.

Case 2: $|J| = 1$ *and* $f = 0$. We have $J = \{p\}$, and thus $|W_p^0| < m-k$ and $|W_i^0| = m-k$ ($W_i^1 = \emptyset$) for every $i \in I_0 \setminus \{p\}$. Recall that $l_p \geq 2^{m-1} - (k-2)$ and $l_i \geq 2$ for every $i \in I_0$.

Procedure DPC-F2 // See Fig. 8(b).

1: Decompose $(l_1, \ldots, l_{k_0})$ into $(l'_1, \ldots, l'_{k_0}) + (l''_1, \ldots, l''_{k_0})$ subject to the two conditions:
  F2a: $l'_i = l_i - 1$ ($l''_i = 1$) for all $i \in I_0 \setminus \{p\}$;
  F2b: $l'_p = 2^{m-1} - \sum_{i \in I_0 \setminus \{p\}} l'_i$.
2: Find a k_0-DPC$[\{(s_i, l'_i, \emptyset) : i \in I_0\} \,|\, G_0, \emptyset]$. Let x_i denote the sink of the s_i-path, P_i, in the k_0-DPC.
3: Let $P'_i = (P_i, \overline{x_i})$ for every $i \in I_0 \setminus \{p\}$.
4: In $G_1 \setminus \{\overline{x_i} : i \in I_0 \setminus \{p\}\}$, find a Hamiltonian $\overline{x_p}$–y path P_h for some $y \notin W_p^1$. Let $P'_p = (P_p, P_h)$.
5: Then, $\{P'_i : i \in I_0\}$ is the desired k-DPC of $G_0 \oplus G_1$.

We can see that $l'_p \geq 1$ because $\sum_{i \in I_0 \setminus \{p\}} l'_i = \sum_{i \in I_0 \setminus \{p\}} l_i - (k-1) = (2^m - l_p) - (k-1) \leq 2^{m-1} - 1$. Also, we have $l'_p = 2^{m-1} - \sum_{i \in I_0 \setminus \{p\}} l'_i \leq 2^{m-1} - (k-1) < l_p$. Thus, it holds that $1 \leq l'_i, l''_i < l_i$ for every $i \in I_0$ and moreover, $\sum_{i \in I_0} l'_i = 2^{m-1}$. The existence of the k_0-DPC in Step 2 is obvious because $f_0 + k_0 \leq m-2$. The Hamiltonian path P_h of Step 4 exists by Lemma 2(a) when $k \leq m-3$, and by Lemma 20 when $k = m-2$ (where $|W_i| \leq 2$ for every $i \in I$). Thus, the procedure is correct.

Case 3: $|J| = 2$ *and* $f = 0$. Let $J = \{p, q\}$. It holds that $l_p = l_q = 2^{m-1} - (k-2)$ and $l_i = 2$ for each $i \neq p, q$ from the fact that $\sum_{i \in I_0} l_i = (l_p + l_q) + \sum_{i \in I_0 \setminus \{p,q\}} l_i \geq 2(2^{m-1} - (k-2)) + (k-2) \cdot 2 = 2^m$. Thus, we have $J = \{1, 2\}$ from our assumption that $l_1 \geq \cdots \geq l_{k_0}$.

Procedure DPC-F3 // See Fig. 8(c).

1: Let $P_i = (s_i, \overline{s_i})$ for every $i \in I_0 \setminus \{1, 2\}$.
2: In $G_0 \setminus \{s_i : i \in I_0 \setminus \{1\}\}$, find a Hamiltonian s_1–x path P_h for some x with $\bar{x} \notin W_1^1$. Let $P_1 = (P_h, \bar{x})$.
3: In $G_1 \setminus \{\bar{x}, \overline{s_3}, \ldots, \overline{s_k}\}$, find a Hamiltonian $\overline{s_2}$–y path P'_h for some y with $y \notin W_2^1$. Let $P_2 = (s_2, P'_h)$.
4: Then, $\{P_i : i \in I_0\}$ is the desired k-DPC of $G_0 \oplus G_1$.

The Hamiltonian paths P_h and P'_h of Steps 2 and 3 exist due to Lemma 2(a) when $k \leq m-3$, and due to Lemma 20 when $k = m-2$ (where $|W_i| \leq 2$ for every $i \in I$). Thus, the procedure works correctly. This completes the proof. □

4.4. *Case when* $k_0 \geq 2$, $\Delta \geq 2$, *and* $f_0 + k_0 = f + k = m-1$

This is one of the three exceptional cases that were deferred from Section 3.1. It holds that $f_0 = f$ ($f_1 = f_2 = 0$), $k_0 = k$ ($k_1 = 0$), and $|W_i| \leq 1$ for every $i \in I$. We note that the induction hypothesis does not guarantee the existence of a k_0-DPC in $G_0 \setminus F_0$, even if $W_i = \emptyset$ for all $i \in I$. This is because $f_0 + k_0 \not\leq (m-1) - 1$. Instead, we will pick up a source $s_i \in S_0$ and find a $(k_0 - 1)$-DPC for the sources of $S_0 \setminus \{s_i\}$ in $G_0 \setminus F_0$, and then construct the desired k-DPC using the $(k_0 - 1)$-DPC. Recall the assumption that $l_1 \geq \cdots \geq l_{k_0}$.

Lemma 23. *When* $k_0 \geq 2$, $\Delta \geq 2$, *and* $f_0 + k_0 = f + k = m-1$, *there exists a* k*-DPC*$[\{(s_i, l_i, W_i) : i \in I\} \,|\, G_0 \oplus G_1, F]$.

Proof. Two cases arise depending on the size of l_1. Note that if $f = 0$, we have $k = m - 1 - f = m - 1 \geq 4$ from the assumption of $m \geq 5$.

Case 1: $l_1 \leq 2^{m-1}$.

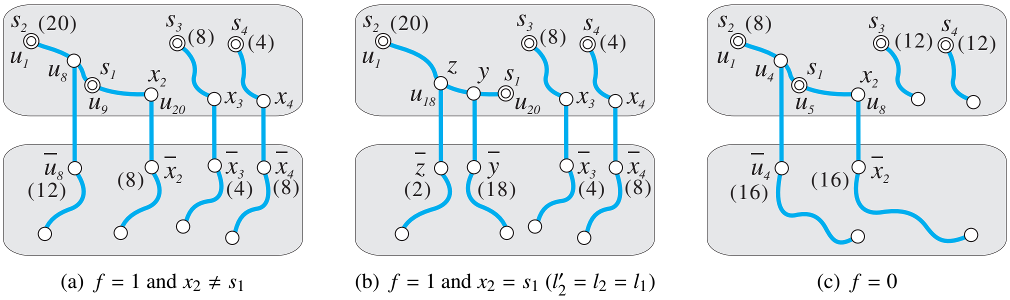


Fig. 9: Illustrations of Procedure DPC-G1, where $m = 6$, $k = 4$, $(l_1, l_2, l_3, l_4) = (20, 20, 12, 12)$, and $r = 2$.

Case 1.1: either $f \geq 1$ or $f = 0$ and $(k-3) + l_{k-1} + l_k \leq 2^{m-1}$. Assume w.l.o.g. that either $W_1^1 = \emptyset$ or $W_i^1 \neq \emptyset$ for all $i \in I_0$ with $l_i = l_1$. Observe that $l_1 \geq 2^m/k \geq 2^m/(m-1) \geq 8$. The procedure shown below utilizes a (k_0-1)-DPC of $G_0 \setminus F_0$ and a k'_0-DPC of G_1 for some k'_0, where $k'_0 \leq k_0$ if $f \geq 1$; $k'_0 < k_0$ if $f = 0$.

Procedure DPC-G1 // See Fig. 9.

1: Decompose $(l_2, \ldots, l_{k_0})$ into $(l'_2, \ldots, l'_{k_0}) + (l''_2, \ldots, l''_{k_0})$ subject to the following conditions:
   G1a: $1 \leq l'_i \leq l_i$ for every $i \in I_0 \setminus \{1\}$;
   G1b: $\sum_{i \in I_0 \setminus \{1\}} l'_i = 2^{m-1}$;
   G1c: $l'_i = l_i$ ($l''_i = 0$) if $l_i = 2$ for $i \in I_0 \setminus \{1\}$;
   G1d: $l'_{k_0-1} = l_{k_0-1}$ and $l'_{k_0} = l_{k_0}$ ($l''_{k_0-1} = l''_{k_0} = 0$) if $f = 0$.

2: Regarding s_1 as a non-source vertex virtually, find a (k_0-1)-DPC$[\{(s_i, l'_i, W'_i) : i \in I_0 \setminus \{1\}\} \,|\, G_0, F_0]$, where

$$W'_i = \begin{cases} \overline{W_1^1} & \text{if } W_i^0 = \emptyset \text{ and } l'_i = l_i = l_1, \\ \{s_1\} & \text{if } W_i^0 = \emptyset \text{ and } l'_i = l_i < l_1, \\ \overline{W_i^1} & \text{if } l'_i = l_i - 1, \\ W_i^0 & \text{otherwise.} \end{cases}$$

   Let x_i denote the sink of the s_i-path P_i in the (k_0-1)-DPC.

3: Let P_r in the (k_0-1)-DPC pass through s_1, i.e., $P_r = (u_1, \ldots, u_{l'_r})$, where $s_1 = u_{j+1}$ for some $j \geq 1$. Let $I'_0 = \{i \in I_0 : l'_i < l_i\} \cup \{1, r\}$ and $k'_0 = |I'_0|$.

4: Unless $l'_r = l_r = l_1$ and $s_1 = u_{l'_r}$, find k'_0-DPC$[\{(\overline{x_r}, l_1 - l'_r + j, W_1^1), (\overline{u_j}, l_r - j, W_r^1)\} \cup \{(\overline{x_i}, l''_i, W_i^1) : i \in I'_0 \setminus \{1, r\}\} \,|\, G_1, \emptyset]$, and then merge the two DPCs with edges $(x_r, \overline{x_r})$, $(u_j, \overline{u_j})$, and $(x_i, \overline{x_i})$ for $i \in I'_0 \setminus \{1, r\}$.

5: If $l'_r = l_r = l_1$ and $s_1 = u_{l'_r}$, find k'_0-DPC$[\{(\bar{y}, l_1 - 2, W_1^1), (\bar{z}, 2, W_r^1)\} \cup \{(\overline{x_i}, l''_i, W_i^1) : i \in I'_0 \setminus \{1, r\}\} \,|\, G_1, \emptyset]$, where $y = u_{l'_r - 1}$ and $z = u_{l'_r - 2}$, and then merge the two DPCs with edges $(y, \bar{y})$, $(z, \bar{z})$, and $(x_i, \overline{x_i})$ for $i \in I'_0 \setminus \{1, r\}$.

The decomposition of Step 1 is possible. This is because (i) $\sum_{i \in I_0 \setminus \{1\}} l_i \geq 2^{m-1}$; (ii) if either $f \geq 1$ or $f = 0$ and $l_{k_0-1} \leq 2$, then $\sum_{i \in I_0 \setminus \{1\} \text{with } l_i = 2} 2 + \sum_{i \in I_0 \setminus \{1\} \text{with } l_i \neq 2} 1 \leq 2(k-1) \leq 2(m-2) < 2^{m-1}$; (iii) if $f = 0$ and $l_{k_0-1} \geq 3$, then $l_{k_0-1} + l_{k_0} + \sum_{i \in I_0 \setminus \{1, k_0-1, k_0\}} 1 = (k-3) + l_{k_0-1} + l_{k_0} \leq 2^{m-1}$ by the hypothesis of Case 1.1. The (k_0-1)-DPC of Step 2 exists because $f_0 + (k_0 - 1) = f + k - 1 \leq (m-1) - 1$ and $f_0 + (k_0-1) + |W'_i| \leq m - 1$ for all $i \in I_0 \setminus \{1\}$. It is straightforward to check that for each $i \in I_0 \setminus \{1, r\}$, the final s_i-path of length l_i has its sink not contained in W_i. Concerning the s_1-path completed in Step 4, we claim that either $l_1 - l'_r + j \geq 2$ or $l_1 - l'_r + j = 1$ and $\overline{x_r} \notin W_1^1$. It suffices to show that $\overline{x_r} \notin W_1^1$ if $l_1 - l'_r + j = 1$ (i.e., $l'_r = l_r = l_1$ and $j = 1$). If $W_r^0 = \emptyset$, we have $W'_r = \overline{W_1^1}$ and thus $\overline{x_r} \notin W_1^1$; if $W_r^0 \neq \emptyset$, we have $W_r^1 = \emptyset$ and, moreover, $W_1^1 = \emptyset$ from our assumption that $W_1^1 = \emptyset$ or $W_i^1 \neq \emptyset$ for all $i \in I_0$ with $l_i = l_1$, implying $\overline{x_r} \notin W_1^1$. Thus, the claim is proved.

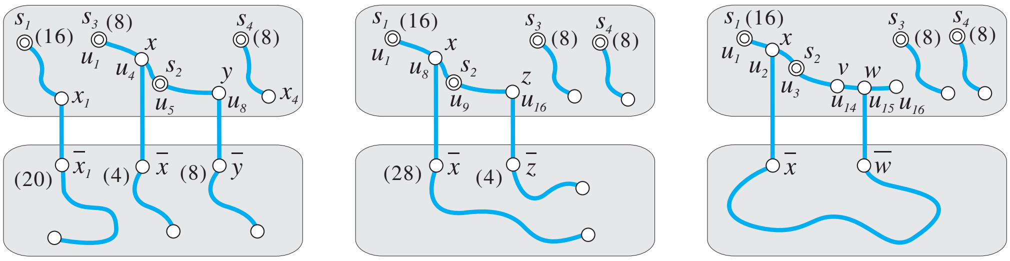
In addition, we claim that either $l_r - j \geq 2$ or $l_r - j = 1$ and $\overline{u_j} \notin W_r^1$, concerned with the s_r-path completed in Step 4. We will show that $\overline{u_j} \notin W_r^1$ if $l_r - j = 1$ (i.e., $l'_r = l_r$ and $j = l'_r - 1$ ($s_1 = u_{l'_r}$)). It follows that $W'_r \neq \{s_1\}$, because $s_1 = u_{l'_r}$. From the definition of W'_r and the fact that $l'_r = l_r$, we can see that $W_r^0 \neq \emptyset$ or $l'_r = l_r = l_1$. If $W_r^0 \neq \emptyset$, then $W_r^1 = \emptyset$ and thus $\overline{u_j} \notin W_r^1$. The case $l'_r = l_r = l_1$ does not occur due to the hypothesis of Step 4. Thus, the claim is proved. Also, the s_1-path and s_r-path completed in Step 5 have their sinks that are not contained in W_1^1 and W_r^1, respectively, as their subpaths embedded in G_1 have lengths at least two. If $f = 0$, the s_{k-1}-path or s_k-path is embedded wholly within G_0, and thus $k'_0 < k$. The k'_0-DPCs of Steps 4 and 5 exist because (i) for $f \geq 1$, it holds that $f_1 + k'_0 \leq k = (m-1) - f \leq (m-1) - 1$, and (ii) for $f = 0$, it holds that $f_1 + k'_0 \leq k - 1 \leq (m-1) - 1$. Therefore, the procedure is correct.

Case 1.2: $f = 0$ and $(k-3)+l_{k-1}+l_k > 2^{m-1}$. We can see that $k = 4$ because for any $k \geq 5$, we have $(k-3)+l_{k-1}+l_k \leq (m-4)+2\cdot 2^m/5 \leq 2^{m-1}$. Furthermore, it holds that $(k-3)+l_{k-1}+l_k > 2^{m-1}$ only when $k = 4$ ($m = 5$) and $l_1 = l_2 = l_3 = l_4 = 2^{m-2} = 8$. We let $G_0 = H_0 \oplus H_1$, where H_0 and H_1 are isomorphic to the 3-dimensional RHL graph $G(8, 4)$. For a vertex v in H_i, $i \in \{0, 1\}$, the vertex in H_{1-i} adjacent to v is denoted by $\hat{v}$. When all the four sources are contained in H_0, we first find a Hamiltonian s_1–s_4 path, $(u_1, u_2, \ldots, u_8)$, in H_0, where $s_1 = u_1$ and $s_4 = u_8$. Assume w.l.o.g. that $s_2 = u_{i+1}$ and $s_3 = u_{j+1}$ for some $i < j$. There exists a Hamiltonian cycle $C = (x_1, x_2, \ldots, x_7, x_1)$ in $H_1 \setminus \{\hat{u}_i\}$. Assuming w.l.o.g. that $x_1 = \hat{s}_4$ and $x_7 \notin W_4^0$, we have an s_4-path $P_4 = (s_4, x_1, \ldots, x_7)$. Also, there exists a 3-DPC$[\{(\overline{\hat{u}_i}, 8 - i - 1, W_1^1), (\overline{u_j}, 8 - j + i, W_2^1), (\overline{u_7}, j + 1, W_3^1)\} \,|\, G_1, \emptyset]$ by the induction hypothesis. Then, the remaining three paths P_1, P_2, and P_3 are the concatenation of $(u_1, \ldots, u_i, \hat{u}_i)$ and $\overline{\hat{u}_i}$-path in the 3-DPC, the concatenation of $(u_{i+1}, \ldots, u_j)$ and the $\overline{u_j}$-path, and the concatenation of $(u_{j+1}, \ldots, u_7)$ and the $\overline{u_7}$-path, respectively.

When three sources, say s_1, s_2, and s_3, are contained in H_0, we find a Hamiltonian s_1–s_3 path, $(u_1, u_2, \ldots, u_8)$, in H_0 where $s_1 = u_1$, $s_2 = u_{i+1}$ for some $1 \leq i \leq 6$, and $s_3 = u_8$. Then, P_1, P_2, and P_3 are obtained by merging $\{(u_1, \ldots, u_i), (u_{i+1}, \ldots, u_7), (u_8)\}$ and a 3-DPC$[\{(\overline{u_i}, 8 - i, W_1^1), (\overline{u_7}, i + 1, W_2^1), (\overline{u_8}, 7, W_3^1)\} \,|\, G_1, \emptyset]$ with edges $(u_i, \overline{u_i})$, $(u_7, \overline{u_7})$, and $(u_8, \overline{u_8})$. P_4 will be a Hamiltonian s_4–x path in H_1 for some $x \notin W_4^0$. Finally, when two sources, say s_1 and s_2, are contained in H_0, we can find a 2-DPC$[\{(s_1, 5, \{\hat{s}_3, \hat{s}_4\}), (s_2, 3, \emptyset)\} \,|\, H_0, \emptyset]$ by Lemma 4(a). Let x_i be the sink of the s_i-path for $i \in \{1, 2\}$. We can easily check that there exists a 3-DPC$[\{(\hat{x}_1, 3, W_1^0), (s_3, l'_3, \emptyset), (s_4, l'_4, \emptyset)\} \,|\, H_1, \emptyset]$ for some $(l'_3, l'_4) \in \{(1, 4), (2, 3), (3, 2), (4, 1)\}$. Let y_j be the sink of the s_j-path for $j \in \{3, 4\}$. Also, there exists a 3-DPC$[\{(\overline{x_2}, 5, W_2^1), (\overline{y_3}, l''_3, W_3^1), (\overline{y_4}, l''_4, W_4^1)\} \,|\, G_1, \emptyset]$, where $l''_3 = 8 - l'_3$ and $l''_4 = 8 - l'_4$. The desired 4-DPC can be obtained from three DPCs: the 2-DPC of H_0, the 3-DPC of H_1, and the 3-DPC of G_1.

Case 2: $l_1 > 2^{m-1}$. We will regard s_2 (instead of s_1) as a virtual non-source vertex and find a $(k_0 - 1)$-DPC of G_0, from which a k-DPC of $G_0 \oplus G_1$ will be constructed in a similar way to Case 1. For $I' := \{i \in I_0 : l_i = l_2\}$, it is assumed w.l.o.g. that either $W_2^1 = \emptyset$ or $W_i^1 \neq \emptyset$ for all $i \in I'$. If $l_2 = 3$ and $W_2^1 = \{w\}$ for some $w \in V(G_1)$, we let $I'' = \{i \in I' : (s_i, \overline{w}) \in E(G_0)\}$; let $I'' = \emptyset$ otherwise.

Procedure DPC-G2 // See Fig. 10.



(a) $r = 3$ (b) $r = 1$ and $l'_1 - j < l_2$ ($j = 8$) (c) $r = 1$ and $l'_1 - j \geq l_2$ ($j = 2$)

Fig. 10: Illustrations of Procedure DPC-G2, where $m = 6$, $k = 4$, and $(l_1, l_2, l_3, l_4) = (36, 12, 8, 8)$.

1: Decompose $(l_1, l_3, \ldots, l_{k_0})$ into $(l'_1, l'_3, \ldots, l'_{k_0}) + (l''_1, l''_3, \ldots, l''_{k_0})$ subject to the two conditions:
   G2a: $l'_i = l_i$ for every $i \in I_0 \setminus \{1, 2\}$;
   G2b: $l'_1 = 2^{m-1} - \sum_{i \in I_0 \setminus \{1,2\}} l'_i$.

2: Regarding s_2 as a non-source vertex virtually, find a $(k_0 - 1)$-DPC$[\{(s_i, l'_i, W'_i) : i \in I_0 \setminus \{2\}\} \,|\, G_0, F_0]$, where $W'_1 = W^0_1$ and for $i \in I_0 \setminus \{1, 2\}$,

$$W'_i = \begin{cases} \overline{W^0_i} & \text{if } W^0_i \neq \emptyset, \\ \overline{W^1_2} & \text{if } W^0_i = \emptyset,\ i \in I' \setminus I'', \text{ and } l_i \geq 3, \\ \{s_2\} & \text{otherwise.} \end{cases}$$

   Let x_i denote the sink of the s_i-path, P_i, in the $(k_0 - 1)$-DPC.

3: There exists a path, P_r, in the $(k_0 - 1)$-DPC that passes through s_2.

At this point, the procedure is incomplete. The remainder differs depending on whether $r \geq 3$ or $r = 1$. Let $r \geq 3$ for the first case. We represent P_r as $(s_r, P_x, x, s_2, P_y, y)$. Let l'_r and l'_2 be the lengths of subpaths (s_r, P_x, x) and (s_2, P_y, y), respectively, and let $l''_r = l_r - l'_r$, $l''_2 = l_2 - l'_2$. To construct a k-DPC, it suffices to define three paths: an s_1-path P_1, an s_2-path P_2, and an s_r-path P_r. If (i) either $l''_r \geq 2$ or $l''_r = 1$ and $\bar{x} \notin W^1_r$ and (ii) either $l''_2 \geq 2$ or $l''_2 = 1$ and $\bar{y} \notin W^1_2$, it suffices to find a 3-DPC$[\{(\overline{x_1}, l_1 - l'_1, W^1_1), (\bar{x}, l''_r, W^1_r), (\bar{y}, l''_2, W^1_2)\} \,|\, G_1, \emptyset]$ and merge the two DPCs, the $(k_0 - 1)$-DPC of G_0 and the 3-DPC of G_1. Suppose $l''_r = 1$ and $\bar{x} \in W^1_r$, violating the condition (i). Then, we have $s_2 = y$ ($W'_r \neq \{s_2\}$) and $W^0_r = \emptyset$. By the choice of W'_r, we can see that $r \in I' \setminus I''$ and $l_r \geq 3$ ($l_r = l_2$). If either $l_r \geq 4$ or $l_r = 3$ and $W^1_2 \neq \{\bar{x}\}$, where the s_r-path in the $(k_0 - 1)$-DPC is represented as (s_r, P_z, z, x, s_2), it suffices to construct a 3-DPC$[\{(\overline{x_1}, l_1 - l'_1, W^1_1), (\bar{z}, 2, W^1_r), (\bar{x}, l_2 - 2, W^1_2)\} \,|\, G_1, \emptyset]$ and merge the two DPCs. The remaining possibility of $l_r = 3$ and $W^1_2 = \{\bar{x}\}$ leads to $r \in I''$, which contradicts the fact that $r \in I' \setminus I''$. Now, suppose $l''_2 = 1$ and $\bar{y} \in W^1_2$, violating the condition (ii). Then, we have $l_r = l_2$ ($r \in I'$); whenever $l_2 = 3$, it holds that $(s_r, y) \notin E(G_0)$, meaning $r \in I' \setminus I''$. Note that no RHL graph contains a cycle of length three by Lemma 1(b). Furthermore, we have $W^1_r \neq \emptyset$ ($W^0_r = \emptyset$), because $W^1_2 \neq \emptyset$. (Recall the assumption that $W^1_2 = \emptyset$ or $W^1_i \neq \emptyset$ for all $i \in I'$.) By the choice of W'_r (where $W'_r \neq \overline{W^1_2}$), we have $l_r \leq 2$. This implies that the s_r-path is (s_r, s_2), which is impossible because $W'_r = \{s_2\}$.

For the second case, let $r = 1$, i.e., the s_1-path of the $(k_0 - 1)$-DPC passes through s_2. We represent the s_1-path as $(u_1, u_2, \ldots, u_{l'_1})$, where $s_1 = u_1$, $s_2 = u_{j+1}$ for some $1 \leq j \leq l'_1 - 1$, $x = u_j$, $y = u_{l'_1 - 1}$, and $z = u_{l'_1}$. It suffices to construct two paths, an s_1-path P_1 and an s_2-path P_2, of the desired k-DPC. Let $P_1' = (u_1, \ldots, u_j)$ and $P_2' = (u_{j+1}, \ldots, u_{l_1'})$. By the choice of W_1', we have $z \notin W_1^0$. There are three possibilities. First, if either $l_1' - j \leq l_2 - 2$ or $l_1' - j = l_2 - 1$ and $\bar{z} \notin W_2^1$, there exists a 2-DPC$[\{(\bar{x}, l_1 - j, W_1^1), (\bar{z}, l_2 - l_1' + j, W_2^1)\} \,|\, G_1, \emptyset]$ by the induction hypothesis. Then, P_1 is the concatenation of P_1' and the $\bar{x}$-path in the 2-DPC, and P_2 is the concatenation of P_2' and the $\bar{z}$-path. Second, suppose $l_1' - j = l_2 - 1$ and $\bar{z} \in W_2^1$. If $l_2 \geq 3$, then by Lemmas 2(a) and 18 depending on whether $m \geq 6$ or $m = 5$, it suffices to let $P_2 = (u_{j+1}, \ldots, u_{l_1'-1}, \bar{y}, t_2)$ for some $t_2 \neq \bar{z}$ and let P_1 be the concatenation of three subpaths: P_1', a Hamiltonian $\bar{x}$–$\bar{z}$ path in $G_1 \setminus \{\bar{y}, t_2\}$, and the one-vertex path (z). If $l_2 = 2$, meaning $s_2 = z$, then we let $P_2 = (s_2, x)$ and let P_1 be the concatenation of $(u_1, \ldots, u_{j-1})$ and a Hamiltonian $\overline{u_{j-1}}$–t_1 path in G_1 for some $t_1 \notin W_1^1$. (Note that $l_1' \geq 3$. Suppose $l_1' = 2$ for a contradiction. It follows that $\sum_{i \in I_0 \setminus \{1,2\}} l_i = 2^{m-1} - 2$. From the hypothesis that $l_1 > 2^{m-1}$, we have $l_1 = 2^{m-1} + 1$ and $l_2 = 1$. This implies that $\sum_{i \in I_0 \setminus \{1,2\}} l_i = \sum_{i \in I_0 \setminus \{1,2\}} 1 = k_0 - 2 \leq m - 3$, which is a contradiction.) Finally, suppose $l_1' - j \geq l_2$. If $v \notin W_2^0$, where $v = u_{j+l_2}$, we let $P_2 = (u_{j+1}, \ldots, u_{j+l_2})$. An s_1-path P_1 is constructed as follows: If $v = z$, P_1 is the concatenation of P_1' and a Hamiltonian $\bar{x}$–t_1 path of G_1 for some $t_1 \notin W_1^1$; if $v \neq z$, P_1 is the concatenation of three subpaths, P_1', a Hamiltonian $\bar{x}$–$\bar{w}$ path of G_1, where $w = u_{j+l_2+1}$, and $(u_{j+l_2+1}, \ldots, u_{l_1'})$. If $v \in W_2^0$ (where $l_2 \geq 2$), we let $P_2 = (u_{j+1}, \ldots, u_{j+l_2-1}, \bar{u})$, where $u = u_{j+l_2-1}$, and let P_1 be the concatenation of three subpaths: P_1', a Hamiltonian $\bar{x}$–$\bar{v}$ path in $G_1 \setminus \{\bar{u}\}$, and $(u_{j+l_2}, \ldots, u_{l_1'})$. This completes the entire proof. □

4.5. Exceptional case of Lemma 12

The exceptional case of Lemma 12 occurs when the following four conditions are satisfied simultaneously: (a) $f_1 = 0$; (b) $S_0 \cap \overline{S_1} = \emptyset$, $S_0 \cap \bar{Z} = \emptyset$, and $S_1 \cap Z = \emptyset$; (c) $|W_i^1| = m - f - k$ and $W_i^1 \cap (\overline{S_0} \cup S_1 \cup Z) = \emptyset$ for every $i \in I_0$ with $l_i \geq 3$; (d) $\overline{s_i} \in W_i^1$ for every $i \in I_0$ with $l_i = 2$. Here, $Z = \{z \in V(G_1) : (z, \bar{z}) \in F_2\}$. We have the additional conditions that $k_0 \geq 2$ and $\Delta = 1$ $(k_1 \geq 1)$.

Lemma 24. *In the exceptional case of Lemma 12, there exists a k-DPC$[\{(s_i, l_i, W_i) : i \in I\} \,|\, G_0 \oplus G_1, F]$.*

Proof. There are two cases.

Case 1: $f_0 + f_2 + k_0 \geq 3$. Let $r = k_0 + 1$. By assumption, we have $l_r \geq l_j$ for all $j \in I_1$.

Procedure DPC-H // See Fig. 11.

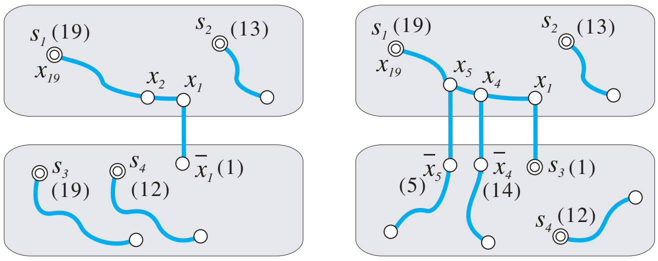
1: Decompose $(l_1, \ldots, l_{k_0})$ into $(l_1', \ldots, l_{k_0}') + (l_1'', \ldots, l_{k_0}'')$ so that $l_1' = l_1 - 1$ and $l_i' = l_i$ for every $i \in I_0 \setminus \{1\}$.

2: Find a k_0-DPC$[\{(s_i, l_i', W_i') : i \in I_0\} \,|\, G_0, F_0]$, where $W_1' = (\bar{Z} \cup \overline{S_1} \cup \overline{W_1^1}) \setminus \{\overline{s_r}\}$ and $W_i' = W_i^0$ for all $i \in I_0 \setminus \{1\}$. Let the s_1-path in the k_0-DPC be $(x_{l_1'}, x_{l_1'-1}, \ldots, x_1)$, where $s_1 = x_{l_1'}$.

3: Case when $\overline{x_1} \neq s_r$: Find a $(k_1 + 1)$-DPC$[\{(\overline{x_1}, 1, \emptyset)\} \cup \{(s_j, l_j, W_j^1) : j \in I_1\} \,|\, G_1, \emptyset]$ and merge the two DPCs.

4: Case when $\overline{x_1} = s_r$: Pick up an edge (x_q, x_{q+1}), $2 \leq q < l_1'$, of the s_1-path such that $\{(x_q, \overline{x_q}), (x_{q+1}, \overline{x_{q+1}})\} \cap F_2 = \emptyset$, $\{\overline{x_q}, \overline{x_{q+1}}\} \cap S_1 = \emptyset$, and $l_r \geq q + 3$. Then, find a $(k_1 + 2)$-DPC$[\{(\overline{x_{q+1}}, q + 1, W_1^1), (\overline{x_q}, l_r - q - 1, W_r^1), (s_r, 1, \emptyset)\} \cup \{(s_j, l_j, W_j^1) : j \in I_1 \setminus \{r\}\} \,|\, G_1, \emptyset]$ and merge the two DPCs.

The k_0-DPC of Step 2 exists because (i) $f_0 + k_0 \leq f + (k - 1) \leq (m - 1) - 1$, (ii) $f_0 + k_0 + |W_1'| \leq f_0 + k_0 + (f_2 + k_1 + (m - f - k) - 1) = m - 1$, and (iii) for $i \in I_0 \setminus \{1\}$, it holds that $f_0 + k_0 + |W_i'| \leq f + (k - 1) + |W_i| \leq m - 1$. The $(k_1 + 1)$-DPC of Step 3 also exists because $k_1 + 1 = (k - k_0) + 1 \leq k - 1$. For the existence of such an edge (x_q, x_{q+1}) in Step 4, it suffices to pick up one in the set of $f_2 + k_1$ independent edges, $\{(x_2, x_3), (x_4, x_5), \ldots, (x_{2(f_2+k_1)}, x_{2(f_2+k_1)+1})\}$,



(a) $\overline{x_1} \neq s_3$ (b) $\overline{x_1} = s_3$ and $q = 4$

Fig. 11: Illustrations of Procedure DPC-H, where $m = 6$, $k = 4$, $r = 3$, and $(l_1, l_2, l_3, l_4) = (20, 13, 19, 12)$.

where $q \in \{2, 4, \ldots, 2(f_2 + k_1)\}$. (Recall $\overline{x_1} = s_r \in S_1$.) The edge set exists because $l'_1 = l_1 - 1 \geq \lceil L_0/k_0 \rceil - 1 \geq \lceil (2^{m-1} + 1)/(m-2) \rceil - 1 \geq 2(m-3) + 1 \geq 2(f_2 + k_1) + 1$. (Note that $f_2 + k_1 \leq f + (k - k_0) \leq f + k - 2 \leq m - 3$.) Furthermore, we have $l_r \geq \lceil L_1/k_1 \rceil \geq \lceil (2^{m-1} - 1)/(m-3) \rceil > 2(m-3) + 3 \geq 2(f_2 + k_1) + 3 \geq q + 3$ for every possible q. The $(k_1 + 2)$-DPC of Step 4 exists because (i) $f_1 + (k_1 + 2) = f - (f_0 + f_2) + (k - k_0) + 2 = (f + k) - (f_0 + f_2 + k_0) + 2 \leq f + k - 1 \leq (m-1) - 1$ and (ii) $f_1 + (k_1 + 2) + |W_j^1| \leq f + k - 1 + |W_j^1| \leq m - 1$ for every $j \in I_1 \cup \{1\}$. Thus, the procedure is correct.

Case 2: $f_0 + f_2 + k_0 \leq 2$, i.e., $f = 0$ and $k_0 = 2$. We claim that there exist two disjoint paths in G_0, an s_1-path P_1 of length $l_1 - 1$ and an s_2-path P_2 of length l_2, such that $\overline{t_1} \notin S_1 \cup W_1^1$ and $t_2 \notin W_2^0$, where t_i is the sink of P_i. Provided the claim is proved, the desired k-DPC can be constructed straightforwardly using a $(k_1 + 1)$-DPC of G_1 for the sources $S_1 \cup \{\overline{t_1}\}$ (because $k_1 + 1 \leq k - 1$). It remains to prove the claim. If either $l_2 = 1$ or $l_2 = 2$ and $m \geq 6$, the claim is obvious from Lemma 2(a). Hereafter, assume that either $l_2 \geq 3$ or $l_2 = 2$ and $m = 5$. Notice that $|W_2^0| \leq 1$ by the conditions (c) and (d), and recall that $|W_2^0| = 1$ only if $l_2 = 2$, $m = 5$, and $k = 3$ ($k_1 = 1$). Let $W_2^0 = \{w_2\}$ if $|W_2^0| = 1$. For the proof, we utilize the recursive structure of $G_0 = H_0 \oplus H_1$, where H_i, $i \in \{0, 1\}$, is an $(m-2)$-dimensional RHL graph. Recall that $\hat{v}$ denotes the unique neighbor of $v \in V(H_i)$ contained in H_{1-i}.

Case 2.1: $s_1, s_2 \in V(H_0)$. When $l_2 \leq 2^{m-2} - 1$, we first find a Hamiltonian s_1–s_2 path P_h in $H_0 \setminus F'$, where $F' = \{(s_2, w_2)\}$ if $|W_2^0| = 1$ and $(s_2, w_2) \in E(H_0)$; $F' = \emptyset$ otherwise. The path P_h exists due to Lemma 2(a) if $m \geq 6$, and due to Lemma 14(a) if $m = 5$. Let $P_h = (s_1, P_x, x, y, P_y, s_2)$, where the length of subpath (y, P_y, s_2) is l_2. Then, P_2 is the reverse of (y, P_y, s_2) and $P_1 = (s_1, P_x, x, P'_h)$, where P'_h is a Hamiltonian $\hat{x}$–t_1 path of H_1 for some $t_1 \in V(H_1)$ with $\overline{t_1} \notin S_1 \cup W_1^1$. Note that $y \notin W_2^0$ even if $W_2^0 \neq \emptyset$ (where $l_2 = 2$), by the construction of P_h. When $l_2 = 2^{m-2}$, there are two subcases. If $m \geq 6$, we find a Hamiltonian $\hat{s_2}$–t_1 path, $(\hat{s_2}, P', t_2, t_1)$, in H_1 for some t_1 with $\hat{t_1} \neq s_1$ and $\overline{t_1} \notin S_1 \cup W_1^1$. Then, $P_2 = (s_2, \hat{s_2}, P', t_2)$ and $P_1 = (P'_h, t_1)$, where P'_h is a Hamiltonian s_1–$\hat{t_1}$ path of $H_0 \setminus \{s_2\}$. If $m = 5$, we find a Hamiltonian s_1–s_2 path, $(s_1 = u_1, u_2, \ldots, u_8 = s_2)$, in H_0. From Lemma 4(a), there exists a 2-DPC$[\{(\hat{u_3}, 5, W'_1), (\hat{u_4}, 3, \emptyset)\} \,|\, H_1, \emptyset]$, where $W'_1 = \overline{S_1} \cup \overline{W_1^1}$. Note that $|W'_1| \leq k_1 + (m - f - k) = m - f - k_0 = 3$. Then, P_1 is the concatenation of (u_1, u_2, u_3) and the $\hat{u_3}$-path in the 2-DPC, and P_2 is the concatenation of $(u_8, u_7, \ldots, u_4)$ and the $\hat{u_4}$-path.

Case 2.2: $s_1 \in V(H_0)$ and $s_2 \in V(H_1)$. When $l_2 = 2^{m-2}$, it suffices to let P_1 be a Hamiltonian s_1–t_1 path in H_0 for some $t_1 \in V(H_0)$ with $\overline{t_1} \notin S_1 \cup W_1^1$, and P_2 be a Hamiltonian s_2–t_2 path in

H_1 for some $t_2 \in V(H_1)$. When $l_2 = 2^{m-2} - 1$, we find a Hamiltonian s_2–t_1 path, (s_2, P, t_2, t_1), in H_1 for some t_1 with $\hat{t_1} \neq s_1$ and $\overline{t_1} \notin S_1 \cup W_1^1$. Then, $P_2 = (s_2, P, t_2)$ and $P_1 = (P'_h, t_1)$, where P'_h is a Hamiltonian s_1–$\hat{t_1}$ path of H_0. When $l_2 \leq 2^{m-2} - 2$, there are two subcases. If $m \geq 6$, we find a Hamiltonian s_2–y path P_h in H_1 for some $y \in V(H_1)$ with $\bar{y} \notin S_1 \cup W_1^1$. Let $P_h = (s_2, P', t_2, x, P'', y)$, where the length of (s_2, P', t_2) is l_2. Then, $P_2 = (s_2, P', t_2)$. If $\hat{x} \neq s_1$, P_1 is the concatenation of a Hamiltonian s_1–$\hat{x}$ path of H_0 and (x, P'', y); if $\hat{x} = s_1$, P_1 is the concatenation of (s_1, x, P'', y) and a Hamiltonian $\hat{y}$–z path in $H_0 \setminus \{s_1\}$ for some $z \in V(H_0)$ with $\bar{z} \notin S_1 \cup W_1^1$.

Finally, if $m = 5$, H_0 is isomorphic to the 3-dimensional RHL graph $G(8, 4)$, shown in Fig. 2. It is assumed w.l.o.g. that $V(H_0) = \{v_0, \ldots, v_7\}$, $E(H_0) = \{(v_i, v_j) : i+1 \text{ or } i+4 \equiv j \pmod 8\}$, and $s_1 = v_0$. Pick up a vertex y in H_1 such that (i) $\hat{y} \neq v_3, v_5$, (ii) $\bar{y} \notin S_1 \cup W_1^1$, and (iii) $y \neq s_2$. Such a vertex y exists because there are eight candidates whereas at most six $(= 2 + 3 + 1)$ of them could be blocked. From Lemma 14(a), there exists a Hamiltonian s_2–y path P_h in $H_1 \setminus F'$, where $F' = \{(s_2, w_2)\}$ if $|W_2^0| = 1$ and $(s_2, w_2) \in E(H_1)$; $F' = \emptyset$ otherwise. Let $P_h = (s_2, P, t_2, x, P', y)$, where the length of (s_2, P, t_2) is $l_2 (\leq 6)$. Then, $P_2 = (s_2, P, t_2)$. Note that $t_2 \notin W_2^0$ by the construction of P_h. If $\hat{x} \neq s_1$, P_1 is the concatenation of a Hamiltonian s_1–$\hat{x}$ path of H_0 and (x, P', y). Let us now assume that $\hat{x} = s_1$. From Lemma 14(b), there exist at least three vertices y_j, $j \in \{1, 2, 3\}$, such that $\hat{y}$ and y_j are joined by a Hamiltonian path P'_j in $H_0 \setminus \{s_1\}$. (Recall that $\hat{y} \neq v_3, v_5$.) If $\overline{y_j} \notin S_1 \cup W_1^1$ for some j, then $P_1 = (s_1, x, P', y, P'_j)$; otherwise, it suffices to let P_1 be the concatenation of a Hamiltonian s_1–$\hat{y}$ path of H_0 and the reverse of (x, P', y) (because $S_1 \cup W_1^1 = \{\overline{y_1}, \overline{y_2}, \overline{y_3}\}$). This completes the proof. □

4.6. Exceptional case of Lemma 13

We now consider the exception of Lemma 13, the case when $f_0 = f = m - 3$ or $|W_j^1| \geq m - f_1 - k$ for some $j \in I_1$. Note that the condition $|W_j^1| \geq m - f_1 - k$ is equivalent to $f_1 = f$ and $|W_j^1| = m - f - k$, because $m - f - k \leq m - f_1 - k \leq |W_j^1| \leq |W_j| \leq m - f - k$. There are two additional conditions: $k_0 = 1$ and $f_1 + k \leq m - 2$.

Lemma 25. *In the exceptional case of Lemma 13, where $k_0 = 1$, $f_1 + k \leq m - 2$, and either $f_0 = f = m - 3$ or $f_1 = f$ and $|W_j^1| = m - f - k$ for some $j \in I_1$, there exists a k-DPC$[\{(s_i, l_i, W_i) : i \in I\} \,|\, G_0 \oplus G_1, F]$.*

Proof. Part A: Let $k_0 = 1$, $f_1 + k \leq m - 2$, and $f_0 = f = m - 3$ for the first part. Then, we have $k_0 = k_1 = 1$ and $|W_i| \leq 1$ for every $i \in I$ (because $k \geq 2$, $f + k \leq m - 1$, and $f + k + |W_i| \leq m$ for all $i \in I$). From Lemma 20, there exist three vertices x_j, $j \in \{1, 2, 3\}$, such that $G_0 \setminus F_0$ has a Hamiltonian s_1–x_j path for every j. Thus, for at least one x_j, it holds that $\overline{x_j} \notin \{s_2\} \cup W_1^1$. It suffices to find a 2-DPC$[\{(\overline{x_j}, \Delta, W_1^1), (s_2, l_2, W_2^1)\} \,|\, G_1, \emptyset]$, and merge the Hamiltonian path and the 2-DPC with edge $(x_j, \overline{x_j})$.

Part B: We now assume that $k_0 = 1$, $f_1 + k \leq m - 2$, $f_1 = f$ ($f_0 = f_2 = 0$), and $|W_j^1| = m - f - k$ for some $j \in I_1$. Let $r \in I_1$ be an index such that $|W_r^1| = m - f - k$ and $l_r \geq l_j$ for all $j \in I_1$ with $|W_j^1| = m - f - k$. We assume w.l.o.g. that $\overline{s_r} \neq s_1$ or $l_r > l_j$ for any $j \in I_1 \setminus \{r\}$ with $|W_j^1| = m - f - k$. Then, we have $W_r^0 = \emptyset$ and, moreover, $l_r \geq 2$ from the assumption that $W_i = \emptyset$ whenever $l_i = 1$. Contrary to all the previous cases, a k_1-DPC of G_1 will be utilized for the construction of a k-DPC in $G_0 \oplus G_1$. For $l'_r = l_r + \Delta$, there exists a k_1-DPC$[\{(s_r, l'_r, W_1^1)\} \cup \{(s_j, l_j, W_j^1) : j \in I_1 \setminus \{r\}\} \,|\, G_1, F_1]$ by the induction hypothesis (because $k_1 = k - 1$). Let the s_r-path in the k_1-DPC be $(v_1, v_2, \ldots, v_{l'_r})$, where $s_r = v_1$. Let $z = v_{l_r}$, $x = v_{l_r+1}$, $y = v_{l'_r}$, and $v = v_{l_r-1}$. In addition, let $u = v_{l_r-2}$ if $l_r \geq 3$. Note that $y \notin W_1^1$. We denote by P_z and R_z, respectively, the subpaths $(v_1, \ldots, v_{l_r})$ and $(v_{l_r+1}, \ldots, v_{l'_r})$ of the s_r-path so that the s_r-path is the concatenation of P_z and R_z. Similarly, let $P_v = (v_1, \ldots, v_{l_r-1})$ and $R_v = (v_{l_r}, \ldots, v_{l'_r})$, etc. To obtain the desired k-DPC, it suffices to construct an s_r-path P_r of length l_r and an s_1-path P_1 of length l_1.

Case 1: $z \notin W_r^1$. If $\bar{x} \neq s_1$, we let $P_r = P_z$ and P_1 be the concatenation of a Hamiltonian s_1–$\bar{x}$ path of G_0 and R_z. If $\bar{x} = s_1$, it suffices to let $P_r = (P_v, \bar{v})$ and P_1 be the concatenation of a Hamiltonian s_1–$\bar{z}$ path of $G_0 \setminus \{\bar{v}\}$ and R_v. Note that $\bar{v} \notin W_r^0$, because $W_r^0 = \emptyset$.

Case 2: $z \in W_r^1$.

Case 2.1: $\bar{v} \neq s_1$. Let $P_r = (P_v, \bar{v})$. If $\bar{z} \neq s_1$, we let P_1 be the concatenation of a Hamiltonian s_1–$\bar{z}$ path of $G_0 \setminus \{\bar{v}\}$ and R_v. We now assume that $\bar{z} = s_1$. If either $m \geq 6$ or $m = 5$ and $|W_1^0| \leq 2$, we let P_1 be the concatenation of (s_1, R_v) and a Hamiltonian $\bar{y}$–w path of $G_0 \setminus \{s_1, \bar{v}\}$ for some $w \notin W_1^0$. The Hamiltonian $\bar{y}$–w path exists due to Lemma 2(a) if $m \geq 6$, and due to Lemma 20 if $m = 5$ and $|W_1^0| \leq 2$. Finally, if $m = 5$ and $|W_1^0| = 3$, it suffices to let P_1 be the concatenation of a Hamiltonian s_1–$\bar{y}$ path of $G_0 \setminus \{\bar{v}\}$ and the reverse of R_v. Note that $|W_1| \leq 3$ if $m = 5$.

Case 2.2: $\bar{v} = s_1$ and $l_r \geq 3$. Suppose $W_1^1 = \emptyset$ for the first subcase. For some neighbor $w \notin \{s_1, \bar{y}\}$ of $\bar{u}$ in G_0, there exists a Hamiltonian s_1–$\bar{y}$ path in $G_0 \setminus \{\bar{u}, w\}$ by Lemma 2(a) if $m \geq 6$, and by Lemma 18 if $m = 5$. Then, we let $P_r = (P_u, \bar{u}, w)$ and P_1 be the concatenation of the s_1–$\bar{y}$ path and the reverse of R_u. Suppose $|W_1^0| \leq m - 4$ for the second subcase. There exists a 3-DPC$[\{(\bar{u}, 2, \emptyset), (s_1, 1, \emptyset), (\bar{y}, 2^{m-1} - 3, W_1^0)\} \,|\, G_0, \emptyset]$ by the induction hypothesis, because (i) $f_0 + 3 \leq (m-1) - 1$ and (ii) $f_0 + 3 + |W_1^0| \leq m - 1$. Then, we let P_r be the concatenation of P_u and the $\bar{u}$-path of the 3-DPC, and let P_1 be the concatenation of (s_1, R_u) and the $\bar{y}$-path. There remains a subcase where $|W_1^0| \geq m - 3$ and $|W_1^1| \geq 1$, i.e., $f = 0$, $k = 2$, $|W_1^0| = m - 3$, and $|W_1^1| = 1$. Let $(\overline{s_r}, P, t_r, w, P', s_1)$ be a Hamiltonian $\overline{s_r}$–s_1 path of G_0, where the length of $(\overline{s_r}, P, t_r)$ is $l_r - 1$. (Note that $l_r < 2^{m-1}$.) It suffices to let $P_r = (s_r, \overline{s_r}, P, t_r)$ and P_1 be the concatenation of the reverse of (w, P', s_1) and a Hamiltonian $\bar{w}$–t_1 path of $G_1 \setminus \{s_r\}$ for some $t_1 \notin W_1^1$.

Case 2.3: $\bar{v} = s_1$ and $l_r = 2$, i.e., $s_r = v$ and $\overline{s_r} = s_1$. By the choice of s_r, we have $|W_j^1| < m - f - k$ for every $j \in I_1 \setminus \{r\}$. The condition $f_1 + k \leq m - 2$ will be applied to this case (whereas this condition has not been used so far in this proof). If $k = 2$, pick up a neighbor $t_r \in V(G_1)$ of s_r such that $(s_r, t_r) \notin F_1$ and $t_r \notin W_r^1$. Then, we let $P_r = (s_r, t_r)$ and P_1 be a Hamiltonian s_1–t_1 path in $G_0 \oplus G_1 \setminus (F \cup \{s_r, t_r\})$ for some $t_1 \notin W_1$. The Hamiltonian path exists by Lemma 2(a) if $f \leq m - 5$, and by Lemma 20 if $f = m - 4$ (where $|W_1|, |W_2| \leq 2$). Note that $f \leq m - 4$ because $f_1 + k \leq m - 2$ and $f_1 = f$. Hereafter, assume that $k \geq 3$. If $W_r^1 \setminus W_1^1 \neq \emptyset$, let t_1 be a vertex in $W_r^1 \setminus W_1^1$. For $W'_r = W_r^1 \setminus \{t_1\}$, there exists a k-DPC$[\{(t_1, \Delta, \emptyset), (s_r, l_r, W'_r)\} \cup \{(s_j, l_j, W_j^1) : j \in I_1 \setminus \{r\}\} \,|\, G_1, F_1]$, because (i) $f_1 + k \leq (m-1) - 1$, (ii) $f_1 + k + |W'_r| = f_1 + k + (m - f - k - 1) \leq m - 1$, and (iii) $f_1 + k + |W_j^1| \leq f_1 + k + (m - f - k - 1) \leq m - 1$ for every $j \in I_1 \setminus \{r\}$. Then, P_1 is obtained by concatenating a Hamiltonian s_1–$\bar{w}$ path of G_0, where w is the sink of the t_1-path in the k-DPC, and the reverse of the t_1-path. Now, we consider the case $W_r^1 \setminus W_1^1 = \emptyset$, i.e., $W_r^1 = W_1^1$. If $\Delta \geq 2$, there exists a k-DPC$[\{(w, \Delta, W'_1), (s_r, l_r, W'_r)\} \cup \{(s_j, l_j, W_j^1) : j \in I_1 \setminus \{r\}\} \,|\, G_1, F_1]$, where $w \in W_r^1$, $W'_r = W_r^1 \setminus \{w\}$, and $W'_1 = W_1^1 \setminus \{w\}$. The k-DPC exists because $f_1 + k \leq (m-1) - 1$ and $|W'_1|, |W'_r|, |W_j^1| \leq (m-1) - f - k$ for every $j \in I_1 \setminus \{r\}$. Then, we let P_1 be the concatenation of a Hamiltonian s_1–$\bar{w}$ path of G_0 and the w-path in the k-DPC.

Finally, assume $\Delta = 1$. We let $q \in I_1 \setminus \{r\}$ be an index such that $l_q \geq l_j$ for any $j \in I_1 \setminus \{r\}$. Then, we have $l_q \geq (2^{m-1} - \Delta - l_r)/(k-2) \geq (2^{m-1} - 3)/(m-4) > 5$. There are two subcases depending on the size of m. First, let $m \geq 6$. For $l'_q = l_q + 1$, there exists a k_1-DPC$[\{(s_q, l'_q, W_1^1)\} \cup \{(s_j, l_j, W_j^1) : j \in I_1 \setminus \{q\}\} \,|\, G_1, F_1]$ by the induction hypothesis, as $k_1 < k$.

Let the s_q-path in the DPC be $(w_1, \dots, w_{l_q}, w_{l_q+1})$, and let $u = w_{l_q-2}$, $v = w_{l_q-1}$, $y = w_{l_q+1}$. If $w_{l_q} \notin W_q^1$, it suffices to let $P_q = (w_1, \dots, w_{l_q})$ and P_1 be the concatenation of a Hamiltonian s_1–$\bar{y}$ path of G_0 and the one-vertex path (y). Suppose otherwise, i.e., $w_{l_q} \in W_q^1$. There exists a neighbor $w \in V(G_0)$ of $\bar{u}$ such that $w \notin \{s_1, \bar{v}\} \cup W_q^0$. Note that $|W_q^0| < m - f - k \leq m - 3$ because $w_{l_q} \in W_q^1$. Then, we let $P_q = (w_1, \dots, w_{l_q-2}, \bar{u}, w)$ and P_1 be the concatenation of a Hamiltonian s_1–$\bar{v}$ path in $G_0 \setminus \{\bar{u}, w\}$ and $(w_{l_q-1}, w_{l_q}, w_{l_q+1})$. Now suppose that $m = 5$. It follows that $f = 0$ and $k = 3$. Let t_r be a neighbor of s_r in G_1 such that $t_r \notin \{s_q\} \cup W_r^1$. Note that $|W_1|, |W_r|, |W_q| \leq 2$. Let $P_r = (s_r, t_r)$. By Lemma 19, there exists a vertex $w \in V(G_1) \setminus (\{s_r, t_r, s_q\} \cup W_1^1)$ such that $G_1 \setminus \{s_r, t_r, w\}$ has a Hamiltonian cycle C. Then, we let P_1 be the concatenation of a Hamiltonian s_1–$\bar{w}$ path of G_0 and the one-vertex path (w); P_q is obtained by removing one of the two edges on C incident to s_q because $|W_q^1| < m - f - k = 2$. The proof is complete. □

4.7. *Case when $k_0 = 1$ and $f_1 + k = m - 1$*

This is the last exceptional case that was deferred from Section 3.3. The condition $f_1 + k = m - 1$ is equivalent to that $f_1 = f$ and $f + k = m - 1$, because $f_1 + k \leq f + k \leq m - 1$.

Lemma 26. *When $k_0 = 1$, $f_1 = f$, and $f + k = m - 1$, there exists a k-DPC$[\{(s_i, l_i, W_i) : i \in I\} \,|\, G_0 \oplus G_1, F]$.*

Proof. It holds that $f_0 = f_2 = 0$ and $|W_i| \leq m - f - k = 1$ for every $i \in I$. Similar to the second part of the proof of Lemma 25, we will utilize a k_1-DPC for the sources of S_1 to construct the desired k-DPC of $G_0 \oplus G_1$. There are two cases.

Case 1: $|W_j^1| = 0$ for every $j \in I_1$. For $l_2' = l_2 + \Delta$, there exists a k_1-DPC$[\{(s_2, l_2', W_1^1)\} \cup \{(s_j, l_j, W_j^1) | j \in I_1 \setminus \{2\}\} \,|\, G_1, F_1]$ by the induction hypothesis, as $k_1 = k - 1$. Let the s_2-path in the k_1-DPC be $(v_1, v_2, \dots, v_{l_2'})$, where $s_2 = v_1$, and let $x = v_{l_2+1}$ and $y = v_{l_2'}$. To obtain the desired k-DPC, it suffices to construct an s_1-path P_1 and an s_2-path P_2. Let $P_2 = (v_1, \dots, v_{l_2})$. If $\bar{x} \neq s_1$, we let P_1 be the concatenation of a Hamiltonian s_1–$\bar{x}$ path of G_0 and $(v_{l_2+1}, \dots, v_{l_2'})$. If $\bar{x} = s_1$ and $\Delta \geq 2$ (where $x \neq y$), we let P_1 be the concatenation of $(s_1, v_{l_2+1}, \dots, v_{l_2'})$ and a Hamiltonian $\bar{y}$–t_1 path in $G_0 \setminus \{s_1\}$ for some vertex $t_1 \notin W_1^0$. Finally, suppose $\bar{x} = s_1$ and $\Delta = 1$ (where $x = y$). It follows that $l_2 \geq (2^{m-1} - \Delta)/(k - 1) \geq (2^{m-1} - 1)/(m - 2) \geq 5$. For $u = v_{l_2-2}$ and $v = v_{l_2-1}$, there exists a 3-DPC$[\{(\bar{u}, 2, W_2^0), (\bar{v}, 2^{m-1} - 3, W_1^0), (s_1, 1, \emptyset)\} \,|\, G_0, \emptyset]$ by the induction hypothesis. It suffices to redefine P_2 as the concatenation of $(v_1, \dots, v_{l_2-2})$ and the $\bar{u}$-path in the 3-DPC, and let P_1 be the concatenation of $(s_1, v_{l_2'}, v_{l_2}, v_{l_2-1})$ and the $\bar{v}$-path.

Case 2: $|W_j^1| = 1$ for some $j \in I_1$. The precondition of this case is very similar to that of Part B of the proof of Lemma 25, in that $k_0 = 1$, $f_1 = f$, and $|W_j^1| = m - f - k$ for some $j \in I_1$. (Note that $m - f - k = 1$ in this case.) In fact, the two differ only in the value of $f + k$; $f + k = m - 1$ in this case, whereas $f + k \leq m - 2$ in Part B. Fortunately, the condition $f + k \leq m - 2$ is applied only to Case 2.3 when we prove Part B, so the proof of Part B excluding Case 2.3 may be recycled. Refer to the second paragraph and Cases 1 through 2.2 of the proof of Lemma 25. Thus, under the condition $f + k = m - 1$, we will provide an alternative proof for Case 2.3, where $\overline{s_r} = s_1$, $l_r = 2$, $|W_r^1| = 1$, and $|W_j^1| = 0$ for every $j \in I_1 \setminus \{r\}$. First, suppose $k = 2$, implying $f_1 = m - 3$. Then, by Lemma 2(b), $G_1 \setminus F_1$ has a Hamiltonian cycle, $(u_1, u_2, \dots, u_{2^{m-1}}, u_1)$, where $s_r = u_1$. At least one of u_2 and $u_{2^{m-1}}$, say u_2, is not contained in W_r^1. Let $P_r = (s_r, u_2)$. Assuming w.l.o.g. that $u_{2^{m-1}} \notin W_1^1$, it suffices to let P_1 be the concatenation of a Hamiltonian s_1–$\overline{u_3}$ path of G_0 and $(u_3, \dots, u_{2^{m-1}})$. Now, suppose $k \geq 3$. Let q be an arbitrary index in $I_1 \setminus \{r\}$, so that $W_q^1 = \emptyset$. For $l_q' = l_q + \Delta$, there exists a k_1-DPC$[\{(s_q, l_q', W_1^1)\} \cup \{(s_j, l_j, W_j^1) : j \in I_1 \setminus \{q\}\} \,|\, G_1, F_1]$

by the induction hypothesis. Let the s_q-path in the k_1-DPC be $(v_1, \ldots, v_{l'_q})$, where $s_q = v_1$, and let $x = v_{l_q+1}$. Then, it suffices to let $P_q = (v_1, \ldots, v_{l_q})$ and P_1 be the concatenation of a Hamiltonian s_1–$\bar{x}$ path of G_0 and $(v_{l_q+1}, \ldots, v_{l'_q})$. This completes the proof. □

5. Conclusion

In this paper, we have proved that every m-dimensional RHL graph is f_b-fault k-path partitionable in a strong sense for any $f_b \geq 0$ and $k \geq 1$ subject to $f_b + k \leq m - 1$, where $m \geq 4$. The proofs are constructive, so they may lead to an efficient algorithm for constructing a prescribed-source-and-length disjoint path cover in an RHL graph with faults. Furthermore, the bounds $m-1$ and m on f_b+k and $f_b+k+|W_i|$, respectively, are both the best possible. The techniques suggested in this paper may be applicable to other interconnection networks including hypercubes.

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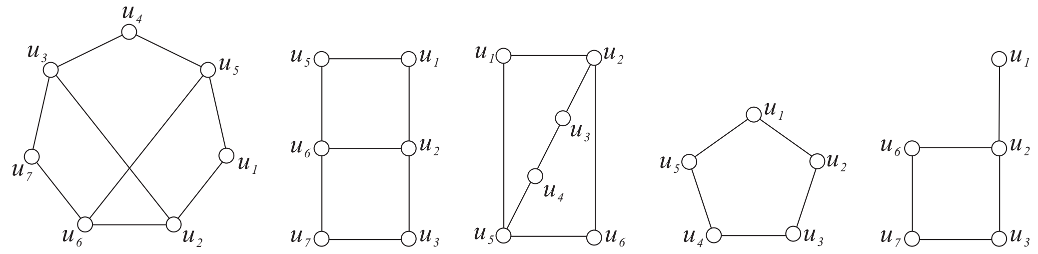

(a) H_1 (b) H_2^a (c) H_2^b (d) H_3^a (e) H_3^b

Fig. A.12: The graph $G(8,4) \setminus V(P)$, where P is a v_0-path with at most three vertices.

Appendix A. 3-Dimensional RHL Graph

In this section, we give proofs for Lemmas 3, 4, 14, and 16, which deal with the 3-dimensional RHL graph $G(8,4)$ shown in Fig. 2. Let G denote the graph $G(8,4)$ and P be a v_0-path of G. We begin with the graph $G \setminus V(P)$ obtained from G by deleting the vertices of P. If P is a one-vertex path, $G \setminus V(P)$ is isomorphic to the graph H_1 of Fig. A.12(a). If P is a two-vertex path, $G \setminus V(P)$ is isomorphic to H_2^a of Fig. A.12(b), derived from $P = (v_0, v_4)$, or isomorphic to H_2^b of Fig. A.12(c), derived from $P = (v_0, v_7)$. In addition, if P is a three-vertex path, $G \setminus V(P)$ is isomorphic to H_3^a of Fig. A.12(d) or H_3^b of Fig. A.12(e).

Lemma 27. *(a) The graph H_2^a has a Hamiltonian u_1–x path for each $x \in \{u_2, u_5, u_7\}$ and a Hamiltonian u_2–y path for each $y \in \{u_1, u_3\}$.*
(b) The graph H_2^b has a Hamiltonian u_1–x path for each $x \in \{u_3, u_4, u_6\}$ and a Hamiltonian u_3–y path for each $y \in \{u_1, u_6\}$.
(c) The graph H_3^b has a Hamiltonian u_1–x path for each $x \in \{u_3, u_6\}$.
(d) The graph H_2^b has a 2-DPC$[\{(s_1, l_1, \emptyset), (s_2, l_2, \emptyset)\} \,|\, H_2^b, \emptyset]$ if $s_1 \in \{u_1, u_6\}$, $s_2 \in V(H_2^b) \setminus \{s_1\}$, and $(l_1, l_2) \in \{(3,3), (4,2)\}$.

Proof. The proof is by an immediate inspection. □

Proof of Lemma 3. Due to the symmetric structure of G, we may assume w.l.o.g. that $l_2 \le l_1$, $s_2 = v_0$, and $s_1 \in \{v_1, v_2, v_3, v_4\}$. There are four cases depending on the length, l_2, of an s_2-path P_2. If $l_2 = 1$, then $G \setminus \{s_2\}$ (which is isomorphic to H_1) has a Hamiltonian cycle, from which we can extract an s_1-path, P_1, whose sink is not contained in W_1. For the second case, suppose $l_2 = 2$. If $v_4 \ne s_1$ and $W_2 \ne \{v_4\}$, it suffices to let $P_2 = (v_0, v_4)$ and extract P_1 from the Hamiltonian cycle of $G \setminus V(P_2)$, which is isomorphic to H_2^a. If $v_4 = s_1$, then it suffices to let $P_2 = (v_0, v_7)$ or (v_0, v_1), depending on whether $v_7 \notin W_2$ or not, and find the desired Hamiltonian s_1-path of $G \setminus V(P_2)$ by Lemma 27(b). If $W_2 = \{v_4\}$ (and $v_4 \ne s_1$), we let $P_2 = (v_0, v_7)$ for $s_1 \in \{v_1, v_3\}$ and let $P_2 = (v_0, v_1)$ for $s_1 = v_2$; P_1 can be constructed in $G \setminus V(P_2)$ by Lemma 27(b). For the third case, suppose $l_2 = 3$. If $W_2 \ne \{v_6\}$, we let $P_2 = (v_0, v_7, v_6)$ and extract P_1 from the Hamiltonian cycle of $G \setminus V(P_2)$. Now, let $W_2 = \{v_6\}$. If $s_1 = v_1$, it suffices to let $P_2 = (v_0, v_4, v_5)$ and extract P_1 from $G \setminus V(P_2)$ by Lemma 27(c). If $s_1 = v_2$, either $\{(v_0, v_4, v_3), (v_2, v_1, v_5, v_6, v_7)\}$ or $\{(v_0, v_1, v_5), (v_2, v_6, v_7, v_3, v_4)\}$ will be the desired 2-DPC. If $s_1 \in \{v_3, v_4\}$, we let $P_2 = (v_0, v_1, v_2)$ and extract P_1 from the Hamiltonian cycle of $G \setminus V(P_2)$. For the last case, suppose $l_2 = 4$. If $s_1 = v_4$, either $\{(v_0, v_1, v_2, v_3), (v_4, v_5, v_6, v_7)\}$ or $\{(v_0, v_7, v_6, v_5), (v_4, v_3, v_2, v_1)\}$ is the desired

2-DPC unless either $W_1 = \{v_1\}$ and $W_2 = \{v_3\}$ or $W_1 = \{v_7\}$ and $W_2 = \{v_5\}$, i.e., unless the configuration is equivalent to the exception of this lemma. If $s_1 \in \{v_1, v_2\}$, it suffices to extract P_2 and P_1, respectively, from the length-four cycles $(v_0, v_7, v_3, v_4, v_0)$ and $(v_1, v_2, v_6, v_5, v_1)$. If $s_1 = v_3$, P_2 and P_1 are extracted from pairwise disjoint cycles $(v_0, v_1, v_5, v_4, v_0)$ and $(v_2, v_3, v_7, v_6, v_2)$. This completes the proof. □

Proof of Lemma 4. Proof for (a): Assume w.l.o.g. that $s_2 = v_0$ and $s_1 \in \{v_1, v_2, v_3, v_4\}$. First, let $(l_1, l_2) = (6, 2)$. If $s_1 = v_1$, $G \setminus V(P_2)$, where $P_2 = (v_0, v_4)$, has a Hamiltonian s_1–x path for $x \in \{v_2, v_5, v_7\}$ by Lemma 27(a) and, moreover, $G \setminus V(P_2)$, where $P_2 = (v_0, v_7)$, has a Hamiltonian s_1–y path for $y \in \{v_3, v_4, v_6\}$ by Lemma 27(b). Thus, the desired 2-DPC exists because $|W_1| \leq 3$ and there are more than three possible sink positions of P_1. The same argument applies to the remaining cases. If $s_1 = v_2$, $G \setminus V(P_2)$, where $P_2 = (v_0, v_4)$, has a Hamiltonian s_1–x path for $x \in \{v_1, v_3\}$, and $G \setminus V(P_2)$, where $P_2 = (v_0, v_1)$, has a Hamiltonian s_1–y path for $y \in \{v_4, v_5, v_7\}$. If $s_1 = v_3$, $G \setminus V(P_2)$, where $P_2 = (v_0, v_4)$, has a Hamiltonian s_1–x path for $x \in \{v_2, v_5, v_7\}$, and $G \setminus V(P_2)$, where $P_2 = (v_0, v_7)$, has a Hamiltonian s_1–y path for $y \in \{v_1, v_6\}$. If $s_1 = v_4$, $G \setminus V(P_2)$, where $P_2 = (v_0, v_7)$, has a Hamiltonian s_1–x path for $x \in \{v_1, v_6\}$, and $G \setminus V(P_2)$, where $P_2 = (v_0, v_1)$, has a Hamiltonian s_1–y path for $y \in \{v_2, v_7\}$.

Now, let $(l_1, l_2) = (5, 3)$. If we let $P_2 = (v_0, v_7, v_6)$, then $G \setminus V(P_2)$ has a Hamiltonian cycle, from which we can extract two Hamiltonian s_1-paths whose sinks are different from each other. It suffices to provide two more sink positions of P_1 other than the two obtained just before. If $s_1 = v_1$, $G \setminus V(P_2)$, where $P_2 = (v_0, v_4, v_5)$, has a Hamiltonian s_1–y path for $y \in \{v_3, v_6\}$ by Lemma 27(c). The desired 2-DPC exists because we have four possible sink positions, $\{v_2, v_5\} \cup \{v_3, v_6\}$, of P_1. If $s_1 = v_2$, $G \setminus V(P_2)$, where $P_2 = (v_0, v_4, v_3)$, has a Hamiltonian s_1–v_7 path, and $G \setminus V(P_2)$, where $P_2 = (v_0, v_1, v_5)$, has a Hamiltonian s_1–v_4 path. If $s_1 = v_3$, $G \setminus V(P_2)$, where $P_2 = (v_0, v_1, v_2)$, has a Hamiltonian s_1–v_7 path, and $G \setminus V(P_2)$, where $P_2 = (v_0, v_4, v_5)$, has a Hamiltonian s_1–v_1 path. If $s_1 = v_4$, $G \setminus V(P_2)$, where $P_2 = (v_0, v_7, v_3)$, has a Hamiltonian s_1–y path for $y \in \{v_1, v_6\}$. The proof of (a) is complete.

Proof for (b): Assume w.l.o.g. that $s_1 = v_0$ and $s_2 \in \{v_1, v_2, v_3, v_4\}$. If $v_6 \notin W_1$, it suffices to let $P_1 = (v_0, v_7, v_6)$ and find a Hamiltonian s_2-path in $G \setminus V(P_1)$. Also, if $v_5 \notin W_1$, we let $P_1 = (v_0, v_4, v_5)$ and $P_2 = (v_1, v_2, v_6, v_7, v_3)$ or its reversal $(v_3, v_7, v_6, v_2, v_1)$ for $s_2 \in \{v_1, v_3\}$, and let $P_1 = (v_0, v_1, v_5)$ and $P_2 = (v_2, v_6, v_7, v_3, v_4)$ or its reversal for $s_2 \in \{v_2, v_4\}$. There remains the case where $W_1 = \{v_5, v_6\}$. If $s_2 = v_1$, we have $P_1 = (v_0, v_7, v_3)$ and $P_2 = (v_1, v_2, v_6, v_5, v_4)$. If $s_2 = v_2$, we have $P_1 = (v_0, v_4, v_3)$ and $P_2 = (v_2, v_1, v_5, v_6, v_7)$. Finally, if $s_2 \in \{v_3, v_4\}$, it suffices to let $P_1 = (v_0, v_1, v_2)$ and find a Hamiltonian s_2-path in $G \setminus V(P_1)$, which completes the proof for (b). □

Proof of Lemma 14. Proof for (a): We assume w.l.o.g. that $(x, y) = (v_0, v_4)$ or (v_0, v_7). First, let $(x, y) = (v_0, v_4)$ and assume further that $s = v_0$ and $t \in \{v_1, v_2, v_3, v_4\}$. For each t, there exists a Hamiltonian s–t path in $G \setminus F$ as follows: $(v_0, v_7, v_6, v_5, v_4, v_3, v_2, v_1)$, $(v_0, v_1, v_5, v_4, v_3, v_7, v_6, v_2)$, $(v_0, v_7, v_6, v_2, v_1, v_5, v_4, v_3)$, and $(v_0, v_1, v_2, v_3, v_7, v_6, v_5, v_4)$. Second, let $(x, y) = (v_0, v_7)$ and assume further that $s = v_0$. Then, $G \setminus F$ has a Hamiltonian cycle, $(v_0, v_1, v_2, v_3, v_7, v_6, v_5, v_4, v_0)$, from which we can extract a Hamiltonian s–t path for each $t \in \{v_1, v_4\}$. For each of the remaining t, $G \setminus F$ has a Hamiltonian s–t path, as shown below: $(v_0, v_4, v_3, v_7, v_6, v_5, v_1, v_2)$, $(v_0, v_4, v_5, v_1, v_2, v_6, v_7, v_3)$, $(v_0, v_4, v_3, v_7, v_6, v_2, v_1, v_5)$, $(v_0, v_4, v_5, v_1, v_2, v_3, v_7, v_6)$, and $(v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7)$.

Proof for (b): Recall that $G \setminus \{v_0\}$ is isomorphic to H_1 shown in Fig. A.12(a). The graph $G \setminus \{v_0\}$ has a (unique) Hamiltonian cycle, $C = (v_3, v_4, v_5, v_1, v_2, v_6, v_7, v_3)$, from which a Hamiltonian s–t path can be extracted if $(s, t) \in E(C)$. For each of the remaining pairs of s and t, we will construct a Hamiltonian s–t path. First, let $s = v_4$. It suffices to consider $t \in \{v_1, v_2\}$ due to the symmetry, where we have $(v_4, v_5, v_6, v_7, v_3, v_2, v_1)$ and $(v_4, v_3, v_7, v_6, v_5, v_1, v_2)$. Second, let $s = v_1$ and $t \in \{v_4, v_7\}$. For $t = v_4$, the desired path is the reverse of the aforementioned Hamiltonian v_4–v_1 path; for $t = v_7$, we have $(v_1, v_2, v_6, v_5, v_4, v_3, v_7)$. Third, let $s = v_2$ and $t = v_4$. The desired path is the reverse of the Hamiltonian v_4–v_2 path. Finally, let $s = v_3$. We have nothing to prove, as the desired Hamiltonian paths are obtained from C. Therefore, the lemma is proved. □

Proof of Lemma 16. First, let $(l_1, l_2, l_3) = (3, 3, 2)$. Assume w.l.o.g. that $s_3 = v_0$. If $\{v_1, v_7\} = \{s_1, s_2\}$, we have a 3-DPC $\{(v_0, v_4), (v_1, v_2, v_3), (v_7, v_6, v_5)\}$. If $|\{v_1, v_7\} \cap \{s_1, s_2\}| = 1$, say $s_1 = v_1$, it suffices to let $P_3 = (v_0, v_7)$ and find s_1- and s_2-paths in $G \setminus V(P_3)$ by Lemma 27(d). Now, we have $\{v_1, v_7\} \cap \{s_1, s_2\} = \emptyset$. If $\{v_2, v_6\} \cap \{s_1, s_2\} \neq \emptyset$, say $s_1 = v_2$, it suffices to let $P_3 = (v_0, v_1)$ and apply Lemma 27(d) again to $G \setminus V(P_3)$. There remain only two cases up to symmetry: If $\{s_1, s_2\} = \{v_3, v_5\}$, we have $\{(v_0, v_4), (v_3, v_2, v_1), (v_5, v_6, v_7)\}$; if $\{s_1, s_2\} = \{v_3, v_4\}$, we have $\{(v_0, v_7), (v_3, v_2, v_1), (v_4, v_5, v_6)\}$.

Second, let $(l_1, l_2, l_3) = (4, 2, 2)$. Assume w.l.o.g. that $s_1 = v_0$. If $\{v_1, v_2\} = \{s_2, s_3\}$, we have a 3-DPC $\{(v_0, v_7, v_3, v_4), (v_1, v_5), (v_2, v_6)\}$. If $|\{v_1, v_2\} \cap \{s_2, s_3\}| = 1$, say $s_3 \in \{v_1, v_2\}$ and $s_2 \notin \{v_1, v_2\}$, it suffices to let $P_3 = (v_1, v_2)$ or its reversal and find s_1- and s_2-paths in $G \setminus V(P_3)$ by Lemma 27(d). A 3-DPC can be constructed symmetrically if $|\{v_6, v_7\} \cap \{s_2, s_3\}| \geq 1$. Now, we have $s_2, s_3 \in \{v_3, v_4, v_5\}$. Then, there remain only two cases up to symmetry: If $\{s_2, s_3\} = \{v_3, v_5\}$, we have $\{(v_0, v_7, v_6, v_2), (v_3, v_4), (v_5, v_1)\}$; if $\{s_2, s_3\} = \{v_4, v_5\}$, we have $\{(v_0, v_7, v_6, v_2), (v_4, v_3), (v_5, v_1)\}$.

Finally, let $(l_1, l_2, l_3) = (5, 2, 1)$. Assume w.l.o.g. that $s_3 = v_0$. The graph $G \setminus \{s_3\}$ is isomorphic to H_1 of Fig. A.12(a), so it suffices to prove that there exists a 2-DPC$[\{(s_1, 5, \emptyset), (s_2, 2, \emptyset)\} \mid H_1, \emptyset]$ for any s_1 and s_2. Suppose $s_1 \in \{u_4, u_5, u_1, u_2\}$ due to the symmetric structure of H_1. The graph H_1 has a Hamiltonian cycle, $C = (u_3, u_4, u_5, u_1, u_2, u_6, u_7, u_3)$, from which the desired 2-DPC can be extracted if $(s_1, s_2) \in E(C)$ or if $(s_1, u_i), (s_2, u_i) \in E(C)$ for some $u_i \in V(C)$. Then, there remain four cases up to symmetry: (i) If $s_1 = u_4$ and $s_2 = u_2$, we have a 2-DPC $\{(u_4, u_5, u_6, u_7, u_3), (u_2, u_1)\}$. (ii) If $s_1 = u_5$ and $s_2 \in \{u_6, u_7\}$, we have a 2-DPC $\{(u_5, u_1, u_2, u_3, u_4), P_2\}$, where $P_2 = (u_6, u_7)$ if $s_2 = u_6$; $P_2 = (u_7, u_6)$ otherwise. (iii) If $s_1 = u_1$ and $s_2 \in \{u_7, u_3\}$, we have a 2-DPC $\{(u_1, u_2, u_6, u_5, u_4), P_2\}$, where $P_2 = (u_7, u_3)$ or its reversal depending on whether $s_2 = u_7$ or not. (iv) If $s_1 = u_2$ and $s_2 \in \{u_3, u_4\}$, we have a 2-DPC $\{(u_2, u_1, u_5, u_6, u_7), P_2\}$, where $P_2 = (u_3, u_4)$ or its reversal depending on whether $s_2 = u_3$ or not. This completes the entire proof. □