Ore-Type Degree Conditions for Disjoint Path Covers in Simple Graphs

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Abstract

A *many-to-many k-disjoint path cover* of a graph joining two disjoint vertex subsets S and T of equal size k is a set of k vertex-disjoint paths between S and T that altogether cover every vertex of the graph. It is classified as *paired* if each source in S is required to be paired with a specific sink in T, or *unpaired* otherwise. In this paper, we develop Ore-type sufficient conditions for the existence of many-to-many k-disjoint path covers joining arbitrary vertex subsets S and T. Also, an Ore-type degree condition is established for the one-to-many k-disjoint path cover, a variant derived by allowing to share a single source. The bounds on the degree sum are all best possible.

Keywords: Ore's theorem, degree condition, disjoint path cover, path partition, Hamiltonian-connected.

1. Introduction

Let G be a simple undirected graph, whose vertex and edge sets are denoted by $V(G)$ and $E(G)$, respectively. The *order* of G is the number of vertices in G. If $(u, v) \in E(G)$, u is *adjacent* to v or u is a *neighbor* of v. A *path* from v_1 to v_m is a sequence of vertices, $(v_1, v_2, \ldots, v_m)$, such that v_j is adjacent to v_{j-1} for every $j \in \{2, \ldots, m\}$. A *disjoint path cover* (DPC for short) of G is a set of paths in G such that every vertex of G belongs to one and only one path.

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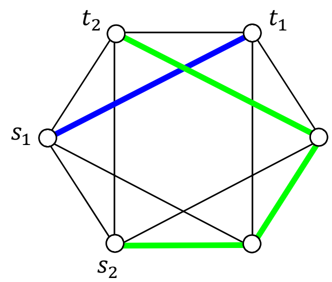
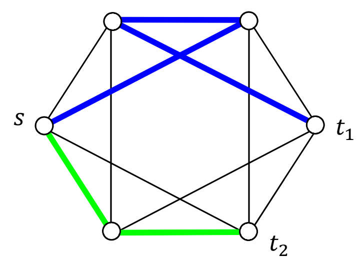


(a) Paired many-to-many 2-DPC.



(b) One-to-many 2-DPC.

Figure 1: Examples of disjoint path covers.

Definition 1 (Many-to-many k-disjoint path cover). *Given two disjoint vertex subsets $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$, each representing k* sources *and* sinks*, the* many-to-many k-disjoint path cover *of G is a disjoint path cover of size k, each of whose paths joins a pair of a source and a sink.*

The disjoint path cover is *paired* if every source s_i must be joined with a specific sink t_i. On the other hand, it is *unpaired* if, for some permutation σ on $\{1, \ldots, k\}$, P_i is a path from s_i to $t_{\sigma(i)}$ for all $i \in \{1, \ldots, k\}$. The sources and sinks are referred to as *terminals*. By these definitions, a paired k-disjoint path cover is always an unpaired k-disjoint path cover. An example of the paired many-to-many DPC is shown in Figure 1(a).

Definition 2. *A graph G is* paired *(resp.* unpaired*)* many-to-many k-coverable *if $|V(G)| \geq 2k$ and there exists a paired (resp. unpaired) many-to-many k-DPC for any disjoint source set S and sink set T of size k each.*

Two simpler variants of the many-to-many k-disjoint path cover can be derived by allowing to share a single source and/or a single sink. The first one is of *one-to-many* type with $S = \{s\}$ and $T = \{t_1, \ldots, t_k\}$, in which all paths start from the unique source s. The second one is of *one-to-one* type with $S = \{s\}$ and $T = \{t\}$, where all internally disjoint paths connect the unique source and sink. Refer to Figure 1(b) for an example of the one-to-many DPC.

Definition 3. *A graph G is* one-to-many *(resp.* one-to-one*)* k-coverable *if*

$|V(G)| \geq k+1$ and G has a one-to-many (resp. one-to-one) k-DPC for any disjoint source set $S = \{s\}$ and sink set $T = \{t_1, \ldots, t_k\}$ (resp. $T = \{t\}$).

The disjoint path cover problem is strongly related to the well-known Hamiltonian problem, which is a fundamental one in graph theory. Actually, a Hamiltonian path joining a pair of vertices in a graph forms a many-to-many, one-to-many, and one-to-one 1-disjoint path covers of the graph. A graph of order $n \geq 3$ is one-to-many 2-coverable if and only if it is Hamiltonian-connected. Moreover, a graph of order $n \geq 3$ is one-to-one 2-coverable if and only if it is Hamiltonian.

One of the core subjects in Hamiltonian graph theory is to develop sufficient conditions for a graph to have a Hamiltonian path/cycle (refer to [10] for a survey). The approaches taken to develop sufficient conditions usually involve degree conditions for providing enough edges to overcome any obstacle to the existence of a Hamiltonian path/cycle. Dirac [4] proved that a graph G of order $n \geq 3$ is Hamiltonian if $d_G(v) \geq n/2$ for every vertex v of G, where $d_G(v)$ denotes the degree of a vertex v in G. Ore [14, 15] improved Dirac's condition as follows:

Theorem 1 (Ore [14, 15]). *(a) A graph G of order $n \geq 3$ is Hamiltonian if $d_G(u) + d_G(v) \geq n$ for all distinct nonadjacent vertices u and v.*
(b) A graph G of order $n \geq 2$ is Hamiltonian-connected if $d_G(u) + d_G(v) \geq n+1$ for all distinct nonadjacent vertices u and v.

The close relationship between the disjoint path cover problem and the Hamiltonian problem motivates the study of developing degree conditions for a graph to have disjoint path covers. In this paper, we establish Ore-type conditions for the existence of k-disjoint path covers in a simple graph as follows:

1. A graph G of order $n \geq 2k$, where $k \geq 1$, is unpaired many-to-many k-coverable if $d_G(u) + d_G(v) \geq n + k$ for every pair of nonadjacent vertices u and v.
2. A graph G of order $n \geq 2k$, where $k \geq 2$, is paired many-to-many k-coverable if $d_G(u) + d_G(v) \geq n + 3k - 4$ for every pair of nonadjacent vertices u and v.
3. A graph G of order $n \geq k+1$, where $k \geq 2$, is one-to-many k-coverable if $d_G(u) + d_G(v) \geq n + k - 1$ for every pair of nonadjacent vertices u and v.

Also, we show that all the above three bounds on the degree sum $d_G(u) + d_G(v)$ are the minimum possible. Note that the results for unpaired many-to-many and one-to-many disjoint path covers are generalizations of Ore's theorem, Theorem 1(b). For the one-to-one disjoint path cover problem, an Ore-type condition was derived by Lin *et al.* [12]: *A graph G of order $n \geq k+1$, where $k \geq 2$, is one-to-one k-coverable if $d_G(u)+d_G(v) \geq n+k-2$ for every pair of nonadjacent vertices u and v. Moreover, the bound on the degree sum is tight.* A one-to-one k-coverable graph is also known as a k^**-connected* graph.

2. Related Works and Definitions

The disjoint path cover problem finds applications in many areas such as software testing, database design, and code optimization [1, 13]. In addition, the problem is concerned with applications where full utilization of network nodes is important [18]. It has been studied with respect to various graphs such as hypercubes [3, 5, 7], recursive circulants [8, 9, 18, 19], hypercube-like graphs [6, 11, 18, 19], cubes of connected graphs [16, 17], k-ary n-cubes [20, 21], and connected graphs [12]. Unfortunately, it is NP-complete to determine whether or not there exists a many-to-many, one-to-many, or one-to-one k-DPC for a given pair of terminal sets in a general graph for any fixed $k \geq 1$ [18, 19].

A *Hamiltonian path* of a graph G is a path that contains all the vertices of G. A graph is said to be *Hamiltonian-connected* if every pair of distinct vertices are joined by a Hamiltonian path. A *cycle* is a closed path of three or more vertices. A *Hamiltonian cycle* of G is a closed Hamiltonian path, i.e., a cycle that contains all the vertices of G. A graph is called *Hamiltonian* if it has a Hamiltonian cycle. An *s–t path* refers to a path from s to t; An *s-path* refers to a path starting at vertex s. For a path $P = (v_1, v_2, \ldots, v_m)$, the *reverse* of P is the path $(v_m, v_{m-1}, \ldots, v_1)$. The vertex and edge sets of path P are denoted by $V(P)$ and $E(P)$, respectively.

The *neighborhood* of a vertex v of G, denoted by $N_G(v)$, is the set of neighbors of v in G, i.e., $N_G(v) = \{u \in V(G) : (v, u) \in E(G)\}$. The *degree* of vertex v in G is the number of its neighbors, i.e., $d_G(v) = |N_G(v)|$. For a vertex subset $W \subseteq V(G)$, the *subgraph of G induced by W* is a graph whose vertex set is W and for every pair of vertices $u, v \in W$, (u, v) is an edge of the graph if and only if $(u, v) \in E(G)$. For a vertex subset $W \subseteq V(G)$, we denote by $G \setminus W$ the resultant subgraph obtained from G by deleting all the vertices in W (including the edges incident to them). Note that $G \setminus W$ is the subgraph of G induced by $V(G) \setminus W$. Graph theoretic terms not defined here can be found in [2].

We will often abbreviate the terms paired many-to-many k-DPC and unpaired many-to-many k-DPC as paired k-DPC and unpaired k-DPC, respectively.

3. Unpaired Many-to-Many Disjoint Path Covers

An unpaired many-to-many k-DPC of a graph G, in which a source set $S = \{s_1, \ldots, s_k\}$ and a sink set $T = \{t_1, \ldots, t_k\}$ such that $S \cap T = \emptyset$ are given, is a set of k pairwise disjoint paths, $\{P_1, \ldots, P_k\}$, such that each P_j joins s_j and $t_{\sigma(j)}$ for some permutation σ on $\{1, \ldots, k\}$ and $\bigcup_{j=1}^{k} V(P_j) = V(G)$. The first main result of this paper is the following.

Theorem 2. *Let G be a graph of order $n \geq 2k$, where $k \geq 1$. If $d_G(u) + d_G(v) \geq n + k$ for all distinct nonadjacent vertices u and v, then G is unpaired many-to-many k-coverable.*

Proof. The proof is by induction on k. For the base step of $k = 1$, the theorem holds by Ore's theorem, Theorem 1(b). Note that G is unpaired many-to-many 1-coverable if and only if G is Hamiltonian-connected. Suppose $k \geq 2$ for the inductive step. Given a source set $S = \{s_1, \ldots, s_k\}$ and a sink set $T = \{t_1, \ldots, t_k\}$ in G such that $S \cap T = \emptyset$, we will show that there always exists an unpaired k-DPC joining S and T in G. Let G' denote the subgraph of G induced by $V(G) \setminus \{t_k\}$, i.e., $G' = G \setminus \{t_k\}$. For any pair of nonadjacent vertices u and v of G', we have $d_{G'}(u) + d_{G'}(v) \geq (d_G(u) - 1) + (d_G(v) - 1) \geq n + k - 2 = |V(G')| + (k - 1)$. Thus, by the induction hypothesis, G' is unpaired many-to-many $(k-1)$-coverable. We are going to construct an unpaired k-DPC of G using some unpaired $(k-1)$-DPC of G'.

Let $\{P_1, \ldots, P_{k-1}\}$ be an unpaired $(k-1)$-DPC of G' joining $S \setminus \{s_k\}$ and $T \setminus \{t_k\}$, in which each path P_j joins s_j and t_{i_j} where $\{i_1, \ldots, i_{k-1}\} = \{1, \ldots, k-1\}$. The path P_j for $1 \leq j \leq k-1$ may be represented as $(v_0^j, \ldots, v_{l_j}^j)$, where $s_j = v_0^j$ and $t_{i_j} = v_{l_j}^j$. There is a path, say P_1, in the unpaired $(k-1)$-DPC that passes through the source s_k as an intermediate vertex. Then, $s_k = v_p^1$ for some $1 \leq p < l_1$, and let $x = v_{p-1}^1$, possibly $x = s_1$. If we delete the edge (x, s_k) from P_1, then P_1 is divided into two paths: the s_1–x path and s_k–t_{i_1} path. It follows that G' has a partition made of k pairwise disjoint s_j-paths for $1 \leq j \leq k$, where each s_j-path for $j \geq 2$ runs to a sink whereas the s_1-path runs to $x \notin T$. Also, the partition of G' along with an additional one-vertex path (t_k) forms a partition of G. We rename v^1_j as v^k_{j-p} for every $p \le j \le l_1$, so that like the other paths, the s_k-path is represented as $(v^k_0, \dots, v^k_{l_k})$ where $l_k = l_1 - p$.

In the remaining part of this proof, we will show that the s_1–x path and the path (t_k) can be combined into an s_1–t_k path; Or the s_1–x path, the path (t_k), and one s_q-path for some $2 \le q \le k$ can be modified into two paths, an s_1–$v^q_{l_q}$ path and s_q–t_k path, each of which joins a source and a sink. If $(x, t_k) \in E(G)$, then it suffices to concatenate the s_1–x path and the path (t_k) into an s_1–t_k path, $(v^1_0, \dots, v^1_{p-1}, t_k)$. So, we assume $(x, t_k) \notin E(G)$ hereafter. If there exists a vertex-pair (v^1_r, v^1_{r+1}) such that $v^1_r \in N_G(x)$ and $v^1_{r+1} \in N_G(t_k)$ for some $0 \le r \le p-3$, it suffices to divide the s_1–x path into two paths, $(v^1_0, \dots, v^1_r)$ and $(v^1_{r+1}, \dots, v^1_{p-1})$, and then concatenate the three paths, $(v^1_0, \dots, v^1_r)$, the reverse of $(v^1_{r+1}, \dots, v^1_{p-1})$, and the path (t_k) into an s_1–t_k path, as illustrated in Figure 2(a). Similarly, if there is a vertex-pair $(v^q_{r'}, v^q_{r'-1})$ for some $2 \le q \le k$ such that $v^q_{r'} \in N_G(x)$ and $v^q_{r'-1} \in N_G(t_k)$ for some $1 \le r' \le l_q$, it suffices to concatenate $(v^1_0, \dots, v^1_{p-1})$ and $(v^q_{r'}, \dots, v^q_{l_q})$ into an s_1–$v^q_{l_q}$ path, and then concatenate $(v^q_0, \dots, v^q_{r'-1})$ and the path (t_k) into an s_q–t_k path, as shown in Figure 2(b). We claim that there always exists such vertex-pair, (v^1_r, v^1_{r+1}) or $(v^q_{r'}, v^q_{r'-1})$, that satisfies the aforementioned conditions.

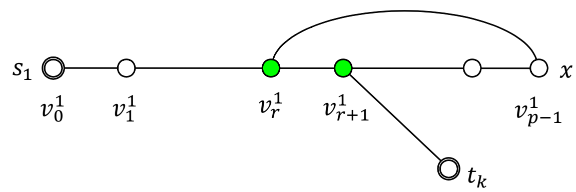
It remains to prove the claim. We let $U = U_1 \cup U_2$, where $U_1 := \{v^1_0, \dots, v^1_{p-3}\}$ and $U_2 := \bigcup_{i=2}^{k} \{v^i_1, \dots, v^i_{l_i}\} = \bigcup_{i=2}^{k} (V(P_i) \setminus \{s_i\})$. Define $X = N_G(x) \cap U$ and $Y = Y_1 \cup Y_2$, where

$$Y_1 := \{v^1_{j-1} : v^1_j \in N_G(t_k),\, 1 \le j \le p-2\} \subseteq U_1, \text{ and}$$
$$Y_2 := \{v^i_{j+1} : v^i_j \in N_G(t_k),\, 2 \le i \le k,\, 0 \le j \le l_i - 1\} \subseteq U_2.$$

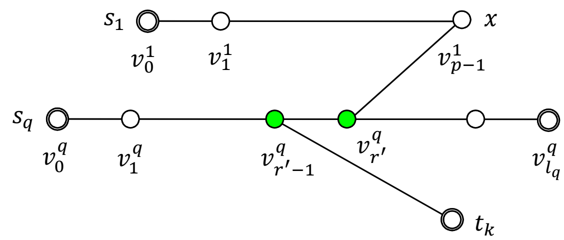
Then, the claim holds if and only if $X \cap Y \ne \emptyset$. Keeping the relationship of $X \cup Y \subseteq U$ in mind, we will derive an upper bound on $|U|$ and lower bounds on $|X|$ and $|Y|$. Consider $|U|$ first. We have $|U_1| = p - 2$ if $p \ge 2$; $|U_1| = 0$ otherwise. Also, we have $|U_2| = n - (p+1) - (k-1) = n - p - k$. It follows that

$$|U| = |U_1| + |U_2| \le \max\{(p-2) + (n-p-k), n-p-k\} \le n-k-1$$

because $p \ge 1$. Consider $|X|$ and $|Y|$ now. Recall $(x, t_k) \notin E(G)$. Observe that among the vertices of $N_G(x)$, there are at most k neighbors (of x) that are not included in X; The possible neighbors are v^1_{p-2} and $\{s_2, \dots, s_k\}$. Also, there are at most k neighbors of t_k, $\{s_1\} \cup \{t_1, \dots, t_{k-1}\}$, that contribute no element to Y. Note that $\{v^i_{l_i} : 2 \le i \le k\} = \{t_1, \dots, t_{k-1}\}$. Thus,



(a) $v_r^1 \in N_G(x)$ and $v_{r+1}^1 \in N_G(t_k)$ for some $0 \leq r \leq p-3$.



(b) $v_{r'}^q \in N_G(x)$ and $v_{r'-1}^q \in N_G(t_k)$ for some $2 \leq q \leq k$ and $1 \leq r' \leq l_q$.

Figure 2: Construction of an unpaired k-DPC of G from an unpaired $(k-1)$-DPC of G'.

we have

$$|X| \geq d_G(x) - k \quad \text{and} \quad |Y| \geq d_G(t_k) - k.$$

Therefore,

$$\begin{aligned} |X \cap Y| &= |X| + |Y| - |X \cup Y| \geq |X| + |Y| - |U| \\ &\geq (d_G(x) - k) + (d_G(t_k) - k) - (n - k - 1) \\ &= d_G(x) + d_G(t_k) - n - k + 1 \\ &\geq (n + k) - n - k + 1 = 1, \end{aligned}$$

proving the claim. This completes the entire proof. ■

Remark 1. The bound, $n + k$, on the degree sum $d_G(u) + d_G(v)$ in Theorem 2 is the minimum possible, in a sense that not every graph G of order $n \geq 2k$ such that $d_G(u) + d_G(v) \geq n + k - 1$ for all distinct nonadjacent vertices u and v, is unpaired many-to-many k-coverable. Consider a graph G of order $n = 3k+1$ defined as the *join* $G_1 + G_2$ of a complete graph G_1 of order $2k$ and a null graph G_2 of order $k+1$. That is, $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{(x, y) : x \in V(G_1),\ y \in V(G_2)\}$. Then, for all distinct nonadjacent vertices u and v of G, we have $d_G(u) + d_G(v) = 2k + 2k =$

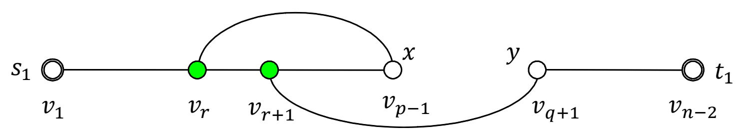
$(3k+1)+(k-1) = n+k-1$. The graph G, however, does not admit an unpaired k-DPC joining S and T if $S, T \subseteq V(G_1)$, i.e., $S \cup T = V(G_1)$. This is because no path in the unpaired k-DPC may pass through two or more vertices of G_2 as intermediate vertices.

4. Paired Many-to-Many Disjoint Path Covers

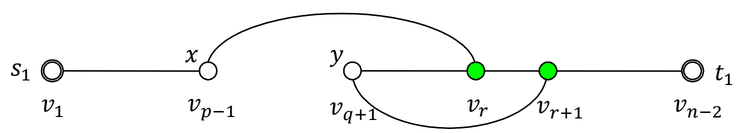
In this section, we prove that a graph G of order $n \geq 2k$, where $k \geq 2$, is paired many-to-many k-coverable if $d_G(u) + d_G(v) \geq n + 3k - 4$ for all distinct nonadjacent vertices u and v of G. On the contrary to the unpaired k-DPC problem, it is not possible to rely on Ore's theorem, Theorem 1(b). So, we begin with the paired 2-DPC problem, as a base step of our inductive proof, in the following two lemmas. Note that if the degree sum condition for $k = 2$ is satisfied, then the graph G of order $n \in \{4, 5\}$ is a complete graph, and thus is paired 2-coverable. (Suppose for a contradiction G is not a complete graph. For some nonadjacent vertices x and y of G, we have $d_G(x) + d_G(y) \leq (n-2) + (n-2) = 2n - 4$, which contradicts the fact that $d_G(x) + d_G(y) \geq n + 3 \cdot 2 - 4 = n + 2$.)

Lemma 1. *Let G be a graph of order $n \geq 6$ in which four distinct terminals s_1, t_1, s_2, and t_2 are given. If $d_G(u) + d_G(v) \geq n + 2$ for all distinct nonadjacent vertices u and v and moreover, there exists a Hamiltonian s_1–t_1 path in $G \setminus \{s_2, t_2\}$, then there exists a paired many-to-many 2-DPC $\{P_1, P_2\}$ in G, where P_1 is an s_1–t_1 path and P_2 is an s_2–t_2 path.*

Proof. Let $P_h = (v_1, v_2, \ldots, v_{n-2})$ denote a Hamiltonian s_1–t_1 path of $G \setminus \{s_2, t_2\}$, where $s_1 = v_1$ and $t_1 = v_{n-2}$. If (s_2, t_2) is an edge of G, then P_h and the two-vertex path (s_2, t_2) form a desired 2-DPC and we are done. So, we assume $(s_2, t_2) \notin E(G)$ hereafter. Then, there exists a common neighbor of s_2 and t_2 in $P'_h := P_h \setminus \{s_1, t_1\} = (v_2, \ldots, v_{n-3})$; Suppose otherwise, $d_G(s_2) + d_G(t_2) \leq (n-4) + 2 + 2 = n$, which contradicts the hypothesis of this lemma that $d_G(s_2) + d_G(t_2) \geq n + 2$. We let $(v_p, \ldots, v_q)$, where $p \leq q$, be a maximal contiguous subsequence of P'_h whose members are all common neighbors of s_2 and t_2, in a sense that there exists no proper contiguous supersequence of $(v_p, \ldots, v_q)$ whose members are all contained in $N_G(s_2) \cap N_G(t_2)$. That is, we have (i) $v_j \in N_G(s_2) \cap N_G(t_2)$ for every $p \leq j \leq q$, (ii) $v_{p-1} \notin N_G(s_2) \cap N_G(t_2)$ or $p = 2$, and (iii) $v_{q+1} \notin N_G(s_2) \cap N_G(t_2)$ or $q = n-3$. Also, let $x = v_{p-1}$ and $y = v_{q+1}$. If $(x, v_i) \in E(G)$ for some $p+1 \leq i \leq q+1$, there exists a paired 2-DPC in which P_1 is the concatenation of $(v_1, \ldots, v_{p-1})$ and $(v_i, \ldots, v_{n-2})$, and P_2 is $(s_2, v_p, \ldots, v_{i-1}, t_2)$. Symmetrically, if $(y, v_j) \in E(G)$ for some $p-$



(a) $v_r \in N_G(x)$ and $v_{r+1} \in N_G(y)$ for some $1 \le r \le p-3$.



(b) $v_r \in N_G(x)$ and $v_{r+1} \in N_G(y)$ for some $q+2 \le r \le n-3$.

Figure 3: Hamiltonian s_1–t_1 paths in the subgraph induced by $\{v_1, \ldots, v_{p-1}\} \cup \{v_{q+1}, \ldots, v_{n-2}\}$.

$1 \le j \le q-1$, there also exists a paired 2-DPC, where P_1 is the concatenation of $(v_1, \ldots, v_j)$ and $(v_{q+1}, \ldots, v_{n-2})$, and P_2 is $(s_2, v_{j+1}, \ldots, v_q, t_2)$. So, we assume $(x, v_i) \notin E(G)$ for every $p+1 \le i \le q+1$ and $(y, v_j) \notin E(G)$ for every $p-1 \le j \le q-1$. Note that $(x, y) \notin E(G)$.

In the remainder of this proof, we will show that there exists a Hamiltonian s_1–t_1 path P_1 in the subgraph induced by $\{v_1, \ldots, v_{p-1}\} \cup \{v_{q+1}, \ldots, v_{n-2}\}$, which implies that P_1 and $P_2 := (s_2, v_p, \ldots, v_q, t_2)$ form a desired 2-DPC. If there is a vertex-pair (v_r, v_{r+1}) such that $v_r \in N_G(x)$ and $v_{r+1} \in N_G(y)$ for some $1 \le r \le p-3$, it suffices to let P_1 be the concatenation of $(v_1, \ldots, v_r)$, the reverse of $(v_{r+1}, \ldots, v_{p-1})$, and $(v_{q+1}, \ldots, v_{n-2})$, as illustrated in Figure 3(a). Similarly, if there is a vertex-pair (v_r, v_{r+1}) such that $v_r \in N_G(x)$ and $v_{r+1} \in N_G(y)$ for some $q+2 \le r \le n-3$, it suffices to let P_1 be the concatenation of $(v_1, \ldots, v_{p-1})$, the reverse of $(v_{q+1}, \ldots, v_r)$, and $(v_{r+1}, \ldots, v_{n-2})$, as shown in Figure 3(b). We claim that there always exists such vertex-pair (v_r, v_{r+1}), where $v_r \in N_G(x)$, $v_{r+1} \in N_G(y)$, and either $1 \le r \le p-3$ or $q+2 \le r \le n-3$.

It remains to prove the claim. We let $U = U_1 \cup U_2$, where $U_1 := \{v_1, \ldots, v_{p-3}\}$ and $U_2 := \{v_{q+2}, \ldots, v_{n-3}\}$. Define $X = N_G(x) \cap U$ and $Y = Y_1 \cup Y_2$, where

$$Y_1 := \{v_{j-1} : v_j \in N_G(y),\ 2 \le j \le p-2\} \subseteq U_1, \text{ and}$$
$$Y_2 := \{v_{j-1} : v_j \in N_G(y),\ q+3 \le j \le n-2\} \subseteq U_2.$$

Then, the claim holds if and only if $X \cap Y \ne \emptyset$. To show $|X \cap Y| \ge 1$ as in the proof of Theorem 2, we will derive an upper bound on $|U|$ and lower bounds on $|X|$ and $|Y|$. Consider $|U|$ first. We have $|U_1| = p-3$ if $p \geq 3$; $|U_1| = 0$ otherwise. Also, we have $|U_2| = n-q-4$ if $q \leq n-4$; $|U_2| = 0$ otherwise. It follows that

$$\begin{aligned}
|U| &= |U_1| + |U_2| \\
&= \begin{cases} (p-3)+(n-q-4) & \text{if } p \geq 3 \text{ and } q \leq n-4, \\ (p-3)+0 & \text{if } p \geq 3 \text{ and } q = n-3, \\ 0+(n-q-4) & \text{if } p = 2 \text{ and } q \leq n-4, \end{cases} \\
&\leq \begin{cases} n-6 & \text{if } p = q = n-3 \text{ or } p = q = 2, \\ n-7 & \text{elsewhere,} \end{cases}
\end{aligned}$$

because $2 \leq p \leq q \leq n-3$. Note that the case where $p = 2$ and $q = n-3$ never occurs. (Suppose $p = 2$ and $q = n-3$. Then, we have $x = s_1$ and $y = t_1$, so that $d_G(x) + d_G(y) \leq |\{v_2, s_2, t_2\}| + |\{v_{n-3}, s_2, t_2\}| = 6$, which contradicts the hypothesis that $d_G(x)+d_G(y) \geq n+2 \geq 8$.) Consider $|X|$ and $|Y|$ now. Recall the assumption that $(x, v_i) \notin E(G)$ for every $p+1 \leq i \leq q+1$ and $(y, v_j) \notin E(G)$ for every $p-1 \leq j \leq q-1$. Observe that among the vertices of $N_G(x)$, there are at most four that are not included in X. The four are v_{p-2}, v_p, t_1, and one of $\{s_2, t_2\}$ if $p \geq 3$ (i.e., $x \neq s_1$); They are v_p, t_1, s_2 and t_2 otherwise (i.e., $x = s_1$). Furthermore, if $p = q = n-3$ (where $y = t_1$ and $x \neq s_1$), there are at most three vertices, v_{p-2}, v_p, and one of $\{s_2, t_2\}$, in $N_G(x) \setminus X$. Note that $x \notin N_G(s_2) \cap N_G(t_2)$ if $x \neq s_1$; $y \notin N_G(s_2) \cap N_G(t_2)$ if $y \neq t_1$. Also, there are at most four vertices of $N_G(y)$ that contribute no element to Y. The four are s_1, v_q, v_{q+2}, and one of $\{s_2, t_2\}$ if $q \leq n-4$ (i.e., $y \neq t_1$); They are s_1, v_q, s_2, and t_2 otherwise (i.e., $y = t_1$). Furthermore, if $p = q = 2$ (where $x = s_1$ and $y \neq t_1$), there are at most three vertices, v_q, v_{q+2}, and one of $\{s_2, t_2\}$, of $N_G(y)$ that contribute no element to Y. Thus, we have

$$|X| \geq \begin{cases} d_G(x)-3 & \text{if } p = q = n-3, \\ d_G(x)-4 & \text{elsewhere;} \end{cases} \qquad |Y| \geq \begin{cases} d_G(y)-3 & \text{if } p = q = 2, \\ d_G(y)-4 & \text{elsewhere.} \end{cases}$$

Therefore,

$$\begin{aligned}
|X \cap Y| &= |X| + |Y| - |X \cup Y| \geq |X| + |Y| - |U| \\
&\geq \begin{cases} (d_G(x)-3)+(d_G(y)-4)-(n-6) & \text{if } p = q = n-3, \\ (d_G(x)-4)+(d_G(y)-3)-(n-6) & \text{if } p = q = 2, \\ (d_G(x)-4)+(d_G(y)-4)-(n-7) & \text{elsewhere,} \end{cases} \\
&= d_G(x) + d_G(y) - n - 1 \\
&\geq (n+2) - n - 1 = 1,
\end{aligned}$$

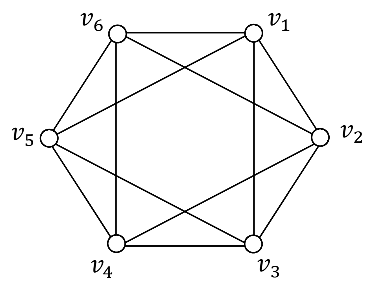


Figure 4: The octahedral graph.

proving the claim. This completes the proof. ■

One might expect that if $d_G(u) + d_G(v) \geq n + 2$ for all distinct nonadjacent vertices u and v in a graph G of order $n \geq 6$, then G is 2-*vertex-fault Hamiltonian-connected*, i.e., for any four distinct vertices s, t, v_f, and w_f, there exists a Hamiltonian s–t path in $G \setminus \{v_f, w_f\}$. Unfortunately, this is not always true. The octahedral graph, shown in Figure 4, is not 2-vertex-fault Hamiltonian-connected (for example, $\{s, t\} = \{v_1, v_4\}$ and $\{v_f, w_f\} = \{v_2, v_5\}$), while the degree sum condition is satisfied. The graph is, in fact, paired many-to-many 2-coverable, which will become clear soon. Instead, we will provide a direct construction of a paired 2-DPC of G satisfying the degree condition in case when $G \setminus \{s_2, t_2\}$ has no Hamiltonian s_1–t_1 path, in the following lemma.

Lemma 2. *Let G be a graph of order $n \geq 6$ in which four distinct terminals s_1, t_1, s_2, and t_2 are given. If $d_G(u) + d_G(v) \geq n + 2$ for all distinct nonadjacent vertices u and v and moreover, there exists no Hamiltonian s_1–t_1 path in $G \setminus \{s_2, t_2\}$, then there exists a paired many-to-many 2-DPC $\{P_1, P_2\}$ in G, where P_1 is an s_1–t_1 path and P_2 is an s_2–t_2 path.*

Proof. Let $G' = G \setminus \{s_2, t_2\}$, the subgraph induced by $V(G) \setminus \{s_2, t_2\}$. Then, there exists a Hamiltonian cycle in G' by Theorem 1(a). This is because for all distinct nonadjacent vertices u and v, $d_{G'}(u) + d_{G'}(v) \geq (d_G(u) - 2) + (d_G(v) - 2) = d_G(u) + d_G(v) - 4 \geq (n + 2) - 4 = |V(G')|$. From the Hamiltonian cycle of G', we can extract a Hamiltonian path, P_h, one of whose end-vertices is t_1. Let P_h be represented as $(v_1, v_2, \ldots, v_{n-2})$, where $t_1 = v_{n-2}$ and $s_1 = v_p$ for some $1 \leq p \leq n - 3$. If $p = 1$, then the path P_h joins s_1 and t_1, which contradicts the hypothesis of this lemma.

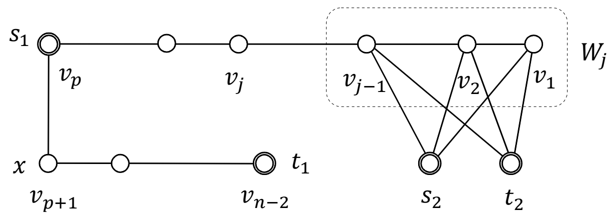


Figure 5: The graph H_j, where $1 \leq j \leq p$.

So, we assume $p \geq 2$. It will be shown that there exists a paired 2-DPC $\{P_1, P_2\}$, where for some $q \in \{2, \ldots, p\}$, P_1 is a Hamiltonian s_1–t_1 path of the subgraph induced by $\{v_q, \ldots, v_{n-2}\}$ and P_2 is a Hamiltonian s_2–t_2 path of the subgraph induced by $\{v_1, \ldots, v_{q-1}\} \cup \{s_2, t_2\}$.

We denote by H_1 the spanning subgraph of G composed of the Hamiltonian path P_h and two isolated vertices s_2 and t_2. Expanding the graph H_1, we define H_j, $1 \leq j \leq p$, to be a graph whose vertex and edge sets, respectively, are

$$V(H_j) = \{v_1, \ldots, v_{n-2}\} \cup \{s_2, t_2\} \text{ and}$$
$$E(H_j) = \{(v_i, v_{i+1}) : 1 \leq i < n-2\} \cup \{(s_2, v_i), (t_2, v_i) : 1 \leq i \leq j-1\},$$

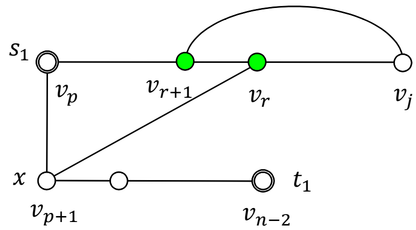
as shown in Figure 5. Let $x = v_{p+1}$ and $W_j = \{v_1, \ldots, v_{j-1}\}$ where $W_1 = \emptyset$.

Claim 1. *If H_j, $j \in \{1, \ldots, p-1\}$, is a spanning subgraph of G such that $N_G(x) \cap W_j = \emptyset$, then*

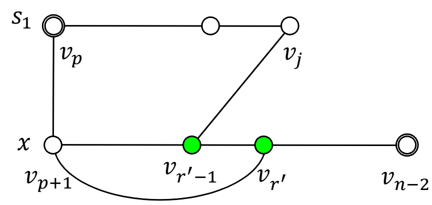
- *the graph G_j, defined as the subgraph of G induced by $\{v_j, \ldots, v_{n-2}\}$, has a Hamiltonian s_1–t_1 path, or*
- *H_{j+1} is a spanning subgraph of G such that $N_G(x) \cap W_{j+1} = \emptyset$.*

Provided the claim is proved, we may conclude that for some $q \in \{1, \ldots, p\}$, the induced subgraph G_q has a Hamiltonian s_1–t_1 path, P_1. Note that H_1 is a spanning subgraph of G such that $N_G(x) \cap W_1 = \emptyset$, and G_p has a Hamiltonian s_1–t_1 path, $(v_p, \ldots, v_{n-2})$. It follows that if $q = 1$, then $G \setminus \{s_2, t_2\}$ has a Hamiltonian s_1–t_1 path, which contradicts the hypothesis of this lemma; If $q \geq 2$, G has a paired 2-DPC made of P_1 and an s_2–t_2 path, $P_2 = (s_2, v_1, \ldots, v_{q-1}, t_2)$. It remains to prove the claim.

Proof of Claim 1. If $(x, v_j) \in E(G)$, then there exists a Hamiltonian s_1–t_1 path, obtained by concatenating the reverse of $(v_j, \ldots, v_p)$ and



(a) $v_r \in N_G(x)$ and $v_{r+1} \in N_G(v_j)$ for some $j+1 \leq r \leq p-1$.



(b) $v_{r'} \in N_G(x)$ and $v_{r'-1} \in N_G(v_j)$ for some $p+3 \leq r' \leq n-2$.

Figure 6: Hamiltonian s_1–t_1 paths in the induced subgraph G_j.

$(v_{p+1}, \ldots, v_{n-2})$, proving the claim. So, we assume $(x, v_j) \notin E(G)$. If there exists a vertex-pair (v_r, v_{r+1}) such that $v_r \in N_G(x)$ and $v_{r+1} \in N_G(v_j)$ for some $j+1 \leq r \leq p-1$, then we have a Hamiltonian s_1–t_1 path of G_j by concatenating the three paths, the reverse of $(v_{r+1}, \ldots, v_p)$, $(v_j, \ldots, v_r)$, and $(v_{p+1}, \ldots, v_{n-2})$, as shown in Figure 6(a). Similarly, if there exists a vertex-pair $(v_{r'}, v_{r'-1})$ such that $v_{r'} \in N_G(x)$ and $v_{r'-1} \in N_G(v_j)$ for some $p+3 \leq r' \leq n-2$, then concatenating the three paths, the reverse of $(v_j, \ldots, v_p)$, the reverse of $(v_{p+1}, \ldots, v_{r'-1})$, and $(v_{r'}, \ldots, v_{n-2})$, results in a Hamiltonian s_1–t_1 path of G_j, as shown in Figure 6(b). Similar to the proof of Lemma 1, we consider the existence of such vertex-pair (v_r, v_{r+1}) or $(v_{r'}, v_{r'-1})$. We let $U = U_1 \cup U_2$, where $U_1 := \{v_{j+1}, \ldots, v_{p-1}\}$ and $U_2 := \{v_{p+3}, \ldots, v_{n-2}\}$. Define $X = N_G(x) \cap U$ and $Y_j = Y_j^1 \cup Y_j^2$, where

$$Y_j^1 := \{v_{i-1} : v_i \in N_G(v_j),\ j+2 \leq i \leq p\} \subseteq U_1, \text{ and}$$
$$Y_j^2 := \{v_{i+1} : v_i \in N_G(v_j),\ p+2 \leq i \leq n-3\} \subseteq U_2.$$

Then, there exists a pair (v_r, v_{r+1}) or $(v_{r'}, v_{r'-1})$ satisfying the aforementioned conditions if and only if $X \cap Y_j \neq \emptyset$. To establish a lower bound on $|X \cap Y_j|$, we will derive inequalities on $|U|$, $|X|$ and $|Y_j|$. We have $|U_1| = p - j - 1$. (Note that $j < p \leq n-3$.) Also, $|U_2| = n - p - 4$ if $p \leq n-4$; $|U_2| = 0$ otherwise. It follows that

$$\begin{aligned} |U| = |U_1| + |U_2| &\leq \max\{(p-j-1) + (n-p-4), (p-j-1) + 0\} \\ &\leq \begin{cases} n-j-5 & \text{if } p \leq n-4 \text{ (i.e., } x \neq t_1), \\ n-j-4 & \text{if } p = n-3 \text{ (i.e., } x = t_1). \end{cases} \end{aligned}$$

At most four vertices, $\{s_1, v_{p+2}, s_2, t_2\}$, among the neighbors of x may not be contained in X if $p \leq n-4$; Otherwise, there are at most three such vertices, $\{s_1, s_2, t_2\}$. Note that $(x, v_j) \notin E(G)$ and $N_G(x) \cap W_j = \emptyset$, i.e., $N_G(x) \cap W_{j+1} = \emptyset$. Also, at most $j+3$ vertices, $\{v_{j+1}, t_1, s_2, t_2\} \cup W_j$, of $N_G(v_j)$ may contribute no element to Y_j if $p \leq n-4$; Otherwise, there are at most $j+2$ such vertices, $\{v_{j+1}, s_2, t_2\} \cup W_j$. Thus, we have

$$|X| \geq \begin{cases} d_G(x) - 4 & \text{if } p \leq n-4, \\ d_G(x) - 3 & \text{if } p = n-3; \end{cases} \qquad |Y_j| \geq \begin{cases} d_G(v_j) - j - 3 & \text{if } p \leq n-4, \\ d_G(v_j) - j - 2 & \text{if } p = n-3. \end{cases}$$

Therefore,

$$\begin{aligned} |X \cap Y_j| &= |X| + |Y_j| - |X \cup Y_j| \geq |X| + |Y_j| - |U| \\ &\geq \begin{cases} (d_G(x) - 4) + (d_G(v_j) - j - 3) - (n - j - 5) & \text{if } p \leq n-4, \\ (d_G(x) - 3) + (d_G(v_j) - j - 2) - (n - j - 4) & \text{if } p = n-3, \end{cases} \\ &= \begin{cases} d_G(x) + d_G(v_j) - n - 2 \geq (n+2) - n - 2 = 0 & \text{if } p \leq n-4, \\ d_G(x) + d_G(v_j) - n - 1 \geq (n+2) - n - 1 \geq 1 & \text{if } p = n-3. \end{cases} \end{aligned}$$

If $|X \cap Y_j| \geq 1$, then there exists a desired pair (v_r, v_{r+1}) or $(v_{r'}, v_{r'-1})$ and the graph G_j has a Hamiltonian s_1–t_1 path, proving the claim. So, we assume $|X \cap Y_j| = 0$. It follows that $p \leq n-4$, $|Y_j| = d_G(y) - j - 3$ and moreover, $\{v_{j+1}, t_1, s_2, t_2\} \cup W_j \subseteq N_G(v_j)$. This implies that H_{j+1} is a spanning subgraph of G such that $N_G(x) \cap W_{j+1} = \emptyset$, also proving the claim. □ ■

Theorem 3. *Let G be a graph of order $n \geq 4$. If $d_G(u) + d_G(v) \geq n+2$ for all distinct nonadjacent vertices u and v, then G is paired many-to-many 2-coverable.*

Proof. If $n \in \{4, 5\}$, then G is a complete graph and thus is paired many-to-many 2-coverable. For $n \geq 6$, the proof is a direct consequence of Lemmas 1 and 2. ■

Now, we are ready to consider the paired k-DPC problem for general $k \geq 2$.

Theorem 4. *Let G be a graph of order $n \geq 2k$, where $k \geq 2$. If $d_G(u) + d_G(v) \geq n + 3k - 4$ for all distinct nonadjacent vertices u and v, then G is paired many-to-many k-coverable.*

Proof. The proof proceeds by induction on k. The base case of $k = 2$ holds due to Theorem 3. Let $k \geq 3$ for the inductive step. We first claim that $(s_k, t_k) \in E(G)$, or there exists a nonterminal vertex $w \in N_G(s_k) \cap N_G(t_k)$.

To prove the claim, suppose $(s_k, t_k) \notin E(G)$. Then, we have $d_G(s_k) + d_G(t_k) \geq n + 3k - 4$ by the hypothesis of the theorem. This implies that there are at least $(n+3k-4)-(n-2) = 3k-2$ vertices in $N_G(s_k) \cap N_G(t_k)$ and among them, there are at least $(3k-2)-2(k-1) = k \geq 3$ nonterminal vertices. Note that $s_k, t_k \notin N_G(s_k) \cup N_G(t_k)$. Thus, the claim is proved.

We let an s_k–t_k path, P_k, be the two-vertex path (s_k, t_k) if $(s_k, t_k) \in E(G)$; let $P_k = (s_k, w, t_k)$ otherwise. It suffices to prove that $G \setminus V(P_k)$, the subgraph of G induced by $V(G) \setminus V(P_k)$, is paired many-to-many $(k-1)$-coverable. Let G' be the induced subgraph, and let $n_k = |V(P_k)|$ where $n_k = 2$ or 3. For any two nonadjacent vertices u and v of G',

$$\begin{aligned} d_{G'}(u) + d_{G'}(v) &\geq (d_G(u) - n_k) + (d_G(v) - n_k) = d_G(u) + d_G(v) - 2n_k \\ &\geq (n + 3k - 4) - 2n_k = (n - n_k) + (3k - n_k) - 4 \\ &\geq |V(G')| + 3(k-1) - 4. \end{aligned}$$

Therefore, G' is paired $(k-1)$-coverable by the induction hypothesis, completing the proof. ∎

Remark 2. The bound, $n + 3k - 4$, on the degree sum $d_G(u) + d_G(v)$ in Theorem 4 is the best possible. Consider a graph G of order $n = 3k - 1$, obtained from a complete graph, K_n, of order n by deleting k pairwise nonadjacent edges (u_i, v_i) for $1 \leq i \leq k$. That is, $V(G) = V(K_n)$ and $E(G) = E(K_n) \setminus M$, where M is a matching of K_n of size k. Then, for any two nonadjacent vertices, u_i and v_i for some i, of G, we have $d_G(u_i) + d_G(v_i) = (n-2) + (n-2) = n + (n-4) = n + 3k - 5$. The graph G, however, does not admit a paired k-DPC joining $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ if $s_i = u_i$ and $t_i = v_i$ for all i, because every s_i–t_i path should pass through at least one nonterminal vertex as an intermediate vertex, which is impossible. Therefore, not every graph G of order $n \geq 2k$ such that $d_G(u) + d_G(v) \geq n + 3k - 5$ for all distinct nonadjacent vertices u and v, is paired many-to-many k-coverable.

5. One-to-Many Disjoint Path Covers

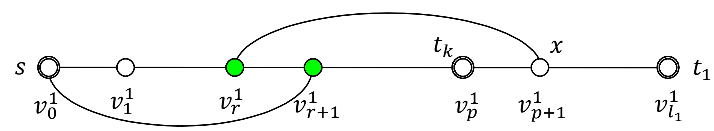
In this section, an Ore-type degree condition will be established for the one-to-many k-disjoint path covers. Given a single source s and a set of k distinct sinks $T = \{t_1, \ldots, t_k\}$ in a graph G such that $s \notin T$, a one-to-many k-DPC joining s and T refers to a set of k paths, $\{P_1, \ldots, P_k\}$, where each path P_i runs from the source s to sink t_i such that $\bigcup_{i=1}^{k} V(P_i) = V(G)$ and $V(P_i) \cap V(P_j) = \{s\}$ for all $i \neq j$.

Theorem 5. *Let G be a graph of order $n \geq k+1$, where $k \geq 2$. If $d_G(u) + d_G(v) \geq n+k-1$ for all distinct nonadjacent vertices u and v, then G is one-to-many k-coverable.*

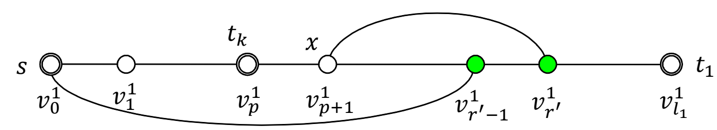
PROOF. The proof is by induction on k. The base case of $k = 2$ holds due to Ore's theorem, Theorem 1(b). Note that a graph of order at least three is one-to-many 2-coverable if and only if it is Hamiltonian-connected. Suppose $k \geq 3$ for the inductive step. Given a source s and a sink set $T = \{t_1, \ldots, t_k\}$ in G such that $s \notin T$, we will show that G has a one-to-many k-DPC joining s and T. By the induction hypothesis, G has a one-to-many $(k-1)$-DPC joining s and $T \setminus \{t_k\}$. Let $\{P_1, \ldots, P_{k-1}\}$ denote the one-to-many $(k-1)$-DPC, in which each path P_i joins s and t_i. The path P_i for $1 \leq i \leq k-1$ may be represented as $(v^i_0, \ldots, v^i_{l_i})$, where $s = v^i_0$ and $t_i = v^i_{l_i}$. There exists a path, say P_1, in the $(k-1)$-DPC that passes through the sink t_k as an intermediate vertex. Then, $t_k = v^1_p$ for some $1 \leq p \leq l_1 - 1$, and let $x = v^1_{p+1}$, possibly $x = t_1$. If $(s, x) \in E(G)$, then it suffices to let $P_k = (v^1_0, \ldots, v^1_p)$ and redefine $P_1 = (s, v^1_{p+1}, \ldots, v^1_{l_1})$, so that $\{P_1, \ldots, P_k\}$ is a desired k-DPC of G. Hereafter, we assume $(s, x) \notin E(G)$, and thus $d_G(s) + d_G(x) \geq n+k-1$.

If there exists a vertex-pair (v^1_r, v^1_{r+1}) such that $v^1_r \in N_G(x)$ and $v^1_{r+1} \in N_G(s)$ for some $1 \leq r \leq p-1$, it suffices to let $P_k = (s, v^1_{r+1}, \ldots, v^1_p)$ and redefine P_1 as the concatenation of $(v^1_0, \ldots, v^1_r)$ and $(v^1_{p+1}, \ldots, v^1_{l_1})$, as illustrated in Figure 7(a). Similarly, if there is a vertex-pair $(v^1_{r'}, v^1_{r'-1})$ such that $v^1_{r'} \in N_G(x)$ and $v^1_{r'-1} \in N_G(s)$ for some $p+3 \leq r' \leq l_1$, then it suffices to let $P_k = (v^1_0, \ldots, v^1_p)$ and redefine P_1 as the concatenation of three path segments, one-vertex path (s), the reverse of $(v^1_{p+1}, \ldots, v^1_{r'-1})$, and $(v^1_{r'}, \ldots, v^1_{l_1})$, as shown in Figure 7(b). In addition, if there exists a vertex-pair (v^i_q, v^i_{q+1}) for some $2 \leq i \leq k-1$ such that $v^i_q \in N_G(x)$ and $v^i_{q+1} \in N_G(s)$ for some $1 \leq q \leq l_i - 1$, it suffices to let $P_k = (v^1_0, \ldots, v^1_p)$ and redefine P_1 and P_i as follows (see Figure 7(c)): P_1 is the concatenation of $(v^i_0, \ldots, v^i_q)$ and $(v^1_{p+1}, \ldots, v^1_{l_1})$, and P_i is the concatenation of the path (s) and $(v^i_{q+1}, \ldots, v^i_{l_i})$. It remains to prove that there always exists at least one such vertex-pair, (v^1_r, v^1_{r+1}), $(v^1_{r'}, v^1_{r'-1})$, or (v^i_q, v^i_{q+1}), that satisfies the aforementioned conditions.

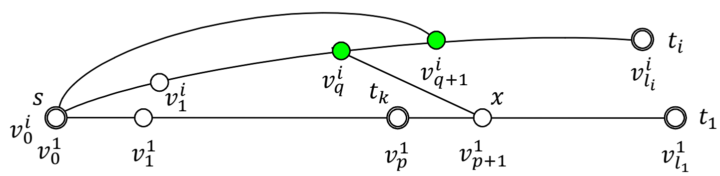
As in the proof of Theorem 2, we let $U = U_1 \cup U_2 \cup U_3$, where $U_1 := \{v^1_1, \ldots, v^1_{p-1}\}$, $U_2 := \{v^1_{p+3}, \ldots, v^1_{l_1}\}$, and $U_3 := \bigcup_{i=2}^{k-1} \{v^i_1, \ldots, v^i_{l_i - 1}\} = \bigcup_{i=2}^{k-1} V(P_i) \setminus (\{s\} \cup \{t_2, \ldots, t_{k-1}\})$. Define $X = N_G(x) \cap U$ and $Y =$



(a) $v_r^1 \in N_G(x)$ and $v_{r+1}^1 \in N_G(s)$ for some $1 \leq r \leq p-1$.



(b) $v_{r'}^1 \in N_G(x)$ and $v_{r'-1}^1 \in N_G(s)$ for some $p+3 \leq r' \leq l_1$.



(c) $v_q^i \in N_G(x)$ and $v_{q+1}^i \in N_G(s)$ for some $2 \leq i \leq k-1$ and $1 \leq q \leq l_i - 1$.

Figure 7: Construction of a one-to-many k-DPC from a one-to-many $(k-1)$-DPC of G.

$Y_1 \cup Y_2 \cup Y_3$, where

$$\begin{aligned}
Y_1 &:= \{v^1_{j-1} : v^1_j \in N_G(s),\, 2 \le j \le p\} \subseteq U_1,\\
Y_2 &:= \{v^1_{j+1} : v^1_j \in N_G(s),\, p+2 \le j \le l_1 - 1\} \subseteq U_2, \text{ and}\\
Y_3 &:= \{v^i_{j-1} : v^i_j \in N_G(s),\, 2 \le i \le k-1,\, 2 \le j \le l_i\} \subseteq U_3.
\end{aligned}$$

It suffices to show that $|X \cap Y| \ge 1$. An upper bound on $|U|$ and lower bounds on $|X|$ and $|Y|$ will be derived first. We have $|U_1| = p - 1$. Also, we have $|U_2| = l_1 - p - 2$ if $p \le l_1 - 2$; $|U_2| = 0$ otherwise. In addition, $|U_3| = (n - l_1) - (1 + (k-2)) = n - l_1 - k + 1$. It follows that

$$\begin{aligned}
|U| &= |U_1| + |U_2| + |U_3|\\
&\le \begin{cases} (p-1) + (l_1 - p - 2) + (n - l_1 - k + 1) = n - k - 2 & \text{if } p \le l_1 - 2,\\ (p-1) + 0 + (n - l_1 - k + 1) = n - k - 1 & \text{if } p = l_1 - 1. \end{cases}
\end{aligned}$$

Consider $|X|$ and $|Y|$ now. Recall the assumption that $(s, x) \notin E(G)$. There are at most k neighbors of x, $\{v^1_p, v^1_{p+2}\} \cup \{t_2, \ldots, t_{k-1}\}$, that are not included in X. Furthermore, if $p = l_1 - 1$ (or equivalently, $x = t_1$), at most $k - 1$ neighbors of x, $\{v^1_p\} \cup \{t_2, \ldots, t_{k-1}\}$, are not included in X. Also, if $p \le l_1 - 2$, there are at most k neighbors of s, $\{v^1_1, t_1\} \cup \{v^2_1, \ldots, v^{k-1}_1\}$, that contribute no element to Y; if $p = l_1 - 1$, there are at most $k - 1$ such neighbors of s, $\{v^1_1\} \cup \{v^2_1, \ldots, v^{k-1}_1\}$. Thus, we have

$$|X| \ge \begin{cases} d_G(x) - k & \text{if } p \le l_1 - 2,\\ d_G(x) - k + 1 & \text{if } p = l_1 - 1; \end{cases} \quad |Y| \ge \begin{cases} d_G(s) - k & \text{if } p \le l_1 - 2,\\ d_G(s) - k + 1 & \text{if } p = l_1 - 1. \end{cases}$$

Therefore,

$$\begin{aligned}
|X \cap Y| = |X| + |Y| - |X \cup Y| &\ge |X| + |Y| - |U|\\
&\ge \begin{cases} (d_G(x) - k) + (d_G(s) - k) - (n - k - 2) & \text{if } p \le l_1 - 2,\\ (d_G(x) - k + 1) + (d_G(s) - k + 1) - (n - k - 1) & \text{if } p = l_1 - 1, \end{cases}\\
&\ge d_G(x) + d_G(s) - (n + k - 2)\\
&\ge (n + k - 1) - (n + k - 2) = 1,
\end{aligned}$$

proving the claim. This completes the entire proof. ■

Remark 3. The bound, $n + k - 1$, on the degree sum $d_G(u) + d_G(v)$ in Theorem 5 is the minimum possible. Consider a graph G of order $n \ge k + 2$ made of a complete subgraph K_{n-1} and a single vertex $w \notin V(K_{n-1})$ that is directly connected, via an edge, to each of some k vertices of K_{n-1}. That is, $V(G) = V(K_{n-1}) \cup \{w\}$ and $E(G) = E(K_{n-1}) \cup \{(w,z) : z \in Z\}$ for some k-vertex subset Z of the subgraph K_{n-1}. Then, we have $d_G(u) + d_G(v) = n + k - 2$ for all distinct nonadjacent vertices u and v of G. The graph G, however, does not admit a one-to-many k-DPC joining s and $T = \{t_1, \ldots, t_k\}$ if $s = w$ and $T = N_G(w)$, because no s–t_i path may pass through a nonterminal vertex. Note that there exists a nonterminal vertex in G since $n \geq k + 2$.

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