Fault-Tolerant Embedding of Starlike Trees into Restricted Hypercube-Like Graphs

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Abstract

A d-*starlike tree* (or a d-*quasistar*) is a subdivision of a star tree of degree d. A family of hypercube-like interconnection networks, called *restricted hypercube-like graphs*, includes most non-bipartite hypercube-like networks found in the literature such as twisted cubes, crossed cubes, Möbius cubes, recursive circulant $G(2^m, 4)$ of odd m, etc. In this paper, we prove that given an arbitrary fault-free vertex r in an m-dimensional restricted hypercube-like graph with a set F of faults (vertex and/or edge faults) and d positive integers, $l_1, l_2, \ldots, l_d$, whose sum is equal to the number of fault-free vertices minus one, there exists a d-starlike tree rooted at r, each of whose subtrees forms a fault-free path on l_i vertices for $i \in \{1, 2, \ldots, d\}$, provided $|F| \leq m-2$ and $|F|+d \leq m$. The bounds on $|F|$ and $|F| + d$ are the maximum possible.

Keywords: Quasistar, spanning tree, RHL graph, disjoint path cover, Hamiltonian path, interconnection network.

1. Introduction

Let G be an undirected graph whose vertex and edge sets are denoted by $V(G)$ and $E(G)$, respectively. A *path* from $s \in V(G)$ to $t \in V(G)$, referred to as an s–t path, is a sequence $\langle u_1, u_2, \ldots, u_l \rangle$ of distinct vertices of G such that $u_1 = s$, $u_l = t$, and $(u_i, u_{i+1}) \in E(G)$ for all $i \in \{1, 2, \ldots, l-1\}$. If $l \geq 3$ and $(u_l, u_1) \in E(G)$, the sequence is called a *cycle*. Also, an s-*path* refers to a path starting at vertex s. A d-*star* is a tree of order $d+1$ which consists of d leaves and a single internal vertex, called the root, of degree d. A d-*starlike tree* (or a d-*quasistar*) is a subdivision of a d-star, where a *subdivision* of G is a graph

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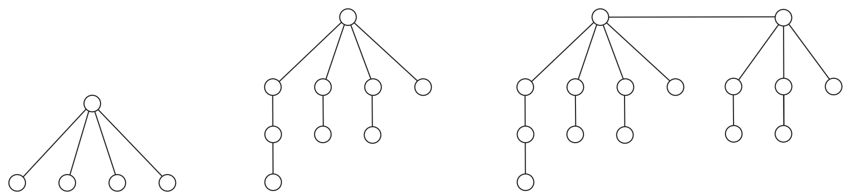


(a) 4-star (b) 4-starlike tree (c) 4-double starlike tree

Fig. 1: Examples of a d-star, a d-starlike tree, and a d-double starlike tree, where $d = 4$.

formed from G by the insertion of vertices of degree two into the edges of G. Let $T(l_1, l_2, \ldots, l_d)$ denote a d-starlike tree (of order $\sum_{i=1}^{d} l_i + 1$) composed of a root and d paths on l_i, $i \in \{1, 2, \ldots, d\}$, vertices as subtrees of the root. In addition, a *d-double starlike tree* is a tree obtained by adding an edge between the roots of a d-starlike tree and a d'-starlike tree with $d' \leq d$. Examples of a d-star, a d-starlike tree, and a d-double starlike tree are shown in Fig. 1.

For two graphs H and G of the same order, an *embedding* of H into G is a one-to-one mapping ϕ of $V(H)$ into $V(G)$ such that $(\phi(u), \phi(v)) \in E(G)$ whenever $(u, v) \in E(H)$. If there is an embedding of H into G (or equivalently, H is isomorphic to a spanning subgraph of G), we say that H *spans* G. Much research has been devoted to investigating if an interconnection network contains a certain class of trees as spanning subgraphs. For spanning trees of hypercubes, one of the well-known interconnection networks, various trees were studied such as binomial trees, caterpillars [5, 9], double-rooted complete binary trees [15], etc. Other containment results can be found in [15, 16].

With regard to the embedding of starlike trees, Nebeskỳ [17] proved that every equitable d-starlike tree of order 2^m with $d \leq m$ spans an m-dimensional hypercube Q_m, where a bipartite graph is said to be *equitable* if the graph has a proper bicoloring in which both color sets have the same cardinality. Note that no non-equitable d-starlike tree of order 2^m spans Q_m because Q_m itself is equitable. Kobeissi and Mollard [13, 14] and Choudum et al. [3] showed that every equitable d-double starlike tree of order 2^m with $d + 1 \leq m$ spans Q_m. Also, it was proved by the authors [24] that every d-starlike tree of order 2^m with $d \leq m$, whether equitable or not, spans an m-dimensional restricted hypercube-like graph.

Meanwhile, an interconnection network is frequently represented as a graph, in which the vertices and edges correspond to nodes and links, respectively. Since node and/or link failure is inevitable in a large network, fault tolerance[8, 11, 20, 21, 22] is essential to the network performance. It is desirable that when nodes and/or links of the network fail, these faulty elements are isolated. Extending the embedding of [24] in this paper, we study the fault-tolerant embedding of starlike trees into Restricted Hypercube-Like graphs (RHL graphs for short) [20], which are a subset of non-bipartite hypercube-like graphs that have received much attention over the past few decades. The class includes most non-bipartite hypercube-like networks found in the literature, as the following examples: twisted cubes [10], crossed cubes [7], Möbius cubes [4], recursive circulant $G(2^m, 4)$ of odd m [19], multiply twisted cubes [6], Mcubes [25], and generalized twisted cubes [1]. An m-dimensional RHL graph (whose definition is deferred to the next section) has 2^m vertices of degree m.

Throughout this paper, let F denote a set of faults (vertices and/or edges) corresponding to the set of node and/or link failures. An embedding into a graph G that contains some faults naturally means an embedding into the graph, $G-F$, with the faults deleted. Our main result on the fault-tolerant embedding of a d-starlike tree into an RHL graph with faults can be stated as follows:

Theorem 1. *Let G be an m-dimensional RHL graph, $m \geq 3$, with a fault set $F \subset V(G) \cup E(G)$ such that $|F| \leq m-2$. Then, for every $r \in V(G) \setminus F$, there exists an embedding of a d-starlike tree of order $|V(G) \setminus F|$ with $|F| + d \leq m$ into $G - F$ that maps the root of the starlike tree to r.*

The bound, m, on $|F|+d$ is the maximum possible; suppose otherwise, $G-F$ would not contain a d-starlike tree rooted at $r \in V(G)\setminus F$ when all the faults are neighbors of r. Also, the bound, $m-2$, on the number of faults is the maximum possible; suppose otherwise, $G-F$ would not contain a 1-starlike tree rooted at $r \in V(G) \setminus F$ when the neighbors of u excluding r are all faults for a neighbor u of r.

The rest of this paper is organized as follows: Section 2 gives preliminaries. Procedures for embedding a d-starlike tree are developed and their correctness is proved in Section 3. Finally, the paper is concluded in Section 4.

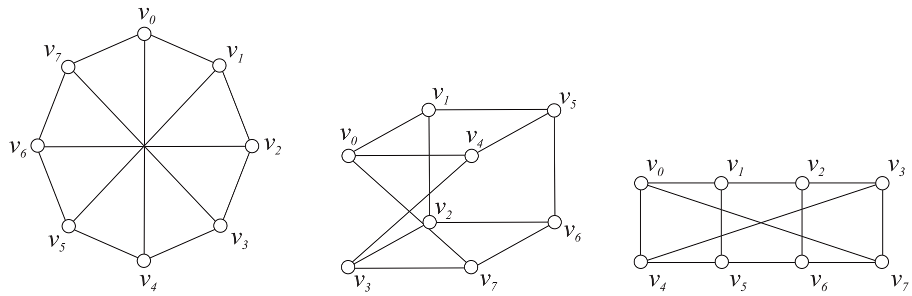
2. Preliminaries

2.1. Restricted hypercube-like graphs

A 3-dimensional RHL graph is isomorphic to recursive circulant $G(8, 4)$ whose vertex and edge sets, respectively, are $\{v_i : 0 \leq i \leq 7\}$ and $\{(v_i, v_j) : i+1 \text{ or } i+4 \equiv j \pmod 8\}$. The 3-dimensional RHL graph is isomorphic to a 3-dimensional twisted cube TQ_3 and to a Möbius ladder with four spokes, as shown in Fig. 2. An m-dimensional RHL graph, $m \geq 4$, is recursively defined with a graph operation $\oplus$. Given two graphs G_0 and G_1 of the same order and a bijection ϕ from $V(G_0)$ to $V(G_1)$, we denote by $G_0 \oplus_\phi G_1$ the graph whose vertex set is $V(G_0) \cup V(G_1)$ and edge set is $E(G_0) \cup E(G_1) \cup \{(v, \phi(v)) : v \in V(G_0)\}$. To simplify the notation, we often omit the bijection ϕ from $\oplus_\phi$.

Definition 1. ([20]) A graph that belongs to RHL_m is called an *m-dimensional RHL graph*, where

- $RHL_3 = \{G(8, 4)\}$, and
- $RHL_m = \{G_0 \oplus_\phi G_1 : G_0, G_1 \in RHL_{m-1}, \phi$ is a bijection from $V(G_0)$ to $V(G_1)\}$ for $m \geq 4$.



(a) $G(8, 4)$ (b) TQ_3 (c) Möbius ladder

Fig. 2: The 3-dimensional RHL graph.

Lemma 1. *([12]) Let G be an m-dimensional RHL graph, where $m \geq 3$.*
(a) G has 2^m vertices of degree m. Moreover, it is non-bipartite.
(b) G has no triangle (cycle of length three).
(c) There are at most two common neighbors for any pair of vertices in G.

For the proof of our theorem, we will utilize the fault-Hamiltonicity of RHL graphs, stated in Lemma 2 below. A path of a graph that visits each vertex exactly once is a *Hamiltonian path*; a cycle that visits each vertex exactly once is a *Hamiltonian cycle*. A graph G is said to be *f_b-fault Hamiltonian* (resp. *f_b-fault Hamiltonian-connected*) if there exists a Hamiltonian cycle (resp. if each pair of vertices are joined by a Hamiltonian path) in $G - F$ for any fault set $F \subset V(G) \cup E(G)$ with $|F| \leq f_b$. If we need to build a Hamiltonian path of the 3-dimensional RHL graph, $G(8, 4)$, with a single fault, we will refer to Lemma 3.

Lemma 2. *([20]) (a) Every m-dimensional RHL graph, $m \geq 3$, is $(m-3)$-fault Hamiltonian-connected.*
(b) Every m-dimensional RHL graph, $m \geq 3$, is $(m-2)$-fault Hamiltonian.

Lemma 3. *([23]) Let $G = G(8, 4)$ and let $F \subset V(G) \cup E(G)$ be its fault set with $|F| = 1$.*
(a) If $F = \{(v_i, v_j)\}$, every pair of vertices $s \in \{v_i, v_j\}$ and $t \in V(G) \setminus \{s\}$ are joined by a Hamiltonian path of $G - F$.
(b) If $F = \{v_0\}$, every pair of vertices $s \in \{v_4\}$ and $t \in V(G) \setminus (F \cup \{s\})$ are joined by a Hamiltonian path of $G - F$. Moreover, there exists a Hamiltonian path joining s and t in $G - F$ if $s = v_1$ and $t \in \{v_2, v_4, v_5, v_7\}$, $s = v_2$ and $t \in \{v_1, v_4, v_6\}$, or $s = v_3$ and $t \in \{v_4, v_7\}$.

2.2. Disjoint path covers

The disjoint-path-cover properties of RHL graphs are also utilized for the embedding of starlike trees. A *k-disjoint path cover* (k-DPC for short) of a graph is a set of k pairwise vertex-disjoint paths that altogether cover every vertex of the graph. Among several variants [17, 18, 21, 26], we are concerned with a disjoint path cover in which sources and path lengths are prescribed [2, 17, 23].

Given a set of k sources, $S = \{s_1, s_2, \ldots, s_k\}$, in a graph G, associated with k positive integers, $l_1, l_2, \ldots, l_k$, such that $\sum_{i=1}^{k} l_i = |V(G)|$, a *prescribed-source-and-length k-DPC* of G is a disjoint path cover composed of k vertex-disjoint paths, each of whose paths is an s_i-path on l_i vertices for $i \in \{1, 2, \ldots, k\}$. Hereafter, a disjoint path cover is assumed to be a prescribed-source-and-length type. A graph G is called *k-path partitionable* if G has a k-DPC for any set of k sources, $s_1, s_2, \ldots, s_k$, associated with any k positive integers, $l_1, l_2, \ldots, l_k$, such that $\sum_{i=1}^{k} l_i = |V(G)|$. A graph G is said to be *f_b-fault k-path partitionable* if $G - F$ is k-path partitionable for any fault set $F \subset V(G) \cup E(G)$ with $|F| \le f_b$.

Definition 2. ([23]) A graph G is called *f_b-fault k-path partitionable in a strong sense* if G is f_b-fault k-path partitionable and, moreover, for any given k subsets W_i of $V(G)$, $i \in \{1, 2, \ldots, k\}$, such that (i) $|W_i| \le \delta(G) - f_b - k$ and (ii) $s_i \notin W_i$ whenever $l_i = 1$, there exists a k-DPC in $G - F$ each of whose paths, P_i, contains l_i vertices and runs from s_i to a sink vertex not contained in W_i for $i \in \{1, 2, \ldots, k\}$. Here, $\delta(G)$ denotes the minimum degree of G.

Given s_i, l_i, and W_i, $i \in \{1, 2, \ldots, k\}$, in a graph G with a fault set F, the disjoint path cover of Definition 2 is denoted by k-DPC$[\{(s_1, l_1, W_1), (s_2, l_2, W_2), \ldots, (s_k, l_k, W_k)\}|G, F]$. The f_b-fault k-path partitionability in a strong sense was studied in [23], as described in Theorem 2. The theorem does not hold true for $m = 3$; nevertheless, the 3-dimensional RHL graph $G(8, 4)$ still has good DPC properties shown in Lemmas 4 and 5.

Theorem 2. *([23]) Every m-dimensional RHL graph is f_b-fault k-path partitionable in a strong sense for any $f_b \ge 0$ and $k \ge 1$ subject to $f_b + k \le m - 1$, where $m \ge 4$.*

Lemma 4. *([23]) There exists a 2-DPC$[\{(s_1, l_1, W_1), (s_2, l_2, W_2)\} \,|\, G(8, 4), \emptyset]$ for every pair of triplets (s_i, l_i, W_i), $i \in \{1, 2\}$, such that $|W_i| \le 1$ and $l_1 + l_2 = 8$, with the unique exception up to symmetry that $s_1 = v_0$, $s_2 = v_4$, $l_1 = l_2 = 4$, $W_1 = \{v_3\}$, and $W_2 = \{v_1\}$.*

Lemma 5. *([23]) If $(l_1, l_2, l_3) = (3, 3, 2)$, $(4, 2, 2)$, or $(5, 2, 1)$, then there exists a 3-DPC$[\{(s_1, l_1, \emptyset), (s_2, l_2, \emptyset), (s_3, l_3, \emptyset)\} \,|\, G(8, 4), \emptyset]$ for any three distinct sources s_1, s_2, and s_3.*

3. Embedding starlike trees into RHL graphs

This section is devoted to proving our main theorem on the embedding of starlike trees into RHL graphs. In fact, we will give a proof for a stronger result, stated in Theorem 3 below, than Theorem 1 asserts, in that no leaf of the starlike tree is allowed to be mapped to a vertex contained in a given vertex subset W of size $m - |F| - d$ or less.

Theorem 3. *Let G be an m-dimensional RHL graph, $m \ge 3$, with a fault set $F \subset V(G) \cup E(G)$ such that $|F| \le m - 2$. Then, for every $r \in V(G) \setminus F$ and $W \subset V(G)$ with $|W| \le m - |F| - d$, there exists an embedding of a d-starlike* *tree of order $|V(G) \setminus F|$ with $|F| + d \leq m$ into $G - F$ that maps the root of the starlike tree to r and maps the leaves to vertices not contained in W.*

The proof will proceed by induction on m. Let $G = G_0 \oplus G_1$, where G_0 and G_1 are $(m-1)$-dimensional RHL graphs. We need some notation for the inductive proof. We denote by F_i the fault set in G_i, $i \in \{0, 1\}$, and by F_2 the set of edge faults between G_0 and G_1. Then, $F = F_0 \cup F_1 \cup F_2$. Let $f = |F|$, $f_0 = |F_0|$, $f_1 = |F_1|$, and $f_2 = |F_2|$, so that $f = f_0 + f_1 + f_2$. In addition, we let $F_i^v = F \cap V(G_i)$, $F_i^e = F \cap E(G_i)$, $f_i^v = |F_i^v|$, $f_i^e = |F_i^e|$, and $f_i = f_i^v + f_i^e$ for $i \in \{0, 1\}$, so that $f^v = f_0^v + f_1^v$, $f^e = f_0^e + f_1^e + f_2$, and $f = f^v + f^e$. Similarly, let $W_0 = W \cap V(G_0)$ and $W_1 = W \cap V(G_1)$, so that $W = W_0 \cup W_1$.

Let $T(l_1, l_2, \ldots, l_d)$ be a d-starlike tree of order $|V(G) \setminus F|$, where $1 + \sum_{i=1}^{d} l_i = |V(G) \setminus F| = 2^m - f^v$. It is assumed without loss of generality that

$$l_1 \geq l_2 \geq \cdots \geq l_d. \tag{1}$$

Given a fault set F, a fault-free vertex r, and a vertex subset W in G subject to

1. $f \leq m - 2$, and
2. $f + d + |W| \leq m$,

we will show that $G - F$ has a spanning tree, isomorphic to $T(l_1, l_2, \ldots, l_d)$, that is rooted at r and moreover, any of whose leaves is not contained in W. It can be assumed that

$$f + d + |W| = m, \tag{2}$$

because it suffices to pick up a superset W' of W, instead of W, such that $|W'| = m - f - d$.

First of all, we consider three extremal cases, (i) $d = 1$, (ii) $f = m - 2$, and (iii) $l_2 = 1$, each of which allows a simple proof from the fault-Hamiltonicity of RHL graphs (not from the induction hypothesis), in the subsequent Lemmas 6, 7, and 8. The three lemmas are followed by Lemma 9 dealing with the base step of $m = 3$.

Lemma 6. *If $d = 1$, there exists a required d-starlike tree in $G - F$.*

Proof. The proof is an immediate consequence of Lemma 2(a) if $f \leq m - 3$, and of Lemma 2(b) if $f = m - 2$ ($|W| = 1$). Notice that if $f = m - 2$ ($|W| = 1$), from a Hamiltonian cycle of $G - F$, we can extract two Hamiltonian r-paths, one of which has a sink vertex not contained in W. ■

Lemma 7. *If $f = m - 2$, there exists a required d-starlike tree in $G - F$.*

Proof. We assume $d = 2$ ($W = \emptyset$) from Lemma 6 and the equality (2). A 2-starlike tree $T(l_1, l_2)$ rooted at r can be extracted easily from a Hamiltonian cycle of $G - F$, which exists by Lemma 2(b). ■

Lemma 8. *If $l_2 = 1$, there exists a required d-starlike tree in $G - F$.*

Proof. We have $l_2 = \cdots = l_d = 1$ from the assumption (1). There are two cases. Suppose $W = \emptyset$ for the first case. There is a set X of $d-2$ neighbors of r such that $x, (r,x) \notin F$ for all $x \in X$, where $|F \cup X| = m-2$ (because $f + d + |W| = m$). So, there exists a Hamiltonian cycle in $G - (F \cup X)$, from which we can extract a 2-starlike tree $T(l_1, l_2)$ rooted at r. Connecting x to $T(l_1, l_2)$ through (r,x) for all $x \in X$ results in a required d-starlike tree. Analogously for the remaining case where $W \neq \emptyset$, we pick up a set X of $d-1$ neighbors of r such that $x, (r,x) \notin F$ and $x \notin W$ for all $x \in X$, which exists because $f + d + |W| = m$. A 1-starlike tree $T(l_1)$ rooted at r whose leaf is not contained in W can be built from a Hamiltonian cycle of $G-(F \cup X)$ if $|W| = 1$ ($|F \cup X| = m-2$); from a Hamiltonian r–t path of $G - (F \cup X)$ for some $t \notin W$ if $|W| \geq 2$ ($|F \cup X| \leq m-3$). The existence of a Hamiltonian path/cycle is by Lemma 2. A required d-starlike tree is obtained by connecting x to $T(l_1)$ through (r,x) for all $x \in X$. This completes the proof. ∎

Due to the above three lemmas, we further assume that

$$d \geq 2,\ f \leq m-3,\ \text{and}\ l_2 \geq 2. \tag{3}$$

Lemma 9. *If $m = 3$, there exists a required d-starlike tree in $G - F$.*

Proof. By the assumption (3), we have $f = 0$ ($d \geq 2$ and $l_2 \geq 2$). Since the 3-dimensional RHL graph, shown in Fig. 2, is vertex-transitive, we can assume w.l.o.g. $r = v_0$. If $d = 3$ ($|W| = 0$), there exists a 3-starlike tree rooted at v_0 whose subtrees form paths in the following:

$\{\langle v_1, v_2, v_6, v_5\rangle, \langle v_4, v_3\rangle, \langle v_7\rangle\}$ for $(l_1, l_2, l_3) = (4, 2, 1)$,
$\{\langle v_1, v_2, v_3\rangle, \langle v_4, v_5\rangle, \langle v_7, v_6\rangle\}$ for $(l_1, l_2, l_3) = (3, 2, 2)$, and
$\{\langle v_1, v_2, v_3\rangle, \langle v_4, v_5, v_6\rangle, \langle v_7\rangle\}$ for $(l_1, l_2, l_3) = (3, 3, 1)$.

If $d = 2$ ($|W| = 1$), there also exists a 2-starlike tree whose subtrees are

$\{\langle v_1, v_2, v_3, v_4, v_5\rangle, \langle v_7, v_6\rangle\}$ or $\{\langle v_7, v_6, v_5, v_4, v_3\rangle, \langle v_1, v_2\rangle\}$ for $(l_1, l_2) = (5, 2)$,
$\{\langle v_1, v_2, v_3, v_7\rangle, \langle v_4, v_5, v_6\rangle\}$ or $\{\langle v_7, v_6, v_5, v_1\rangle, \langle v_4, v_3, v_2\rangle\}$ for $(l_1, l_2) = (4, 3)$.

The proof is completed. ∎

For the inductive step, we let $m \geq 4$ and assume w.l.o.g. $r \in V(G_0)$. For a vertex u in G_i, we denote by $\bar{u}$ the neighbor of u contained in G_{1-i} for $i \in \{0, 1\}$. This is naturally extended to a vertex set $X \subseteq V(G)$ in which $\bar{X} = \{\bar{x} : x \in X\}$. Let I denote the index set $\{1, 2, \ldots, d\}$. One intuitive approach to building a d-starlike tree $T(l_1, l_2, \ldots, l_d)$ rooted at r in $G_0 \oplus G_1$ would be as follows:

1. Decompose $(l_1, l_2, \ldots, l_d)$ into $(l'_1, l'_2, \ldots, l'_d) + (l''_1, l''_2 \ldots, l''_d)$ subject to $1 + \sum_{i \in I} l'_i = |V(G_0) \setminus F_0^v|$ and $1 \leq l'_i \leq l_i$ for all $i \in I$.
2. Build a d-starlike tree $T(l'_1, l'_2 \ldots, l'_d)$ rooted at r in $G_0 - F_0$. The leaf associated with l'_i is denoted by x_i for $i \in I$.
3. Build a k-DPC in $G_1 - F_1$ for the source set $S := \{\bar{x}_i : l'_i < l_i, i \in I\}$ associated with positive l''_is, where $k = |S|$.
4. Finally, combine the d-starlike tree of $G_0 - F_0$ with the k-DPC of $G_1 - F_1$ through edges $(x_i, \bar{x}_i)$ for $\bar{x}_i \in S$.

Here, the leaf x_i must not be contained in W if $l'_i = l_i$; $\bar{x_i}$ and $(x_i, \bar{x_i})$ must not be contained in F if $l'_i < l_i$. Moreover, the sink vertex of a path in the k-DPC must not be contained in W. If $\bar{r}, (r, \bar{r}) \notin F$, we can use (sometimes, must use) the edge $(r, \bar{r})$ when we build a d-starlike tree rooted at r, in a similar way through a $(d-1)$-starlike tree of $G_0 - F_0$ and a k-DPC of $G_1 - F_1$ for some k.

There are two cases according to whether $\bar{r}, (r, \bar{r}) \notin F$ or not, which are dealt with in the subsequent subsections.

3.1. Case when $\bar{r}, (r, \bar{r}) \notin F$

We will present a basic procedure for constructing a d-starlike tree, which fully exploits the edge $(r, \bar{r})$. To the most of the situations, the procedure will be applicable, leaving a few exceptional cases which will be dealt with separately.

Procedure A($(l_1, l_2, \ldots, l_d)$, r, W, G, F) /* $\bar{r}, (r, \bar{r}) \notin F$. See Fig. 3. */

1: Let
$$p = \begin{cases} 1 & \text{if } l_2 + \cdots + l_d \geq 2^{m-1} - f_0^v - 1, \\ 2 & \text{otherwise.} \end{cases}$$
Let $q = 3 - p$, so that $\{p, q\} = \{1, 2\}$.

2: Find a decomposition of $(l_1, l_2, \ldots, l_d)$ into $(l'_1, l'_2, \ldots, l'_d) + (l''_1, l''_2 \ldots, l''_d)$ that maximizes the number, d', of indices $i \in I$ such that $l'_i = l_i$, subject to
  A1: $l'_p = 0$ and $l''_p = l_p$;
  A2: $1 \leq l'_i \leq l_i$ for every $i \in I \setminus \{p\}$;
  A3: $\sum_{i \in I \setminus \{p\}} l'_i = 2^{m-1} - f_0^v - 1$.
Let $I' = \{i \in I : l'_i < l_i\}$ and $k = |I'|$, so that $p \in I'$ and $k = d - d'$.

3: Build a $(d-1)$-starlike tree $T(l'_q, l'_3, l'_4, \ldots, l'_d)$ rooted at r in $G_0 - F_0$ whose leaves are not contained in W', where
$$W' = W_0 \cup \bar{W}_1 \cup Z$$
for $Z := \{v \in V(G_0) : (v, \bar{v}) \in F \text{ or } \bar{v} \in F\}$. Let x_i denote the leaf of the starlike tree associated with l'_i for $i \in I \setminus \{p\}$.

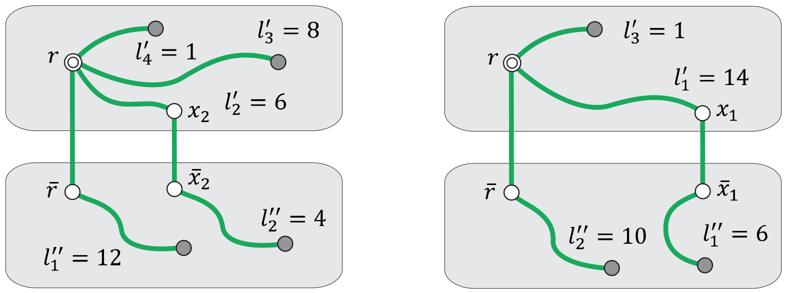
4: Build a k-DPC$[\{(\bar{r}, l''_p, W_1)\} \cup \{(\bar{x_j}, l''_j, W_1) : j \in I' \setminus \{p\}\} | G_1, F_1]$.

5: Merge the starlike tree of $G_0 - F_0$ and the k-DPC of $G_1 - F_1$ with edges $(r, \bar{r})$ and $(x_j, \bar{x_j})$ for $j \in I' \setminus \{p\}$.

The correctness of Procedure A relies on the existence of the decomposition of Step 2, the $(d-1)$-starlike tree of Step 3, and the k-DPC of Step 4. Lemma 10 below derives a condition under which the procedure is correct.

Lemma 10. *Suppose $\bar{r}, (r, \bar{r}) \notin F$. Procedure A is correct if $l_2 \leq 2^{m-1} - f_1^v$, $f_0 + f_2 + d' + |W| \geq 2$, and $f_0 + f_2 + d' + |W_0| \geq 1$.*

Proof. Firstly, we show that the decomposition of Step 2 is possible if the first condition, $l_2 \leq 2^{m-1} - f_1^v$, is satisfied. The decomposition, in fact, can be achieved in a simple, greedy manner if and only if $\sum_{i \in I \setminus \{p\}} 1 \leq 2^{m-1} - f_0^v - 1 \leq \sum_{i \in I \setminus \{p\}} l_i$. Besides, (i) $\sum_{i \in I \setminus \{p\}} 1 = d - 1 \leq m - 1 \leq 2^{m-1} - (m-3) - 1 \leq$



(a) $(l_1, l_2, l_3, l_4) = (12, 10, 8, 1)$, $p = 1$ (b) $(l_1, l_2, l_3) = (20, 10, 1)$, $p = 2$

Fig. 3: Illustrations of Procedure A, where $m = 5$ and $f^v = 0$.

$2^{m-1} - f_0^v - 1$; (ii) if $p = 1$, the inequality $\sum_{i \in I \setminus \{p\}} l_i \geq 2^{m-1} - f_0^v - 1$ holds true by the choice of p in Step 1, and moreover, if $p = 2$, then $\sum_{i \in I \setminus \{p\}} l_i = \sum_{i \in I} l_i - l_2 \geq (2^m - f^v - 1) - (2^{m-1} - f_1^v) = 2^{m-1} - f_0^v - 1$. Thus, the decomposition exists. Next, the $(d-1)$-starlike tree of Step 3 exists by the induction hypothesis, because (i) $f_0 \leq f \leq m - 3 = (m-1) - 2$ and (ii) $f_0 + (d-1) + |W'| \leq f_0 + (d-1) + (|W_0| + |W_1| + f_2 + f_1) = f + d + |W| - 1 = m - 1$. (Recall the assumption (2).)

Finally, we show that the k-DPC of Step 4 exists if the last two conditions of the lemma are satisfied simultaneously. The existence is due to Theorem 2 for $m \geq 5$, and due to Lemma 4 for $m = 4$ and $k = 2$, because (i) $f_1 + k = (f - f_0 - f_2) + (d - d') = f + d - (f_0 + f_2 + d') = m - (f_0 + f_2 + d' + |W|) \leq m - 2 = (m-1) - 1$, and (ii) $f_1 + k + |W_1| = (f - f_0 - f_2) + (d - d') + (|W| - |W_0|) = f + d + |W| - (f_0 + f_2 + d' + |W_0|) \leq m - 1$. (Note that the given configuration for $m = 4$ and $k = 2$ (where $f_1 = 0$ and $|W_1| \leq 1$) does not fall into an exception of Lemma 4, because the set, W_1, that the sink vertex of an $\bar{r}$-path should avoid is the same as the set that the sink vertex of an $\bar{x}_j$-path should avoid.) The existence of the k-DPC for $m = 4$ and $k = 1$ is from Lemma 2, specifically, from Lemma 2(a) if $f_1 = 0$ and $|W_1| \leq 2$, and from Lemma 2(b) if $f_1 = 1$ and $|W_1| \leq 1$. Therefore, the lemma is proved. ■

The decomposition of Step 2 in Procedure A is possible if (and only if) the first condition, $l_2 \leq 2^{m-1} - f_1^v$, of Lemma 10 is satisfied. Based on Lemma 10, Lemmas 11 and 12 below handle the case when $l_2 \leq 2^{m-1} - f_1^v$; whereas, Lemma 13 handles the remaining case when $l_2 > 2^{m-1} - f_1^v$.

Lemma 11. *Suppose $\bar{r}, (r, \bar{r}) \notin F$ and $l_2 \leq 2^{m-1} - f_1^v$. Procedure A builds a required d-starlike tree in $G - F$ with the three exceptions:*
(a) $d = 4$, $l_1 = l_2 = l_3 = 2^{m-2}$, $l_4 = 2^{m-2} - 1$, $f_1^e = f = m - 4$, and $W = \emptyset$;
(b) $d = 3$, $f_1 = f = m - 3$, and $W = \emptyset$;
(c) $d = 2$, $f_1 = f \leq m - 3$, and moreover, $|W| = 1$ or $W = W_1$ ($W_0 = \emptyset$).

Proof. We will prove that the two conditions, $f_0 + f_2 + d' + |W| \geq 2$ and $f_0 + f_2 + d' + |W_0| \geq 1$, of Lemma 10 are satisfied if the given configuration forms neither of the three exceptions of this lemma. There are four cases according to the degree, d. Keep in mind $f + d + |W| = m$.

Case 1: $d \geq 5$. It suffices to show $(d-3) + (l_{d-1} + l_d) \leq 2^{m-1} - f_0^v - 1$, which leads to $d' \geq 2$, hence the two conditions of Lemma 10 are satisfied. The inequality follows from the fact that $(d-3) + (l_{d-1} + l_d) \leq (d-3) + 2 \cdot \frac{2^m}{d} \leq (m-f-3) + \frac{2}{5} \cdot 2^m$ and $(2^{m-1} - f_0^v - 1) - ((m-f-3) + \frac{2}{5} \cdot 2^m) = \frac{1}{10} \cdot 2^m - (m-2) + (f - f_0^v) \geq \frac{1}{10} \cdot 2^m - (m-2) \geq 0$ for $m \geq d \geq 5$.

Case 2: $d = 4$. In this case, we can derive $d' \geq 1$ from $2 + l_4 \leq 2^{m-1} - f_0^v - 1$, which holds true because $(2^{m-1} - f_0^v - 1) - (2 + l_4) \geq (2^{m-1} - (m-4) - 1) - (2^{m-2} + 1) = 2^{m-2} - (m-2) \geq 0$ where $f_0^v \leq f = m - d - |W| \leq m - 4$ and $l_4 \leq 2^{m-2} - 1$. So, if $f_0 + f_2 + |W| \geq 1$, the two conditions of Lemma 10 are satisfied. In addition, if $1 + l_3 + l_4 \leq 2^{m-1} - f_0^v - 1$, then $d' \geq 2$, hence the two conditions are also satisfied. There remains a case where $f_0 + f_2 + |W| = 0$ and $l_3 + l_4 \geq 2^{m-1} - f_0^v - 1$, or equivalently, $f_0 = f_2 = |W| = 0$ and $l_3 + l_4 \geq 2^{m-1} - 1$. Then, $l_3 \geq 2^{m-2}$; moreover, $l_3 \leq 2^{m-2}$. (Suppose otherwise, then $l_1, l_2 \geq 2^{m-2} + 1$, leading to $(l_1 + l_2) + (l_3 + l_4) \geq (2^{m-1} + 2) + (2^{m-1} - 1) > 2^m$, which is a contradiction.) It follows that $l_1 = l_2 = l_3 = 2^{m-2}$ and $l_4 = 2^{m-2} - 1$. Also, $f^v = 0$ and $f_1^e = f = m - d - |W| = m - 4$ from $\sum_{i \in I} l_i = 2^m - 1$ and $f_0 = f_2 = |W| = 0$. Thus, the remaining case is the exception (a) of this lemma.

Case 3: $d = 3$. We can also derive $d' \geq 1$ from $1 + l_3 \leq 2^{m-1} - f_0^v - 1$. (Supposing $l_3 \geq 2^{m-1} - f_0^v - 1$ leads to $l_1 + l_2 + l_3 \geq 3(2^{m-1} - f_0^v - 1) \geq 2^m + 2^{m-1} - 3(m-3) - 3 \geq 2^m$, which is a contradiction.) So, if $f_0 + f_2 + |W| \geq 1$, the two conditions of Lemma 10 are satisfied. There remains a case where $f_0 = f_2 = |W| = 0$, which is the exception (b) of this lemma.

Case 4: $d = 2$. In this case, $|W| \geq 1$. (Suppose otherwise, $f = m - d - |W| = m - 2$, which violates the assumption (3).) The two conditions of Lemma 10 will be satisfied if $f_0 + f_2 + |W| \geq 2$ and $f_0 + f_2 + |W_0| \geq 1$. There remains a case where $f_0 + f_2 + |W| \leq 1$ or $f_0 + f_2 + |W_0| = 0$. Suppose $f_0 + f_2 + |W| \leq 1$, then $f_0 = f_2 = 0$ and $|W| = 1$, and thus $f_1 = f = m - 3$. Suppose $f_0 + f_2 + |W_0| = 0$, then $f_0 = f_2 = |W_0| = 0$, and thus $f_1 = f$ $(\leq m - 3)$. Thus, the remaining case is the exception (c) of this lemma. This completes the proof. ■

Lemma 12. *Suppose $\bar{r}, (r, \bar{r}) \notin F$ and $l_2 \leq 2^{m-1} - f_1^v$. For each of the three exceptions of Lemma 11, there exists a required d-starlike tree in $G - F$.*

Proof. Proof for the exception (a) of Lemma 11: Suppose $m \geq 5$ first. Pick up a neighbor $u \in V(G_0)$ of r and build a cycle $C = \langle r, P, u \rangle$ with $2^{m-1} + 2$ vertices, where P is a Hamiltonian $\bar{r}$–$\bar{u}$ path of $G_1 - F_1$, which exists by Lemma 2(a) because $f_1 = f = m - 4$. If we extract a 2-starlike tree $T(l_3, l_4)$ rooted at r from C, there remains a two-vertex path $\langle x, y \rangle$ for some $x, y \in V(G_1)$. Also, there exists a Hamiltonian $\bar{x}$–$\bar{y}$ path P' in $G_0 - \{u\}$ by Lemma 2(a), from which a cycle $\langle x, P', y \rangle$ with $2^{m-1} + 1$ vertices can be built, from which a 2-starlike tree $T(l_1, l_2)$ rooted at r can be extracted. Thus, there is a 4-starlike tree $T(l_1, l_2, l_3, l_4)$ rooted at r. Suppose $m = 4$ now, where $l_1 = l_2 = l_3 = 4$, $l_4 = 3$, $f = 0$, and $W = \emptyset$. Let $(l_2', l_2'') = (l_3', l_3'') = (2, 2)$. There exists a 3-starlike tree $T(l_2', l_3', l_4)$ rooted at r in G_0 by the induction hypothesis. For the leaves x_2 and x_3 of the starlike tree, associated with l_2' and l_3', respectively, there also exists a 3-DPC$[\{(\bar{r}, l_1, \emptyset), (\bar{x_2}, l_2'', \emptyset), (\bar{x_3}, l_3'', \emptyset)\}|G_1, \emptyset]$ by Lemma 5. Combining the starlike tree of G_0 with the 3-DPC of G_1 through edges $(r, \bar{r})$, $(x_2, \bar{x_2})$, and $(x_3, \bar{x_3})$ results in a 4-starlike tree $T(l_1, l_2, l_3, l_4)$ rooted at r.

Proof for the exception (b) of Lemma 11: If we extract an $\bar{r}$-path with l_3 vertices from a Hamiltonian cycle of $G_1 - F_1$, there remains a path, denoted by a u–v path, where $u \neq v$ because $l_3 \leq 2^{m-1} - f_1^v - 2$. (Supposing $l_3 \geq 2^{m-1} - f_1^v - 1$ leads to $\sum_{i \in I} l_i \geq 3(2^{m-1} - f_1^v - 1) \geq 2^m + 2^{m-1} - 3(m-3) - 3 \geq 2^m$, which is a contradiction.) If we merge the u–v path and a Hamiltonian $\bar{v}$–$\bar{u}$ path of G_0 through edges $(u, \bar{u})$ and $(v, \bar{v})$, we obtain a cycle with $l_1 + l_2 + 1$ vertices, from which we can extract a 2-starlike tree $T(l_1, l_2)$ rooted at r. It suffices to combine the 2-starlike tree with the $\bar{r}$-path through $(r, \bar{r})$.

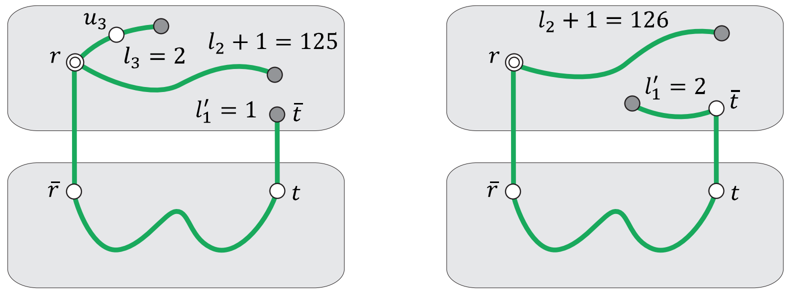
Proof for the exception (c) of Lemma 11: In this case, $l_1 \geq 2^{m-1} - f_1^v$. (Suppose otherwise, $l_2 = (2^m - f^v - 1) - l_1 \geq (2^m - f^v - 1) - (2^{m-1} - f_1^v - 1) = 2^{m-1}$, leading to $l_1 < l_2$, which is a contradiction.) Firstly, we build a Hamiltonian $\bar{r}$–t path, P, in $G_1 - F_1$ for some $t \notin W_1 \cup \bar{W_0}$, which exists by Lemma 2(a) if $f_1 \leq m-4$, and by Lemma 2(b) if $f_1 = m-3$ ($|W| = 1 = |W_1 \cup \bar{W_0}|$). If $l_1 = 2^{m-1} - f_1^v$, it suffices to combine P with a Hamiltonian r–t' path of G_0 for some $t' \notin W_0$, which exists by Lemma 2(a), through $(r, \bar{r})$. Suppose $l_1 > 2^{m-1} - f_1^v$ now. For $l_1' = l_1 - (2^{m-1} - f_1^v)$, we build a 2-DPC$[\{(\bar{t}, l_1', W_0), (r, l_2 + 1, W_0)\}|G_0, F_0]$. The 2-DPC for $m \geq 5$ exists by Theorem 2 because (i) $f_0 + 2 = 2 \leq (m-1) - 1$ and (ii) $f_0 + 2 + |W_0| \leq 3 \leq m-1$; also, the 2-DPC for $m = 4$ exists by Lemma 4 (because the configuration does not fall into an exception of Lemma 4). It suffices to combine the 2-DPC of G_0 with P through edges $(r, \bar{r})$ and $(t, \bar{t})$. Therefore, the lemma is proved. ■

Lemma 13. *Suppose $\bar{r}, (r, \bar{r}) \notin F$ and $l_2 > 2^{m-1} - f_1^v$. There exists a required d-starlike tree in $G - F$.*

Proof. In this case, where $l_1 \geq l_2 \geq 2^{m-1} - f_1^v + 1$, we have $f_1^v \geq 3$ because $0 \leq \sum_{i \in I} l_i - (l_1 + l_2) \leq (2^m - f^v - 1) - 2(2^{m-1} - f_1^v + 1) = f_1^v - f_0^v - 3 \leq f_1^v - 3$. In addition, $m \geq 6$ because $3 \leq f_1^v \leq f \leq m-3$; moreover, $f_0 + d + |W_0| \leq (f + d + |W|) - f_1 = m - f_1 \leq m - 3$. Utilizing a d-DPC of $G_0 - F_0$ and a Hamiltonian path of $G_1 - F_1$, a required d-starlike tree can be constructed as follows (see Fig. 4):

1: Build a Hamiltonian $\bar{r}$–t path in $G_1 - F_1$ for some t with $\bar{t}, (t, \bar{t}) \notin F$ and $\bar{t} \notin W_0$.
2: Pick up $d-2$ neighbors, $u_3, u_4, \ldots, u_d$, of r in G_0 such that for every $i \in I \setminus \{1, 2\}$, (i) $u_i, (r, u_i) \notin F_0$, (ii) $u_i \notin W_0$, and (iii) $u_i \neq \bar{t}$.
3: Build a d-DPC$[\{(\bar{t}, l_1', W_0), (r, l_2 + 1, W_0)\} \cup \{(u_i, l_i, W_0) : i \in I \setminus \{1, 2\}\}|G_0, F_0]$, where $l_1' = l_1 - (2^{m-1} - f_1^v)$.
4: Combine the d-DPC of $G_0 - F_0$ with the Hamiltonian path of $G_1 - F_1$ through edges $(r, \bar{r})$, $(t, \bar{t})$, and (r, u_i) for $i \in I \setminus \{1, 2\}$.

The Hamiltonian path of Step 1 exists by Lemma 2(a) if $f_1 \leq m-4$, and by Lemma 2(b) if $f_1 = m-3$ ($f_0 = f_2 = 0$ and $|W| \leq 1$). Also, there exist such $d-2$



(a) $d = 3$, $(l_1, l_2, l_3) = (124, 124, 2)$ (b) $d = 2$, $(l_1, l_2) = (125, 125)$

Fig. 4: Illustrations of the procedure in the proof of Lemma 13, where $m = 8$ and $f_1^v = 5$.

neighbors of r in Step 2 because $f_0+|W_0|+1+(d-2) = f_0+d+|W_0|-1 \leq m-4$. Finally, the d-DPC of Step 3 exists by Theorem 2, because $f_0+d+|W_0| \leq m-3$, specifically, because (i) $f_0 + d \leq m - 3 < (m - 1) - 1$ and (ii) $f_0 + d + |W_0| \leq m - 3 < m - 1$. Thus, a required d-starlike tree can be built. ■

3.2. *Case when $\bar{r} \in F$ or $(r, \bar{r}) \in F$*

On the contrary to the case when $\bar{r}, (r, \bar{r}) \notin F$ discussed in the previous subsection, the edge $(r, \bar{r})$ cannot be used to construct a d-starlike tree rooted at r in $G - F$. We will develop a basic procedure for constructing a d-starlike tree, through a d-starlike tree of $G_0 - F_0$ and a k-DPC of $G_1 - F_1$ for some k. The procedure will also cover a wide range, leaving a few exceptional cases which will be discussed later.

Procedure B($(l_1, l_2, \ldots, l_d)$, r, W, G, F) /* $\bar{r} \in F$ or $(r, \bar{r}) \in F$. See Fig. 5. */

1: Find a decomposition of $(l_1, l_2, \ldots, l_d)$ into $(l'_1, l'_2, \ldots, l'_d) + (l''_1, l''_2, \ldots, l''_d)$ that maximizes the number, d', of indices $i \in I$ such that $l'_i = l_i$, subject to
  B1: $1 \leq l'_i \leq l_i$ for every $i \in I$;
  B2: $\sum_{i \in I} l'_i = 2^{m-1} - f_0^v - 1$.
  Let $I' = \{i \in I : l'_i < l_i\}$ and $k = |I'|$, so that $k = d - d'$.

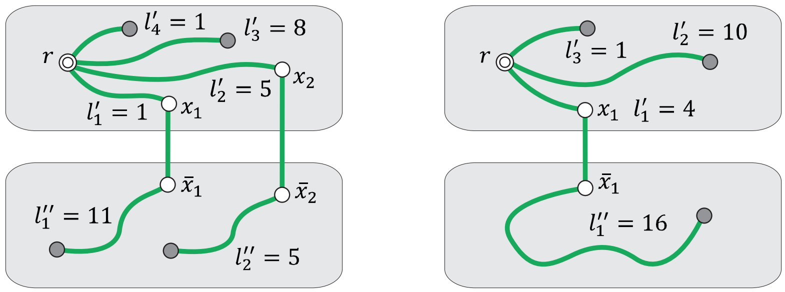
2: Build a d-starlike tree $T(l'_1, l'_2, \ldots, l'_d)$ rooted at r in $G_0 - F_0$ whose leaves are not contained in W', where

$$W' = W_0 \cup \bar{W}_1 \cup (Z \setminus \{r\})$$

  for $Z := \{v \in V(G_0) : (v, \bar{v}) \in F \text{ or } \bar{v} \in F\}$. Let x_i denote the leaf of the starlike tree associated with l'_i for $i \in I$.

3: Build a k-DPC$[\{(\bar{x_j}, l''_j, W_1) : j \in I'\}|G_1, F_1]$.

4: Merge the starlike tree of $G_0 - F_0$ and the k-DPC of $G_1 - F_1$ with edges $(x_j, \bar{x_j})$ for $j \in I'$.



(a) $(l_1, l_2, l_3, l_4) = (12, 10, 8, 1)$ (b) $(l_1, l_2, l_3) = (20, 10, 1)$

Fig. 5: Illustrations of Procedure B, where $m = 5$ and $F = \{(r, \bar{r})\}$.

Lemma 14 below derives a condition under which Procedure B is correct. The correctness of the procedure relies on the existence of the decomposition of Step 1, the d-starlike tree of Step 2, and the k-DPC of Step 3.

Lemma 14. *Suppose $\bar{r} \in F$ or $(r, \bar{r}) \in F$. Procedure B is correct if $f_0 + f_2 + d' + |W| \geq 2$ and $f_0 + f_2 + d' + |W_0| \geq 1$.*

Proof. The decomposition of Step 1 can be achieved by a simple greedy algorithm. Also, the d-starlike tree of Step 2 exists by the induction hypothesis, because (i) $f_0 \leq f \leq m-3 = (m-1)-2$ and (ii) $f_0+d+|W'| \leq f_0+d+(|W_0|+|W_1|+f_2+f_1-1) = f+d+|W|-1 = m-1$. Finally, we show that the k-DPC of Step 3 exists if the two conditions of this lemma are satisfied simultaneously. The existence is due to Theorem 2 for $m \geq 5$, and due to Lemma 4 for $m = 4$ and $k = 2$, because (i) $f_1+k = (f-f_0-f_2)+(d-d') = f+d-(f_0+f_2+d') = m-(f_0+f_2+d'+|W|) \leq m-2 = (m-1)-1$, and (ii) $f_1+k+|W_1| = (f-f_0-f_2)+(d-d')+(|W|-|W_0|) = f+d+|W|-(f_0+f_2+d'+|W_0|) \leq m-1$. (Note that the given configuration for $m = 4$ and $k = 2$ does not fall into an exception of Lemma 4.) The existence of the k-DPC for $m = 4$ and $k = 1$ is from Lemma 2, specifically, from Lemma 2(a) if $f_1 = 0$ and $|W_1| \leq 2$, and from Lemma 2(b) if $f_1 = 1$ and $|W_1| \leq 1$. This completes the proof. ■

In the following lemma, we identify all the exceptional cases to which Procedure B is not applicable, based on Lemma 14.

Lemma 15. *Suppose $\bar{r} \in F$ or $(r, \bar{r}) \in F$. Procedure B builds a required d-starlike tree in $G - F$ with the three exceptions:*
(a) $d = 4$, $l_3 + l_4 = 2^{m-1} - 2$, $f_1 = f = m-4$ ($\bar{r} \in F$), and $W = \emptyset$;
(b) $d = 3$, $l_2 + l_3 \geq 2^{m-1} - 1$, $f_1 = f = m-3$ ($\bar{r} \in F$), and $W = \emptyset$;
(c) $d = 2$, $l_2 = 2^{m-1}-1$, $f_1 = f = m-2-|W|$ ($\bar{r} \in F$), and moreover, $|W| = 1$ or $W = W_1$ ($W_0 = \emptyset$).

Proof. We will prove that the two conditions, $f_0 + f_2 + d' + |W| \geq 2$ and $f_0 + f_2 + d' + |W_0| \geq 1$, of Lemma 14 are satisfied if the given configuration forms neither of the three exceptions of this lemma. The proof follows similar statements made in the proof of Lemma 11. The precondition of this lemma leads to $f \geq f_1^v + f_2 \geq 1$, implying $m = f + d + |W| \geq d + 1$.

Case 1: $d \geq 5$. It suffices to show $(d-2) + (l_{d-1} + l_d) \leq 2^{m-1} - f_0^v - 1$, which leads to $d' \geq 2$, hence the two conditions of Lemma 14 are satisfied. The inequality follows from the fact that $(d-2) + (l_{d-1} + l_d) \leq (d-2) + 2 \cdot \frac{2^m}{d} \leq (m-f-2) + \frac{2}{5} \cdot 2^m$ and $(2^{m-1} - f_0^v - 1) - ((m-f-2) + \frac{2}{5} \cdot 2^m) = \frac{1}{10} \cdot 2^m - (m-1) + (f - f_0^v) \geq \frac{1}{10} \cdot 2^m - (m-1) \geq 0$ for $m \geq d+1 \geq 6$.

Case 2: $d = 4$. We can derive $d' \geq 1$ from $3 + l_4 \leq 2^{m-1} - f_0^v - 1$, which holds true because $(2^{m-1} - f_0^v - 1) - (3 + l_4) \geq (2^{m-1} - (m-4) - 1) - (2^{m-2} + 2) = 2^{m-2} - (m-1) \geq 0$ where $f_0^v \leq f = m - d - |W| \leq m - 4$ and $l_4 \leq 2^{m-2} - 1$. So, if $f_0 + f_2 + |W| \geq 1$, the two conditions of Lemma 14 are satisfied. In addition, if $2 + l_3 + l_4 \leq 2^{m-1} - f_0^v - 1$, then $d' \geq 2$, hence the two conditions are also satisfied. There remains a case where $f_0 + f_2 + |W| = 0$ and $l_3 + l_4 \geq 2^{m-1} - f_0^v - 2$, or equivalently, $f_0 = f_2 = |W| = 0$ and $l_3 + l_4 \geq 2^{m-1} - 2$. It follows that $f_1 = f = m - d - |W| = m - 4$ and $\bar{r} \in F$; moreover, $l_3 + l_4 = 2^{m-1} - 2$. (Supposing $l_3 + l_4 \geq 2^{m-1} - 1$ leads to $l_3 \geq 2^{m-2}$, implying $l_1 + l_2 + (l_3 + l_4) \geq 2^{m-2} + 2^{m-2} + (2^{m-1} - 1) = 2^m - 1$, which contradicts the fact $\sum_{i \in I} l_i = 2^m - f^v - 1 \leq 2^m - 2$.) Thus, the remaining case is the exception (a) of this lemma.

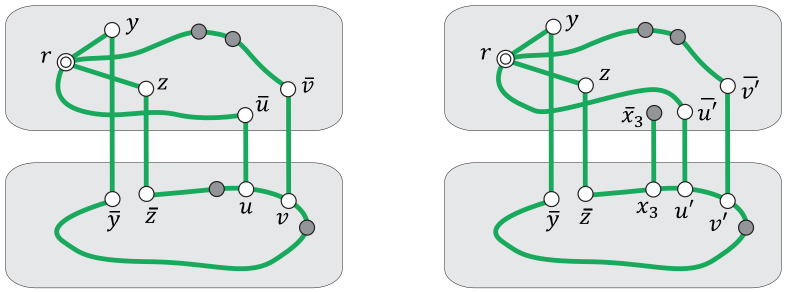
Case 3: $d = 3$. We can also derive $d' \geq 1$ from $2 + l_3 \leq 2^{m-1} - f_0^v - 1$. (Supposing $l_3 \geq 2^{m-1} - f_0^v - 2$ leads to $l_1 + l_2 + l_3 \geq 3(2^{m-1} - f_0^v - 2) \geq 2^m + 2^{m-1} - 3(m-4) - 6 \geq 2^m$, which is a contradiction. Note that $f_0^v \leq f - (f_1^v + f_2) \leq (m-3) - 1$.) So, if $f_0 + f_2 + |W| \geq 1$, the two conditions of Lemma 14 are satisfied. In addition, if $1 + l_2 + l_3 \leq 2^{m-1} - f_0^v - 1$, then $d' \geq 2$, hence the two conditions are also satisfied. There remains a case where $f_0 = f_2 = |W| = 0$ and $l_2 + l_3 \geq 2^{m-1} - 1$, which is the exception (b) of this lemma.

Case 4: $d = 2$. We have $|W| \geq 1$, because $m - d - |W| = f \leq m - 3$. The two conditions of Lemma 14 will be satisfied if $f_0 + f_2 + |W| \geq 2$ and $f_0 + f_2 + |W_0| \geq 1$. In addition, if $1 + l_2 \leq 2^{m-1} - f_0^v - 1$, then $d' \geq 1$, hence the two conditions are also satisfied. There remains a case where (i) $f_0 + f_2 + |W| \leq 1$ (i.e., $f_0 = f_2 = 0$ and $|W| = 1$) or $f_0 + f_2 + |W_0| = 0$ (i.e., $f_0 = f_2 = |W_0| = 0$), and (ii) $l_2 \geq 2^{m-1} - f_0^v - 1 = 2^{m-1} - 1$. Then, $l_2 = 2^{m-1} - 1$. (Supposing $l_2 \geq 2^{m-1}$ leads to $l_1 + l_2 \geq 2^m$, which is a contradiction.) Thus, the remaining case is the exception (c) of this lemma. The proof is completed. ■

In the remaining part of this section, we will construct a required d-starlike tree for each of the three exceptions of Lemma 15. We begin with the exception (a) in Lemma 16 below; the exceptions (b) and (c), respectively, are dealt with later in Lemmas 18 and 19.

Lemma 16. *For the exception (a) of Lemma 15, there exists a required* 4*-starlike tree in* $G - F$.

Proof. We have $1 \leq f_1^v \leq 3$, because $\sum_{i \in I} l_i = 2^m - f_1^v - 1 \geq 2(l_3 + l_4) = 2^m - 4$. Also, $l_4 \geq 2$. (Supposing $l_4 = 1$ leads to $l_3 = 2^{m-1} - 3$, hence $l_1 + l_2 + l_3 + l_4 \geq$



(a) $m \geq 6$ and $f_1^v \leq 2$ (b) $m \geq 7$ and $f_1^v = 3$

Fig. 6: Illustrations of the proof of Lemma 16.

$3(2^{m-1}-3)+1 = 2^m + 2^{m-1} - 8 > 2^m - f_1^v - 1$, which is a contradiction). In addition, $m = f + d + |W| = f_1 + 4 \geq f_1^v + 4 \geq 5$. There are three cases: (i) $f_1^v \leq 2$ and $m \geq 6$, (ii) $f_1^v = 3$ ($m \geq 7$), and (iii) $f_1^v \leq 2$ and $m = 5$ ($F = \{\bar{r}\}$).

Firstly, suppose $m \geq 6$ and $f_1^v \leq 2$. Using Hamiltonian paths in $G_1 - F_1$ and in $G_0 - X$ for some $X \subset V(G_0)$, a required d-starlike tree can be constructed as follows (see Fig. 6(a)):

1: Pick up two neighbors, y and z, of r in G_0 such that $\bar{y}, \bar{z} \notin F$.
2: Build a Hamiltonian $\bar{y}$–$\bar{z}$ path P in $G_1 - F_1$. Let $C = \langle r, y, P, z \rangle$ be a cycle, which has $2^{m-1} - f_1^v + 3$ vertices.
3: Extract a 2-starlike tree $T(l_3, l_4)$ rooted at r from C. Then, there remains a path in C, say a u–v path with $u \neq v$.
4: Build a Hamiltonian $\bar{v}$–$\bar{u}$ path P' in $G_0 - \{y, z\}$. Combining P' with the u–v path through edges $(u, \bar{u})$ and $(v, \bar{v})$ results in a cycle, C'.
5: Extract a 2-starlike tree $T(l_1, l_2)$ rooted at r from C'.
6: Merge $T(l_1, l_2)$ and $T(l_3, l_4)$ into a 4-starlike tree $T(l_1, l_2, l_3, l_4)$ rooted at r.

The Hamiltonian path P of Step 2 exists by Lemma 2(a) because $f_1 = m-4$. The u–v path of Step 3 has $(2^{m-1} - f_1^v + 3) - (l_3 + l_4 + 1) = 4 - f_1^v \geq 2$ vertices, so $u \neq v$. Moreover, $u, v \in V(G_1)$ and $\{\bar{u}, \bar{v}\} \cap \{r, y, z\} = \emptyset$ because $l_4 \geq 2$ and $\bar{r} \in F$. The Hamiltonian path P' of Step 4 exists again by Lemma 2(a). Thus, the procedure successfully produces a required 4-starlike tree.

Secondly, suppose $m \geq 7$ and $f_1^v = 3$. The procedure for the previous case does not work well for this case. This is because extracting a 2-starlike tree $T(l_3, l_4)$ from C in Step 3 leaves a one-vertex path (i.e., $u = v$), leading to that building such cycle C' of Step 4 is not possible. However, the procedure can be recycled with slight changes as follows (see Fig. 6(b)): (i) In Step 3, we extract a 2-starlike tree $T(l_3 - 1, l_4)$, instead of $T(l_3, l_4)$, from C, so that there remains a two-vertex path, say a u'–v' path. For the leaf, x_3, of the 2-starlike tree $T(l_3 - 1, l_4)$ associated with $l_3 - 1$, a 2-starlike tree $T(l_3, l_4)$ is obtained by adding $\bar{x_3}$ to $T(l_3 - 1, l_4)$ through the edge $(x_3, \bar{x_3})$. (ii) In Step 4, we build a Hamiltonian $\bar{v'}$–$\bar{u'}$ path P'' in $G_0 - \{y, z, \bar{x_3}\}$, instead of a Hamiltonian $\bar{v}$–$\bar{u}$ path P' of $G_0 - \{y, z\}$. The Hamiltonian path P'' exists by Lemma 2(a) because $m \geq 7$. Note that $u', v' \in V(G_1)$ and $\{\bar{u'}, \bar{v'}, \bar{x_3}\} \cap \{r, y, z\} = \emptyset$, because $l_3 \geq 2^{m-2} - 1 \geq 4$, $l_4 \geq 2$, and $\bar{r} \in F$. Thus, the modified procedure produces a required 4-starlike tree.

Finally, suppose $m = 5$ and $F = \{\bar{r}\}$. Then, $l_3 + l_4 = 14$, so $l_3 = 7$ or 8. (Supposing $l_3 \geq 9$ leads to $l_1 + l_2 + (l_3 + l_4) \geq 9 + 9 + 14 = 32 > 2^m - f^v - 1 = 30$, which is a contradiction.) Moreover, if $l_3 = 8$, then $(l_1, l_2, l_3, l_4) = (8, 8, 8, 6)$; if $l_3 = 7$, then $(l_1, l_2, l_3, l_4) = (8, 8, 7, 7)$ or $(9, 7, 7, 7)$. Let

$$(l'_1, l'_2, l'_3, l'_4) = \begin{cases} (1, 2, 6, 6) & \text{if } (l_1, l_2, l_3, l_4) = (8, 8, 8, 6), \\ (1, 2, 5, 7) & \text{if } (l_1, l_2, l_3, l_4) = (8, 8, 7, 7), \\ (2, 1, 5, 7) & \text{if } (l_1, l_2, l_3, l_4) = (9, 7, 7, 7). \end{cases}$$

It follows that $(l''_1, l''_2, l''_3, l''_4) := (l_1, l_2, l_3, l_4) - (l'_1, l'_2, l'_3, l'_4) = (7, 6, 2, 0)$ for all the three possibilities. In order to construct a required 4-starlike tree, we now apply Steps 2 through 4 of Procedure B to the specific decomposition of (l_1, l_2, l_3, l_4) into $(l'_1, l'_2, l'_3, l'_4) + (l''_1, l''_2, l''_3, l''_4)$. A 4-starlike tree $T(l'_1, l'_2, l'_3, l'_4)$ rooted at r exists in G_0 by the induction hypothesis (because $W' = \emptyset$). Also, a 3-DPC$[\{(\bar{x_j}, l''_j, \emptyset) : j \in \{1, 2, 3\}\} | G_1, \{\bar{r}\}]$ can be obtained from a 4-DPC$[\{(\bar{r}, 1, \emptyset)\} \cup \{(\bar{x_j}, l''_j, \emptyset) : j \in \{1, 2, 3\}\} | G_1, \emptyset]$, which exists by Lemma 17 below (not by Theorem 2), where $\bar{r}$ is regarded as a fault-free vertex temporarily. It suffices to combine the 4-starlike tree of G_0 with the 3-DPC of $G_1 - F_1$. Therefore, the lemma is proved. ∎

Lemma 17. *Let G be a 4-dimensional RHL graph. If $(l_1, l_2, l_3, l_4) = (7, 6, 2, 1)$, there exists a 4-DPC$[\{(s_i, l_i, \emptyset) : i \in \{1, 2, 3, 4\}\} | G, \emptyset]$ for any distinct sources s_1, s_2, s_3, and s_4.*

Lemma 17 was verified from a computer program that exhaustively searches for DPCs. The source code may be downloaded from `http://tcs.catholic.ac.kr/~jhpark/papers/Lemma_17.zip`. Actually, it was verified that such 4-DPC exists for any distinct sources if (i) $l_i = 2$ for some i and (ii) no two l_i's are equal to 1.

Lemma 18. *For the exception (b) of Lemma 15, there exists a required 3-starlike tree in $G - F$.*

Proof. The proof consists of three cases: (i) $l_3 = 1$, (ii) $l_3 \geq 2$ and $m \geq 5$, and (iii) $l_3 \geq 2$ and $m = 4$. Firstly, suppose $l_3 = 1$. For a neighbor $y \in V(G_0)$ of r, there exists a Hamiltonian cycle, C, in $G - (F \cup \{y\})$ by Lemma 2(b). It suffices to extract a 2-starlike tree $T(l_1, l_2)$ rooted at r from C and then attach y to the root r through the edge (r, y). Secondly, suppose $m \geq 5$ and $l_3 \geq 2$. Pick up a neighbor $y \in V(G_0)$ of r such that $\bar{y} \notin F$. From a Hamiltonian cycle of $G_1 - F_1$, we extract a $\bar{y}$-path having $l_3 - 1$ vertices. Then, there remains a path in the cycle, say a u–v path with $u \neq v$, because $l_3 \leq 2^{m-1} - f_1^v - 1$. (Supposing $l_3 \geq 2^{m-1} - f_1^v$ leads to $l_1 + l_2 + l_3 \geq 3(2^{m-1} - f_1^v) \geq 2^m + 2^{m-1} - 3(m-3) > 2^m - f^v - 1$, which is a contradiction.) Combining a Hamiltonian $\bar{v}$–$\bar{u}$ path of $G_0 - \{y\}$ with the u–v path through edges $(u, \bar{u})$ and $(v, \bar{v})$ results in a cycle, from which we can extract a 2-starlike $T(l_1, l_2)$ rooted at r. Note that

$\{\bar{u}, \bar{v}\} \cap \{r, y\} = \emptyset$ because $l_3 \geq 2$ and $\bar{r} \in F$. It suffices to attach the y-path, obtained by concatenating the one-vertex path $\langle y \rangle$ and the $\bar{y}$-path, to the root r through the edge (r, y).

Finally, suppose $m = 4$ and $l_3 \geq 2$. Then, $F = \{\bar{r}\}$ because $f_1 = f = 1$ and $\bar{r} \in F$. In addition, $l_2 + l_3 \geq 2^{m-1} - 1 = 7$, leading to $l_2 \geq 4$. Moreover, $l_1 \geq 5$ because $l_1 + l_2 + l_3 = 2^m - f^v - 1 = 14$. If $l_1 \geq 6$, a required 3-starlike tree can be constructed in three steps as follows: (i) Build a 3-starlike tree $T(l_1 - 5, l_2 - 2, l_3)$ rooted at r in G_0, where x_1 and x_2, respectively, denote the leaves of the tree associated with $l_1 - 5$ and $l_2 - 2$; (ii) Build a 2-DPC$[\{(\bar{x_1}, 5, \emptyset), (\bar{x_2}, 2, \emptyset)\}|G_1, \{\bar{r}\}]$ from a 3-DPC$[\{(\bar{r}, 1, \emptyset), (\bar{x_1}, 5, \emptyset), (\bar{x_2}, 2, \emptyset)\}|G_1, \emptyset]$, which exists by Lemma 5; (iii) Combine the 3-starlike tree of G_0 with the 2-DPC of $G_1 - F_1$ through edges $(x_1, \bar{x_1})$ and $(x_2, \bar{x_2})$. So, let $l_1 = 5$ hereafter, meaning $(l_1, l_2, l_3) = (5, 5, 4)$. We will show that a required 3-starlike tree $T(5, 5, 4)$ can be constructed from a 3-starlike tree $T(5, 1, 1)$ of G_0 and a 2-DPC of $G_1 - F_1$ made of two paths, respectively, having 3 and 4 vertices. Recall that G_0 and G_1 are isomorphic to the 3-dimensional RHL graph $G(8, 4)$, shown in Fig. 2.

Claim 1. Let r be a vertex of $G(8, 4)$. For any pair of neighbors, y and z, of r, there exists a 3-starlike tree $T(5, 1, 1)$ rooted at r in which y and z each is a leaf and also a child of the root.

Proof. The graph $G(8, 4)$ is vertex-transitive, so we assume w.l.o.g. $r = v_0$. For each of the two possibilities up to symmetry, there exists a 3-starlike tree rooted at v_0 whose subtrees form paths in the following:
$\{\langle v_1 \rangle, \langle v_4 \rangle, \langle v_7, v_3, v_2, v_6, v_5 \rangle\}$ for $\{y, z\} = \{v_1, v_4\}$, and
$\{\langle v_1 \rangle, \langle v_7 \rangle, \langle v_4, v_5, v_6, v_2, v_3 \rangle\}$ for $\{y, z\} = \{v_1, v_7\}$. □

Claim 2. A 3-DPC$[\{(s_1, 1, \emptyset), (s_2, l', \emptyset), (s_3, l'', \emptyset)\}|G(8, 4), \emptyset]$ exists for some $(l', l'') \in \{(3, 4), (4, 3)\}$ if and only if s_1, s_2, s_3 are distinct vertices that do not comprise the neighbor set of a vertex in $G(8, 4)$.

Proof. The necessity part is obvious. For the sufficiency, assume that s_1, s_2, s_3 are distinct and do not comprise the set of neighbors of a vertex. The s_1-path in a 3-DPC is necessarily the one-vertex path $\langle s_1 \rangle$. So, if a Hamiltonian s_2–s_3 path exists in $G(8, 4) - \{s_1\}$, it suffices to divide the Hamiltonian path into two: say, an s_2-path with 3 vertices and an s_3-path with 4 vertices. Now, suppose $G(8, 4) - \{s_1\}$ does not admit a Hamiltonian s_2–s_3 path. If we assume w.l.o.g. $s_1 = v_0$, it can be determined if a Hamiltonian s_2–s_3 path exists in $G(8, 4) - \{s_1\}$, owing to Lemma 3(b). There are 3 possibilities up to symmetry, for each of which there exist required s_2- and s_3-paths as follows:
$\{\langle v_1, v_2, v_6, v_7 \rangle, \langle v_3, v_4, v_5 \rangle\}$ for $\{s_2, s_3\} = \{v_1, v_3\}$,
$\{\langle v_1, v_2, v_3, v_7 \rangle, \langle v_6, v_5, v_4 \rangle\}$ for $\{s_2, s_3\} = \{v_1, v_6\}$, and
$\{\langle v_2, v_6, v_7 \rangle, \langle v_3, v_4, v_5, v_1 \rangle\}$ for $\{s_2, s_3\} = \{v_2, v_3\}$. □

Observe that the neighbor set of a vertex in $G(8, 4)$ forms an independent (vertex) set, no two of which are adjacent. (So, the graph has no triangle, as stated in Lemma 1(b).) Moreover, the maximum independent set of $G(8, 4)$ is of size 3. Thus, we can pick up two, y and z, among the three neighbors of

r in G_0 such that $\{\bar{r}, \bar{y}, \bar{z}\}$ does not form an independent set of G_1. (Suppose otherwise, we would have an independent set of size 4 in G_1.) It follows that $\{\bar{r}, \bar{y}, \bar{z}\}$ does not form the neighbor set of a vertex in G_1, leading to that a 3-DPC$[\{(\bar{r}, 1, \emptyset), (\bar{y}, l', \emptyset), (\bar{z}, l'', \emptyset)\}|G_1, \emptyset]$ exists for some $(l', l'') \in \{(3,4),(4,3)\}$ by Claim 2, from which a 2-DPC$[\{(\bar{y}, l', \emptyset), (\bar{z}, l'', \emptyset)\}|G_1, \{\bar{r}\}]$ is obtained. Also in G_0, there exists a 3-starlike tree $T(5,1,1)$ rooted at r in which y and z each is a leaf and also a child of the root by Claim 1. Combining the 3-starlike tree of G_0 with the 2-DPC of $G_1 - F_1$ through edges $(y, \bar{y})$ and $(z, \bar{z})$ results in a 3-starlike tree $T(5,5,4)$ of $G - F$ rooted at r. Therefore, the lemma is proved. ■

Lemma 19. *For the exception (c) of Lemma 15, there exists a required* 2*-starlike tree in* $G - F$.

Proof. The preconditions $l_2 = 2^{m-1} - 1$ and $\bar{r} \in F$ lead to $l_1 = l_2 = 2^{m-1} - 1$ and $f_1^v = f^v = 1$, because $l_1 = (2^m - f^v - 1) - l_2 = (2^m - f_1^v - 1) - (2^{m-1} - 1) = 2^{m-1} - f_1^v \geq l_2 = 2^{m-1} - 1$. Moreover, $|W| \geq 1$ because $m - 2 - |W| = f \leq m - 3$. Firstly, suppose $|W| = 1$. Then, $f_1 = f = m - 3$.

Claim 3. There is a neighbor $y \in V(G_0)$ of r such that a Hamiltonian r–t path exists in $G_0 - \{y\}$ for all $t \in V(G_0) \setminus \{r, y\}$.

Proof. If $m \geq 5$, it suffices to pick up an arbitrary neighbor of r by Lemma 2(a); if $m = 4$, where G_0 is isomorphic to $G(8,4)$ of Fig. 2(a), it suffices to pick up the neighbor of r that stands opposite to r (i.e., v_0 if $r = v_4$, v_1 if $r = v_5$, etc.) by Lemma 3(b). □

Since $f_1 = m - 3$ and $|W| = 1$, there exists a Hamiltonian cycle, $C = \langle u_1, u_2, \ldots, u_n \rangle$ with $n = 2^{m-1} - 1$, in $G_1 - F_1$ such that $u_1 = \bar{y}$ and $u_{n-1}, u_n \notin W$. Also, there exists a Hamiltonian r–$\bar{u_n}$ path P in $G_0 - \{y\}$ by Claim 3. Merging the one-vertex path $\langle y \rangle$, $C - (u_n, u_1)$, and P through edges (r, y), $(y, \bar{y})$, and $(u_n, \bar{u_n})$ results in a Hamiltonian cycle of $G - F$, from which we can extract a required 2-starlike tree whose leaves are u_{n-1} and u_n.

Now, suppose $|W| \geq 2$. Then, $m \geq 5$, $f_1 = f \leq m - 4$, and $W = W_1$. There exists a vertex $z \in V(G_1) \setminus F_1$ such that neither z nor its neighbors are contained in W_1, because the closed neighborhood, Y, of W_1 (i.e., $Y = W_1 \cup \{v \in V(G_1) : v \text{ is a neighbor of } w \in W_1\}$) is of size $|Y| \leq m|W_1| \leq m(m-3) < 2^{m-1} - 1 = |V(G_1) \setminus F_1|$. Pick up a neighbor $y \in V(G_0)$ of r such that $\bar{y} \neq z$, and build a Hamiltonian $\bar{y}$–z path in $G_1 - F_1$. Let the Hamiltonian path be $\langle u_1 = \bar{y}, u_2, \ldots, u_n = z \rangle$ with $n = 2^{m-1} - 1$. Note that $u_{n-1}, z \notin W_1$ by the choice of z. Also, a Hamiltonian r–$\bar{z}$ path exists in $G_0 - \{y\}$ by Lemma 2(a). Merging the one-vertex path $\langle y \rangle$ and the two Hamiltonian paths through edges (r, y), $(y, \bar{y})$ and $(z, \bar{z})$ results in a Hamiltonian cycle of $G - F$, from which we can extract a required 2-starlike tree rooted at r whose leaves are u_{n-1} and z. Therefore, the lemma is proved. ■

4. Conclusion

In this paper, we proved that for a fault-free vertex r in an m-dimensional RHL graph G with a fault set $F \subset V(G) \cup E(G)$, there exists an embedding of a d-starlike tree of order $|V(G) \setminus F|$ into $G - F$ that maps the root of the starlike tree to r, subject to $|F| \leq m-2$ and $|F| + d \leq m$, where $m \geq 3$. The proofs are constructive, so that they may lead to an efficient algorithm for embedding a starlike tree into an RHL graph with faults. Furthermore, the bounds $m-2$ and m, respectively, on $|F|$ and $|F| + d$ are both the best possible. The techniques suggested in this paper may be applicable to other interconnection networks.

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