Strong Matching Preclusion

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Abstract

The *matching preclusion* problem, introduced by Brigham et al. [Perfect-matching preclusion, Congressus Numerantium 174 (2005) 185-192], studies how to effectively make a graph have neither perfect matchings nor almost perfect matchings by deleting as small a number of edges as possible. Extending this concept, we consider a more general matching preclusion problem, called the *strong matching preclusion*, in which deletion of vertices is additionally permitted. We establish the strong matching preclusion number and all possible minimum strong matching preclusion sets for various classes of graphs.

Keywords: Perfect and almost perfect matching; Matching preclusion; Vertex and/or edge deletion; Restricted hypercube-like graph; Recursive circulant.

1. Introduction

A matching of a graph is a set of pairwise nonadjacent edges. For a graph with n vertices, a matching M is called *perfect* if its size $|M|$ is $n/2$ for even n, or *almost perfect* if $|M| = (n-1)/2$ for odd n. A set F of edges in a graph $G = (V, E)$ is called a *matching preclusion set* (MP set for short) if $G \setminus F$ has neither a perfect matching nor an almost perfect matching. The *matching preclusion number* of G (MP number for short), denoted by $mp(G)$, is defined to be the minimum size of all possible such sets of G. Then the minimum MP set of G is any MP set whose size is $mp(G)$.

Since the problem of matching preclusion was first presented by Brigham *et al.* [4], several classes of graphs have been studied to understand their matching preclusion properties: Petersen, complete, and complete bipartite graphs and hypercubes [4]; Cayley graphs generated by transpositions and (n,k)-star graphs [7]; restricted HL-graphs and recursive circulants $G(2^m, 4)$ [12]. A conditional version of the problem, in which a matching preclusion is not permitted to produce a graph with an isolated vertex, was also discussed by Cheng *et al.* [5], and then was further studied in the following works [6, 16].

An obvious application of the matching preclusion problem was addressed in [4]: when each node of interconnection networks is demanded to have a special partner at any time, those that have larger matching preclusion numbers will be more robust in the event of link failures. Another form of matching obstruction, which is in fact more offensive, is through node failures. The robustness of graphs with respect to the property of having a perfect matching has also been analyzed under vertex deletions, for instance; in [1, 9].

In this article, we move forward one step further, considering a more general matching preclusion problem, which is defined as follows.

Definition 1. *A set F of vertices and/or edges in a graph G is called a* strong matching preclusion set *(SMP set for short) if $G \setminus F$ has neither a perfect matching nor an almost perfect matching. The* strong matching preclusion number *(SMP number for short) of G, denoted by $smp(G)$, is defined to be the minimum size of all possible such sets of G. The* minimum SMP set *of G is any SMP set whose size is $smp(G)$.*

Obviously, when G itself does not contain any matching, whether perfect or almost perfect, both $smp(G)$ and $mp(G)$ are regarded as zero. These numbers are undefined for a trivial graph with only one vertex. Notice that an MP set of a graph is a special SMP set of the graph made of edges only.

Proposition 1. *For every nontrivial graph G, $smp(G) \leq mp(G)$.*

When a set F of vertices and/or edges are removed from a graph, the set is called *fault set*, and their elements are respectively referred to as *fault vertices* and *fault edges*, whose sets are denoted by F_v and F_e, respectively ($F = F_v \cup F_e$). Some fault sets produce a faulty graph, containing isolated vertices. For example, deleting the set $N_G(v)$ of all neighboring vertices adjacent from a given vertex v of G separates v from the remaining graph. Similarly, removing the set $I_G(v)$ of all edges incident on v isolates v. Moreover, a combination of such vertices and edges may isolate a vertex, forming a simple SMP set of G, as described in the next proposition.

Proposition 2. *Given a fault vertex set $X(v) \subseteq N_G(v)$ and a fault edge set $Y(v) \subseteq I_G(v)$, $X(v) \cup Y(v)$ is an SMP set of G if (a)*

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$w \in X(v)$ if and only if $(v, w) \notin Y(v)$ for every $w \in N_G(v)$, and (b) the number of vertices in $G \setminus (X(v) \cup Y(v))$ is even.

Proof. The resultant graph $G \setminus (X(v) \cup Y(v))$, containing an isolated vertex v, has an even number of vertices. Since the graph has an isolated vertex v, it cannot have a perfect matching. □

This proposition suggests an easy way of building SMP sets. Any SMP set constructed as specified in Proposition 2 is called *trivial* and treated specially. It is straightforward to see that, for an arbitrary vertex of degree at least one, there always exists a trivial SMP set that isolates the vertex. This observation leads to the following fact.

Proposition 3. *For any graph G with no isolated vertices, $smp(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G.*

Remark 1. The above proposition is not valid for graphs with isolated vertices. For example, consider a graph G having two connected components K_1 and K_4, where K_n denotes a complete graph with n vertices. Clearly, $\delta(G) = 0$, but $smp(G) = 1$, in which every minimum SMP set is one obtained by deleting a vertex of K_4. Note that $mp(G) = 3$.

In this paper, a path in a graph is defined as a sequence of adjacent vertices, whose *length* refers to the number of vertices in the sequence. A path is called an *even path* if its length is even. Otherwise, it is called an *odd path*. Furthermore, we say that a graph is *matchable* if it has either a perfect matching or an almost perfect matching. Otherwise, the graph is called *unmatchable*. Finally, this section is concluded with one more important proposition that will be frequently referred to afterward.

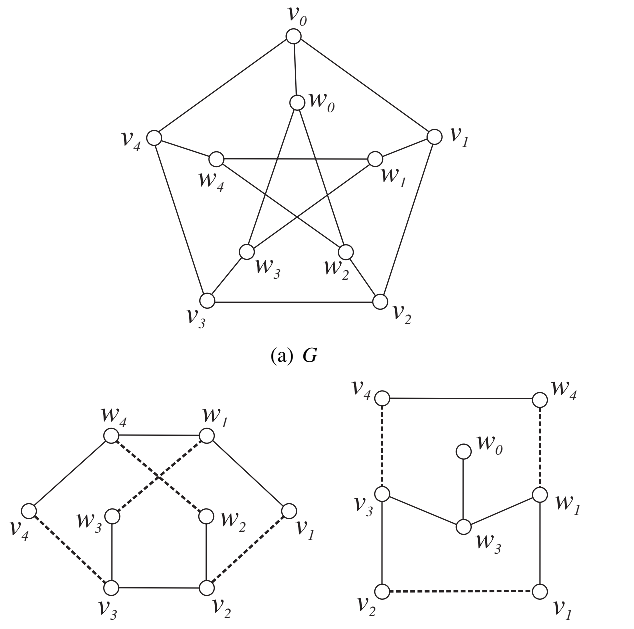
Proposition 4. *Let F be a fault set of a graph G. Then, $G \setminus F$ is matchable if and only if $G \setminus F$ can be spanned by a set of disjoint even paths with at most one exceptional odd path.*

Proof. The necessity is obvious. An even path can be further partitioned into a set of paths of length two, i.e. matchings, while an odd one can be partitioned into matchings plus a single vertex. This implies that the sufficiency holds. □

2. Petersen Graph and Complete Graphs

The Petersen graph is a well-known 3-regular graph (Figure 1(a)). It is *distance-transitive* [2], that is, for any set of four vertices u, v, x, and y satisfying $d(u, v) = d(x, y)$ for the distance function d, there exists an automorphism h such that $h(u) = x$ and $h(v) = y$. Thus, the graph is vertex-transitive and edge-transitive. The Petersen graph is *hypohamiltonian* [3]. In other words, the graph itself does not contain a hamiltonian cycle, but each of its subgraphs obtained by removing a single vertex is hamiltonian.

Theorem 1. *For the Petersen graph G, $smp(G) = 3$. Furthermore, each of its minimum SMP sets is trivial or equivalent to $\{(v_0, w_0), (v_2, v_3), (w_1, w_4)\}$.*



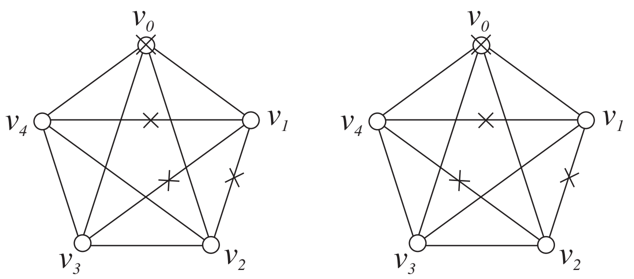
(a) G

(b) $G \setminus \{v_0, w_0\}$ (c) $G \setminus \{v_0, w_2\}$

Figure 1: Petersen graph G and its two subgraphs.

Proof. It was shown in [4] that $mp(G) = 3$ and every minimum MP set is trivial or equivalent to $\{(v_0, w_0), (v_2, v_3), (w_1, w_4)\}$. Since G is hypohamiltonian, any SMP set containing a vertex must have at least three elements. Hence, $smp(G) = 3$ and it suffices to show that, for any SMP set F with $|F| = 3$, containing at least one vertex, F is a trivial SMP set. Suppose that $|F_v| = 1$ or 3 (recall that $F_v = F \cap V(G)$). Consider the hamiltonian cycle in $G \setminus v_f$ for a vertex $v_f \in F$. If we remove the remaining two fault elements from $G \setminus v_f$, the hamiltonian cycle needs to be broken into two path segments, but one of them must be even. Then, by Proposition 4, $G \setminus F$ is matchable, leading to a contradiction. Suppose $|F_v| = 2$. In this case, we can assume w.l.o.g. that F_v is either $\{v_0, w_0\}$ or $\{v_0, w_2\}$ because G is distance-transitive. The first case is impossible because $G \setminus \{v_0, w_0\}$ has two disjoint perfect matchings as illustrated in Figure 1(b), implying that $G \setminus F$ is matchable. When $F_v = \{v_0, w_2\}$, the only possible edge choice is (w_0, w_3), in which case F forms a trivial SMP set. Otherwise, $G \setminus F$ is matchable since $G \setminus (\{v_0, w_2\} \cup \{w_0, w_3\})$ contains two disjoint perfect matchings as shown in Figure 1(c). This completes the proof. □

Now, consider a minimum SMP set F of a complete graph K_n. It is straightforward to see that F_e $(= F \setminus F_v)$ is an MP set of $K_n \setminus F_v$ and its cardinality is the minimum possible. This implies that the SMP number and the minimum SMP sets of K_n may be derived from the MP number and the minimum MP sets of K_p for $p \leq n$. According to [4], $mp(K_n) = n - 1$ for every even $n \geq 2$. Also, every minimum MP set of K_n is trivial for every even n except 4; for $n = 4$, it is trivial or forms a triangle (a cycle of length three). Finaly, $mp(K_n) \geq n$ for every odd $n \geq 3$.



(a) Trivial (b) Nontrivial

Figure 2: Two examples of minimum SMP sets of K_5.

Theorem 2. *For every $n \geq 2$, $smp(K_n) = n - 1$. Furthermore, each of its minimum SMP sets is trivial or is $F_v \cup F_e$, where $|F_v| = n - 4$ and F_e forms a triangle in $K_n \setminus F_v$.*

Proof. For $f_v = |F_v|$, $smp(K_n) = \min_{0 \leq f_v \leq n-2}\{f_v + mp(K_{n-f_v})\} = f_v + (n - f_v - 1) = n - 1$, where the minimum is achieved at even $n - f_v$. Let $F_v \cup F_e$ be a minimum SMP set of K_n. Since F_e is a minimum MP set of $K_n \setminus F_v$, having an even number of vertices, F_e is always trivial in $K_n \setminus F_v$, indicating $F_v \cup F_e$ is also trivial in K_n, except when $K_n \setminus F_v$ is isomorphic to K_4. In that case, $|F_v| = n - 4$ and any nontrivial F_e forms a triangle in $K_n \setminus F_v$. □

Figure 2 illustrates two examples of the minimum SMP sets of K_5, where the symbol × marks the fault elements.

3. Bipartite Graphs and Almost Bipartite Graphs

It is not difficult to imagine that the SMP number of a bipartite graph is small. Let $G = (B \cup W, E)$ be a connected bipartite graph with two nonempty partite sets B and W. If $|B| \geq |W| + 2$, then G itself is not matchable and $smp(G) = 0$; if $|B| = |W| + 1$, then $G \setminus v$ for any $v \in W$ is not matchable, thus $smp(G) \leq 1$; if $|B| = |W|$ and $|B| \geq 2$, then any two vertices of B form an SMP set of G and thus $smp(G) \leq 2$.

For a regular bipartite graph, the SMP number becomes fixed. Let G be a connected m-regular bipartite graph. This graph is known to be *1-factorable* [3]. That is, the edges of G can be partitioned into m disjoint perfect matchings, called *1-factors*, implying that its MP number is m. On the other hand, its SMP number is always two regardless of the degree m.

Theorem 3. *For a connected m-regular bipartite graph G with $m \geq 3$, $smp(G) = 2$. Furthermore, each of its minimum SMP sets is a set of two vertices from the same partite set.*

Proof. Consider a fault set F with $|F| = 2$. If F consists of two edges, $G \setminus F$ is matchable since $mp(G) = m > 2$. Second, suppose that F contains one vertex and one edge. Since there exist m disjoint perfect matchings in G, deletion of the vertex leaves m almost perfect matchings, indicating that $G \setminus F$ is also matchable in spite of the additional edge removal. Now, assume that F is made of two vertices u and v such that $u \in B$ and $v \in W$ for the partite sets B and W of G. We first clam that, for any nonempty *proper* subset B' of B, $|N_G(B')| \geq |B'| + 1$, where the neighbors $N_G(X)$ of a given vertex subset X of graph G is defined to be $\bigcup_{x \in X} N_G(x)$. Suppose the claim is wrong, i.e. $|N_G(B')| = |B'|$ for some B'. Then, $N_G(N_G(B')) = B'$ for the proper subset B' of B, which contradicts the condition that G is connected. Therefore, for any nonempty subset B' of $B \setminus u$, $|N_G(B') \setminus v| \geq |B'|$. By the Hall's marriage theorem, $G \setminus F$ has a perfect matching. Finaly, the remaining case of F made of two vertices from the same partite set becomes the only possible SMP set of G. □

A set S of vertices and/or edges in a graph G is called a *bipartization set* if $G \setminus S$ becomes bipartite. A *bipartization number* $b(G)$ of G is the minimum cardinality among all the bipartization sets of G [8]. Since $smp(G \setminus S) \leq 2$, the following theorem holds.

Theorem 4. *For any graph G, $smp(G) \leq b(G) + 2$.*

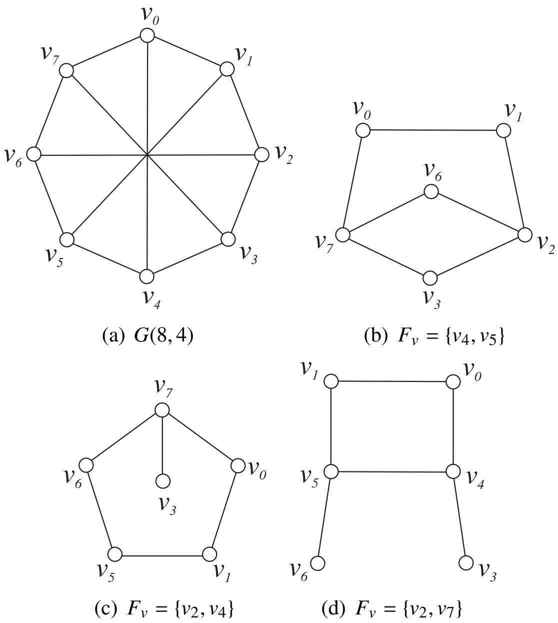
From this theorem, we can see that, for an 'almost bipartite' graph with a small bipartization number, its SMP number is also small, whereas its MP number may be large.

4. Restricted HL-Graphs

The next class of graphs we study in this section is defined using a special graph construction operator. Given two graphs G_0 and G_1, consider a set $\Phi(G_0, G_1)$, made of all bijections from $V(G_0)$ to $V(G_1)$. Then, given a bijection $\phi \in \Phi(G_0, G_1)$, we denote by $G_0 \oplus_\phi G_1$ a graph whose vertex set is $V(G_0) \cup V(G_1)$ and edge set is $E(G_0) \cup E(G_1) \cup \{(v, \phi(v)) : v \in V(G_0)\}$. Here, G_0 and G_1 are called the *components* of $G_0 \oplus_\phi G_1$, where every vertex v in one component has a unique neighbor $\bar{v}$ in the other one. To simplify the notation, we often omit the bijection ϕ from $\oplus_\phi$ when it is clear in the context.

Based on the graph constructor, Vaidya *et al.* [18] gave a recursive definition of a class of graphs, called the *hypercube-like graphs* (HL-graphs for short): $HL_0 = \{K_1\}$ and $HL_m = \{G_0 \oplus_\phi G_1 : G_0, G_1 \in HL_{m-1}, \phi \in \Phi(G_0, G_1)\}$ for $m \geq 1$. A graph in a subclass HL_m is made of 2^m vertices of degree m, and is called an *m-dimensional HL-graph*. Their network properties in the presence of faults have been studied in view of applications to parallel computing: hamiltonicity [14, 10], disjoint path covers [15], and diagnosability [11].

An interesting subset of the HL-graphs is the *restricted HL-graphs*, which are defined recursively as follows [14]: $RHL_3 = HL_3 \setminus Q_3 = \{G(8, 4)\}$; $RHL_m = \{G_0 \oplus_\phi G_1 : G_0, G_1 \in RHL_{m-1}, \phi \in \Phi(G_0, G_1)\}$ for $m \geq 4$. Here, Q_3 is the 3-dimensional hypercube, and $G(8, 4)$ is a recursive circulant whose vertex set is $\{v_i : 0 \leq i \leq 7\}$ and edge set is $\{(v_i, v_j) : j \equiv i+1 \text{ or } i+4 \pmod 8\}$ (refer to Figure 3(a)). A graph that belongs to RHL_m is called an *m-dimensional restricted HL-graph* and is denoted by G^m. Note that, as built from $G(8, 4)$ that is nonbipartite, the restricted HL-graphs form a proper subset of all nonbipartite HL-graphs. As addressed in [14], many of the nonbipartite hypercube-like interconnection networks such as crossed cube, Möbius cube, twisted cube, multiply twisted cube, Mcube, generalized twisted cube, etc. are known to be restricted HL-graphs.



(a) $G(8,4)$ (b) $F_v = \{v_4, v_5\}$

(c) $F_v = \{v_2, v_4\}$ (d) $F_v = \{v_2, v_7\}$

Figure 3: Recursive circulant $G(8,4)$ and its subgraphs.

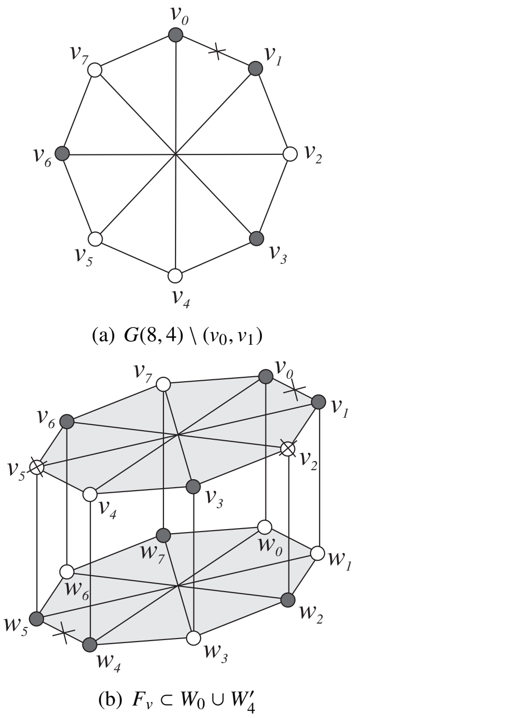
From the following lemma, proven in [14], we can see that G^m with at most $m-1$ fault elements has a hamiltonian path, thus implying that $smp(G^m) \geq m$. Here, a graph G is said to be f*-fault hamiltonian* (resp. f*-fault hamiltonian-connected*) if there exists a hamiltonian cycle (resp. if each pair of vertices are joined by a hamiltonian path) in $G \setminus F$ for any set F of vertices and/or edges with $|F| \leq f$.

Lemma 1. *Every G^m with $m \geq 3$ is $(m-3)$-fault hamiltonian-connected and $(m-2)$-fault hamiltonian.*

In order to establish the minimum SMP sets of G^m, we begin with the 3-dimensional restricted HL-graph, namely $G(8,4)$. It was shown in [12] that $mp(G(8,4)) = 3$ and every minimum MP set of $G(8,4)$ is trivial or equivalent to $\{(v_0,v_1),(v_0,v_4),(v_3,v_4)\}$.

Lemma 2. *$smp(G(8,4)) = 3$ and each of its minimum SMP sets is trivial or equivalent to $\{(v_0,v_1),(v_0,v_4),(v_3,v_4)\}$, $\{(v_0,v_1),v_4,v_5\}$, $\{(v_0,v_1),v_2,v_4\}$, or $\{(v_0,v_1),v_2,v_7\}$.*

Proof. Let F be a fault set of $G(8,4)$ with $|F| = 3$. If F contains no vertex, the lemma holds just as mentioned above. Assume F contains an odd number of vertices. As $G(8,4)$ is 1-fault hamiltonian, $G(8,4) \setminus F$ can be partitioned into at most two disjoint paths that cover all its vertices. Since it also has an odd number of vertices, $G(8,4) \setminus F$ is matchable by Proposition 4. The last case is $|F_v| = 2$, in which F_v is equivalent to either $\{v_4,v_5\}$, $\{v_2,v_4\}$, $\{v_2,v_7\}$, or $\{v_0,v_4\}$. When $F_v = \{v_4,v_5\}$, $G(8,4) \setminus F$ is matchable iff $(v_0,v_1) \notin F$ (Figure 3(b)). When $F_v = \{v_2,v_4\}$, $G(8,4) \setminus F$ is matchable iff (v_3,v_7), (v_0,v_1), and $(v_5,v_6) \notin F$ (Figure 3(c)). In this case, (v_3,v_7) produces a trivial SMP set, while (v_5,v_6) is symmetric to (v_0,v_1). When $F_v = \{v_2,v_7\}$, $G(8,4) \setminus F$ is matchable iff (v_3,v_4), (v_5,v_6), and $(v_0,v_1) \notin F$ (Figure 3(d)). If either (v_3,v_4) or (v_5,v_6) is contained in F, then F is trivial. Finally, if $F_v = \{v_0,v_4\}$, $G(8,4) \setminus \{v_0,v_4\}$ has a hamiltonian cycle $(v_1,v_2,v_3,v_7,v_6,v_5)$ and thus $G(8,4) \setminus F$ is matchable. This completes the proof. □



(a) $G(8,4) \setminus (v_0,v_1)$

(b) $F_v \subset W_0 \cup W_4'$

Figure 4: Black/white vertices and an example of nontrivial minimum SMP sets of G^4.

From now on in this section, we assume that all arithmetic on the indices of vertices is done modulo 8. Let F be a minimum SMP set of $G(8,4)$ with $F_v \neq \emptyset$. If F contains a *diagonal* edge (v_i,v_{i+4}) for some i, F is trivial by Lemma 2. If F contains a *boundary* edge (v_i,v_{i+1}) for some i, F may be nontrivial. Let $W_i = N_{H_i}(v_i) \cup N_{H_i}(v_{i+1})$ and $B_i = V(H_i) \setminus W_i$, where H_i is $G(8,4) \setminus (v_i,v_{i+1})$. Assume w.l.o.g. $(v_0,v_1) \in F$. Then, $W_0 = \{v_2,v_4,v_5,v_7\}$ and $B_0 = \{v_0,v_1,v_3,v_6\}$ (see Figure 4(a)). It can be deduced from Lemma 2 that $\{(v_0,v_1),x,y\}$ is a minimum SMP set for any pair of vertices $x,y \in W_0$, and that $F \setminus (v_0,v_1) \subset W_0$ for any minimum SMP set F including (v_0,v_1) with $F_v \neq \emptyset$. Notice that B_i forms an *independent set*, a set of pairwise non-adjacent vertices, in $G(8,4) \setminus (v_i,v_{i+1})$. In this section, we conveniently call the vertices in W_i and B_i *white* ones and *black* ones with respect to the boundary edge (v_i,v_{i+1}), respectively.

Lemma 3. *Let F' be a set of two vertices in $G(8,4)$. Then, (a) $G(8,4) \setminus (\{(v_0,v_1)\} \cup F')$ has a perfect matching if and only if $F' \not\subset W_0$, and (b) $G(8,4) \setminus (\{(v_0,v_4)\} \cup F')$ has a perfect matching if and only if $F' \neq \{v_1,v_7\}, \{v_3,v_5\}$.*

Proof. The proof is immediate from Lemma 2. □

A 4-dimensional restricted HL-graph G^4 is isomorphic to some $G_0 \oplus G_1$, where G_0 and G_1 are isomorphic to $G(8,4)$.

Let $V(G_0) = \{v_0, v_1, \ldots, v_7\}$ and $V(G_1) = \{w_0, w_1, \ldots, w_7\}$ and assume that v_i is adjacent to both v_{i+1} and v_{i+4} for every i, and similarly for w_i. Also, consider the white vertex set W'_j and black vertex set B'_j w.r.t. a boundary edge (w_j, w_{j+1}) of G_1. For any two boundary edges (v_i, v_{i+1}) in G_0 and (w_j, w_{j+1}) in G_1 of G^4, let F contain them and, additionally, arbitrary two white vertices in $W_i \cup W'_j$ (see Figure 4(b)). If $\bar{W}_i = B'_j$ and $\bar{B}_i = W'_j$, where $\bar{X}$ denotes $\{\bar{x} : x \in X\}$ (recall that $\bar{x}$ is the unique neighbor of x in the other component), then the set of black vertices $B_i \cup B'_j$ forms an independent set in $G^4 \setminus F$. There are 14 fault-free vertices and $|B_i \cup B'_j| = 8$, implying that there exists no perfect matching in $G^4 \setminus F$. Thus, the fault set F is certainly a nontrivial SMP set of G^4. Now, recalling that $smp(G^m) \le m$ by Proposition 3, and $smp(G^m) \ge m$ by Lemma 1, we are ready to present the following theorem.

Theorem 5. *For every $m \ge 3$, $smp(G^m) = m$. Furthermore, (a) for $m \ge 5$, each of its minimum SMP sets is trivial, and (b) for $m = 4$, each of its minimum SMP sets is either trivial or a set consisting of a boundary edge (v_i, v_{i+1}) of G_0, another boundary edge (w_j, w_{j+1}) of G_1, and two white vertices in $W_i \cup W'_j$ such that $\bar{W}_i = B'_j$ and $\bar{B}_i = W'_j$.*

Proof. We proceed by induction on m. First, the base case of $m = 3$ was shown in Lemma 2. In the inductive step, we complete the proof by showing that, given an arbitrary fault set F of $G^m = G_0 \oplus G_1$ with $|F| = m \ge 4$, one of the three cases holds: (i) F is a trivial SMP set; (ii) $m = 4$ and F is such a nontrivial SMP set as stated in (b); (iii) $G_0 \oplus G_1 \setminus F$ is matchable. Let $F = F_0 \cup F_1 \cup F_{01}$, where F_0 and F_1 denote the sets of vertices and/or edges in G_0 and G_1, respectively, and F_{01} represents the set of any edges joining vertices of the two components G_0 and G_1. Here, we assume w.l.o.g. that $|F_0| \ge |F_1|$. Then, there are four cases.

Case 1: $|F_0| \le m - 2$. By Lemma 1, the $(m-1)$-dimensional restricted HL-graphs G_0 and G_1 are $(m-3)$-fault hamiltonian, indicating that each $G_i \setminus F_i$ has a perfect or an almost perfect matching M_i, $i = 0, 1$. If at least one of $G_0 \setminus F_0$ and $G_1 \setminus F_1$ has an even number of vertices, $G_0 \oplus G_1 \setminus F$ is matchable by $M_0 \cup M_1$. Suppose they both have an odd number of vertices, meaning M_0 and M_1 are almost perfect. We claim that *unless the case (ii) is satisfied, there always exists a free edge $(x, \bar{x})$ such that both $G_0 \setminus (F_0 \cup \{x\})$ and $G_1 \setminus (F_1 \cup \{\bar{x}\})$ have perfect matchings M'_0 and M'_1, respectively, again implying that $G_0 \oplus G_1 \setminus F$ is matchable by $M'_0 \cup M'_1 \cup \{(x, \bar{x})\}$*. Here, an edge (v, w) is said to be *free* if v, w, and (v, w) are all fault-free. There are two cases. First, when $|F_0| < m - 2$, there exists a free edge $(x, \bar{x})$ connecting G_0 and G_1 since $2^{m-1} - m > 0$ when $m \ge 4$. Then, together with two perfect matchings in $G_0 \setminus (F_0 \cup \{x\})$ and $G_1 \setminus (F_1 \cup \{\bar{x}\})$ that exist by the inductive hypothesis, the edge forms a perfect matching of G^m. Next, the case of $|F_0| = m - 2$ (and hence $|F_1 \cup F_{01}| = 2$) is subdivided into three cases.

First, suppose $m \ge 6$. The proof is based on the following fact which can be easily verified by induction on m: *G^m has no triangle and there exist at most two common neighbors for each pair of vertices in G^m*. If there exist at least two fault vertices, say v_f and w_f, in F_0, consider the vertex set $N_{G_0}(v_f, w_f) \equiv N_{G_0}(v_f) \cup N_{G_0}(w_f)$. Then, due to the above fact, $F_0 \cup \{x\}$ may not form a trivial SMP set in G_0 for any $x \in N_{G_0}(v_f, w_f)$, indicating $G_0 \setminus (F_0 \cup \{x\})$ always has a perfect matching (note that $F_0 \cup \{x\}$ cannot form a nontrivial SMP set since $m-1 \ge 5$). Furthermore, $|N_{G_0}(v_f, w_f)| \ge 2(m-1) - 2 = 2m - 4$, which is greater than m. Hence, we can choose such a vertex $x \notin F_0$ and hence a free edge $(x, \bar{x})$ that allows a perfect matching of G^m. If F_0 has at most one fault vertex, then it has at least three fault edges. Thus, there exist at least three vertices outside F_0 incident with them. Since at most two of them could be blocked (recall $|F_1 \cup F_{01}| = 2$), we can always find a vertex $x \notin F_0$ such that $(x, \bar{x})$ is the free edge in the claim.

Second, suppose $m = 5$ (and hence $|F_0| = 3$). Recall that F_0 has an odd number of vertices. If F_0 has three fault vertices, we can similarly find a free edge $(x, \bar{x})$ such that $x \in N_{G_0}(v_f, w_f)$. Notice that $F_0 \cup \{x\}$ cannot be a trivial SMP set, as before, and that it cannot be a nontrivial one, either. If there exists one fault vertex $v_f \in F_0$, we need to take care of two subcases. If v_f is incident with some fault edge in F_0, it suffices to pick up a free edge $(x, \bar{x})$ with $x \in V(G_0)$. Otherwise, there exist at least three fault-free vertices in G_0 that are incident to some fault edges in F_0. Again, we can similarly find a desirable free edge whose endvertex in G_0 is one of the three.

Finally, suppose $m = 4$ (and hence $|F_0| = 2$), in which case $G_0 = G(8, 4)$ and F_0 is made of a fault vertex and a fault edge. Due to Lemma 3, there exist at least four vertices $x_p \notin F_0$, such that $G_0 \setminus (F_0 \cup \{x_p\})$ has a perfect matching. If $|F_1| = 1$, it suffices to pick up a free edge $(x, \bar{x})$, where x is an available vertex among those four. In the case of $|F_1| = 2$, i.e. F_1 has a fault vertex and one fault edge, we have to be careful. Similarly, $G_1 \setminus (F_1 \cup \{y_q\})$ has also a perfect matching for at least four vertices $y_q \notin F_1$. If there are two vertices x^*_p and y^*_q such that $\bar{x}^*_p = y^*_q$, then we have found the free edge (x^*_p, y^*_q) in the claim. Suppose not. Then, by Lemma 3, the only possible situation is that F_0 consists of a boundary edge (v_i, v_{i+1}) and a white vertex in W_i, F_1 consists of a boundary edge (w_j, w_{j+1}) and a white vertex in W'_j, $\bar{B}_i = W'_j$, and finally $\bar{B}'_j = W_i$. This exactly satisfies the case (ii). The claim is proved.

Case 2: $|F_0| = m - 1$ and F_0 isolates some vertex z in $G_0 \setminus F_0$. Since G_0 is $(m-3)$-fault hamiltonian, $G_0 \setminus F_0$ has two disjoint paths P_1 and P_2 that cover all the vertices. Let $P_1 = (z)$ and $P_2 = (x_1, x_2, \ldots, x_l)$ with $l \ge 4$. We will show that if $(z, \bar{z})$ is not free, $G_0 \oplus G_1 \setminus F$ is matchable or F is a trivial SMP set, and that if $(z, \bar{z})$ is free, $G_0 \oplus G_1 \setminus F$ is matchable or the case (ii) is satisfied.

First, let $(z, \bar{z})$ be not free, that is, either $\bar{z} \in F$ or $(z, \bar{z}) \in F$. If F has an even number of vertices, then F is a trivial SMP set. If F has an odd number of vertices, we can assume w.l.o.g. that $(x_l, \bar{x}_l)$ is free because $|F_1 \cup F_{01}| = 1$. $G_1 \setminus F_1$ has a hamiltonian cycle and thus it also has a hamiltonian path P_h starting at $\bar{x}_l$. So, P_2 and P_h can be merged into a single path via $(x_l, \bar{x}_l)$, having an even number of vertices, indicating $G_0 \oplus G_1 \setminus F$ has an almost perfect matching.

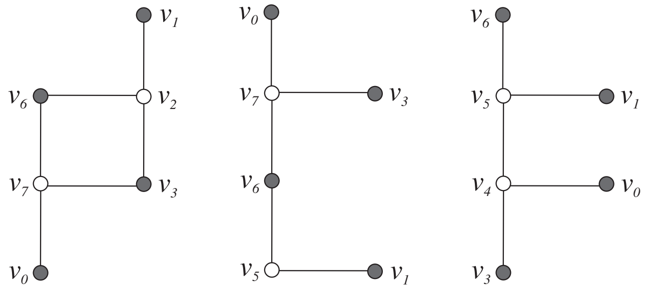
Second, suppose $(z, \bar{z})$ is free. Again, we assume that $(x_l, \bar{x}_l)$ is free. If $m \ge 5$, or $m = 4$ and $|F_1| = 0$, by Lemma 1, there exists a $\bar{z} - \bar{x}_l$ hamiltonian path P'_h in $G_1 \setminus F_1$. By merging the three paths P_1, P'_h, and P_2 via $(z, \bar{z})$ and $(x_l, \bar{x}_l)$, we obtain a hamiltonian path in $G_0 \oplus G_1 \setminus F$. Thus, $G_0 \oplus G_1 \setminus F$ is matchable by Proposition 4.

If $m = 4$ and $|F_1| = 1$ (and hence $|F_{01}| = 0$), the path P_1 and a hamiltonian cycle in $G_1 \setminus F_1$ can be merged into a single path P'_1. Thus, $G_0 \oplus G_1 \setminus F$ is matchable if P'_1 or P_2 has an even number of vertices. Suppose both have odd lengths. The length of P'_1 is odd iff the fault element in F_1 is an edge. The length of P_2 is odd iff F_0 is a trivial SMP set of G_0. If there is no fault vertex in F_0, we have a perfect matching of G^m, which consists of all the edges from G_0 to G_1.

The remaining subcase is that F_0 is a trivial SMP set of G_0 made of two vertices and one edge, and F_1 is a set of one fault edge. In this case, $l = 5$. If F_0 has a diagonal edge, $G_0 \setminus F_0$ should be a union of z and a cycle $(y_1, y_2, \ldots, y_5)$ of length five. From Lemma 3, we can deduce that there exists y_i for some i such that $G_1 \setminus (F_1 \cup \{\bar{z}, \bar{y}_i\})$ has a perfect matching, which implies $G_0 \oplus G_1 \setminus F$ also has a perfect matching containing the free edges $(z, \bar{z})$ and $(y_i, \bar{y}_i)$. Now, we have a boundary edge in F_0 and can assume w.l.o.g. $F_0 = \{(v_0, v_1), v_2, v_5\}$ and $z = v_1$. If F_1 has a diagonal edge, by Lemma 3, for at least one of x_1 and x_l, say x_l, $G_1 \setminus (F_1 \cup \{\bar{z}, \bar{x}_l\})$ has a perfect matching, which implies $G_0 \oplus G_1 \setminus F$ is matchable. Now, let F_1 contain a boundary edge, i.e. $F_1 = \{(w_j, w_{j+1})\}$ for some j. Observe that if $F' = \{v_1, v_0\}$, $\{v_1, v_3\}$, and $\{v_1, v_6\}$, then $G_0 \setminus (F_0 \cup F')$ has a perfect matching $\{(v_3, v_4), (v_6, v_7)\}$, $\{(v_0, v_4), (v_6, v_7)\}$, and $\{(v_3, v_4), (v_7, v_0)\}$, respectively. Thus, if $\bar{v} \in B'_j$ for some v in $B_0 = \{v_0, v_1, v_3, v_6\}$, then there exists a black vertex $w \in B_0$ with $w \neq v$ such that $v_1 \in \{v, w\}$ and $G_0 \setminus (F_0 \cup \{v, w\})$ has a perfect matching. Furthermore, by Lemma 3, $G_1 \setminus (F_1 \cup \{\bar{v}, \bar{w}\})$ has a perfect matching, too. This implies $G_0 \oplus G_1 \setminus F$ is matchable. The other possible case, in which $\bar{v} \in W'_j$ for every v in B_0, satisfies the case (ii).

Case 3: $|F_0| = m - 1$ and F_0 does not isolate any vertex z in $G_0 \setminus F_0$. We first claim that $G_0 \setminus F_0$ has two disjoint paths P_1 and P_2 of lengths at least two, covering all the vertices of the graph. Since G_0 is $(m-3)$-fault hamiltonian, $G_0 \setminus F_0$ always has two vertex covering disjoint paths P'_1 and P'_2. Suppose, for instance, P'_1 is a single vertex path. That is, $P'_1 = (x)$ and $P'_2 = (y_1, y_2, \ldots, y_l)$, $l \geq 4$. As F_0 does not isolate any vertex when deleted, there exists a free edge from x to y_i in $G_0 \setminus F_0$ for some i. Then, by setting $P_2 = (y_1, y_2 \ldots, y_{i-1})$ and $P_1 = (x, y_i, \ldots, y_l)$ if $i \geq 3$, or $P_2 = (y_{i+1}, y_{i+2}, \ldots, y_l)$ and $P_1 = (x, y_i, \ldots, y_1)$ if $i \leq 2$, we can build two paths meeting the claim. Now, let $P_1 = (x_1, \ldots, x_p)$ and $P_2 = (y_1, \ldots, y_q)$ with $p, q \geq 2$. We assume w.l.o.g. that $(x_1, \bar{x}_1)$ and $(y_1, \bar{y}_1)$ are free since $|F_1 \cup F_{01}| = 1$. If $m \geq 5$, or $m = 4$ and $|F_1| = 0$, there exists again an $\bar{x}_1$-$\bar{y}_1$ hamiltonian path P_h in $G_1 \setminus F_1$. Then, via $(x_1, \bar{x}_1)$ and $(y_1, \bar{y}_1)$, the three paths P_1, P_h, and P_2 can be merged into a hamiltonian path in $G_0 \oplus G_1 \setminus F$, implying that $G_0 \oplus G_1 \setminus F$ is matchable.

Now, let $m = 4$ and $|F_1| = 1$ (and hence $|F_{01}| = 0$). If $F_v = \emptyset$, i.e. F does not contain any vertex, there exists a perfect matching $\{(v, \bar{v}) : v \in V(G_0)\}$. Suppose the other case: $F_v \neq \emptyset$. If $G_0 \setminus F_0$ has an odd number of vertices, one of the two paths, say P_1, has an odd length and the other one P_2 has an even length. As in the proof of ***Case 2***, P_1 can be merged through $(x_1, \bar{x}_1)$ with a hamiltonian cycle in $G_1 \setminus F_1$ into a single path. Thus, by Proposition 4, $G_0 \oplus G_1 \setminus F$ is matchable. Suppose $G_0 \setminus F_0$ has an even number of vertices. If $G_0 \setminus F_0$ has a perfect matching, $G_0 \oplus G_1 \setminus F$ is matchable since $G_1 \setminus F_1$ has a hamiltonian path. If $G_0 \setminus F_0$ has no perfect matching, both P_1 and P_2 must have odd lengths. Now, we consider three subcases.



(a) $v_4, v_5 \in F_0$ (b) $v_2, v_4 \in F_0$ (c) $v_2, v_7 \in F_0$

Figure 5: $G(8, 4) \setminus F_0$ with $(v_0, v_1) \in F_0$.

First, if the fault element in F_1 is a vertex, P_1 and a hamiltonian cycle in $G_1 \setminus F_1$ can be merged into an even path, which implies $G_0 \oplus G_1 \setminus F$ is matchable. Second, if F_1 has a diagonal edge, for at least one of y_1 and y_q, say y_1, $G_1 \setminus (F_1 \cup \{\bar{x}_1, \bar{y}_1\})$ has a perfect matching by Lemma 3. Then, we can build a perfect matching of $G_0 \oplus G_1 \setminus F$ containing $(x_1, \bar{x}_1)$ and $(y_1, \bar{y}_1)$.

The third case is that F_1 is a single boundary edge set, in which F_0 with $|F_0| = 3$ must contain two vertices and an edge, and hence the two disjoint paths are $P_1 = (x_1, x_2, x_3)$ and $P_2 = (y_1, y_2, y_3)$. Since F_0 is a nontrivial minimum SMP set of G_0, by Lemma 2, F_0 is equivalent to one of $\{(v_0, v_1), v_4, v_5\}$, $\{(v_0, v_1), v_2, v_4\}$ and $\{(v_0, v_1), v_2, v_7\}$. For each of the three cases, we can observe $\{x_1, x_3, y_1, y_3\} = B_0$. For example, if $v_4, v_5 \in F_0$, then the possible paths are $\{(v_0, v_7, v_3), (v_1, v_2, v_6)\}$ or $\{(v_0, v_7, v_6), (v_1, v_2, v_3)\}$ as shown in Figure 5(a) (the other two cases of $v_2, v_4 \in F_0$ and $v_2, v_7 \in F_0$ are illustrated in Figure 5(b) and (c), respectively). Now, let $F_1 = \{(w_j, w_{j+1})\}$. If there exists an endvertex of the two paths, say x_1 of P_1, such that $\bar{x}_1 \in B'_j$, then for an endvertex of P_2, say y_1, $G_1 \setminus (F_1 \cup \{\bar{x}_1, \bar{y}_1\})$ has a perfect matching and thus $G_0 \oplus G_1 \setminus F$ is matchable. The remaining situation satisfies the case (ii).

Case 4: $|F_0| = m$. $G_0 \setminus F_0$ has three disjoint paths that cover all the vertices. If at least two of them are even, $G_0 \oplus G_1 \setminus F$ is obviously matchable as G_1 has a perfect matching. Otherwise, there exist at least two odd paths. Then, the two odd paths can be merged with a hamiltonian path in G_1 into a single even path. Hence, by Proposition 4, $G_0 \oplus G_1 \setminus F$ is matchable in this case, too, which completes the entire proof. □

Remark 2. There exists a 4-dimensional restricted HL-graph such that every minimum SMP set of the graph is trivial. Let G be a graph in RHL_4 such that $\bar{v}_i = w_{3i}$ for every i. Then, for any set B_i of black vertices, $\bar{B}_i$ is consecutive, that is, $\bar{B}_i = \{w_j, w_{j+1}, w_{j+2}, w_{j+3}\}$ for some j. By Theorem 5, every minimum SMP set of G is trivial.

5. Recursive Circulants

Recursive circulant represents a class of circulant graphs, proposed in [13] to study the interconnection topology of multicomputer networks. An interesting family of the recursive circulant is $G(2^m, 4)$, $m \geq 0$. It is an m-regular graph whose vertex and edge sets are $\{v_0, v_1, \cdots, v_{2^m-1}\}$ and $\{(v_i, v_j) : j \equiv i + 4^k \pmod{2^m}, 0 \leq k < m/2\}$, respectively (see the two examples of $G(2^m, 4)$ in Figures 3(a) and 6).

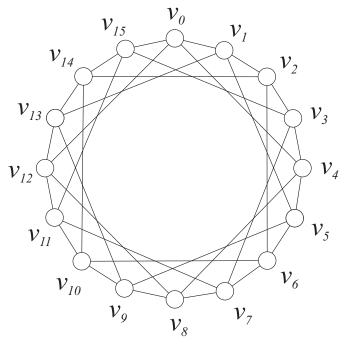
Consider a graph G_i, $0 \leq i < 4$, which is a subgraph of $G(2^m, 4)$ induced by $\{v_j : 0 \leq j < 2^m \text{ and } j \equiv i \pmod 4\}$. Interestingly, each G_i is isomorphic $G(2^{m-2}, 4)$. Furthermore, the subgraph of $G(2^m, 4)$ induced by the vertices of G_i and $G_{(i+1) \bmod 4}$ is isomorphic to the graph product $G(2^{m-2}, 4) \times K_2$ of $G(2^{m-2}, 4)$ and K_2. Thus, $G(2^m, 4)$ belongs to the set of graphs obtained by applying the operation $\oplus$ to the two copies of $G(2^{m-2}, 4) \times K_2$. Of course, $G(2^{m-2}, 4) \times K_2$ is a special case of $G(2^{m-2}, 4) \oplus G(2^{m-2}, 4)$. Note that $G(2^m, 4)$ with odd m is an m-dimensional restricted HL-graph, whose strong matching preclusion properties were analyzed in the previous section. Hence, in this section, we only focus on the properties of $G(2^m, 4)$ with even $m \geq 4$.

First, take a look at $G(2^4, 4)$. It is not bipartite; however, if we discard all the edges joining vertices from G_3 to G_0, then it becomes bipartite as can be understood in Figure 6(b). If we delete the two edges (v_3, v_4) and (v_{11}, v_{12}) from $G(2^4, 4)$, the set of *black* vertices, $B_{3,4} = \{v_3, v_4, v_{11}, v_{12}, v_1, v_9, v_6, v_{14}\}$, forms an independent set. This suggests that there exists a nontrivial SMP set in $G(2^4, 4)$, which consists of the two edges (v_3, v_4) and (v_{11}, v_{12}) and arbitrary two *white* vertices in $W_{3,4} = V(G(2^4, 4)) \setminus B_{3,4}$. The subgraph of $G(2^4, 4)$ induced by the vertices of G_i and $G_{(i+1) \bmod 4}$ is also isomorphic to the 3-dimensional hypercube Q_3, which is bipartite.

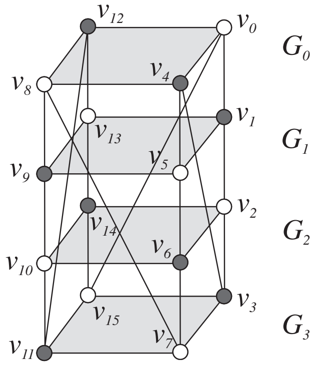
The following lemma describes some fundamental properties of Q_3 with respect to perfect matching, which will be exploited in the proof of the next theorem.

Lemma 4. *Let B and W denote the sets of black and white vertices in Q_3, respectively.*
(a) For any $x \in B$ and $y \in W$, $Q_3 \setminus \{x, y\}$ has a hamiltonian cycle, and thus a perfect matching.
(b) For any $x \in B$, $y \in W$, and $e \in E(Q_3)$, $Q_3 \setminus \{x, y, e\}$ has a perfect matching.
(c) For any $x_1, x_2 \in B$ and $y_1, y_2 \in W$, $Q_3 \setminus \{x_1, x_2, y_1, y_2\}$ has a perfect matching.
(d) Let $F = \{y_1, y_2, y_3\} \subset W$. Then, $Q_3 \setminus F$ has an isolated vertex x. Furthermore, for any set F' of three black vertices including x, $Q_3 \setminus (F \cup F')$ has a perfect matching.
(e) Let $F = \{y_1, y_2, e\}$, where $y_1, y_2 \in W$ and $e \in E(Q_3)$. Then, there exists a black vertex x_i such that for any black vertex $x_j \neq x_i$, $Q_3 \setminus (F \cup \{x_i, x_j\})$ has a perfect matching. Furthermore, if $Q_3 \setminus F$ has no isolated vertex, then at least two such vertices x_i and x'_i exist.

Proof. The proof of (a) is immediate by an inspection of $Q_3 \setminus \{x, y\}$. From (a), the claim (b) trivially follows. By (a), $Q_3 \setminus \{x_1, y_1\}$ has a hamiltonian cycle of length six. Removing



(a) $G(2^4, 4)$

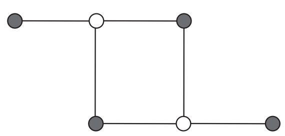


(b) Another view of $G(2^4, 4)$

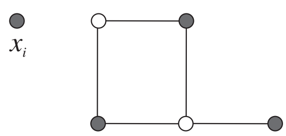
Figure 6: Recursive circulant $G(2^4, 4)$.

x_2 and y_2 further from the cycle results in at most two even paths. Thus, (c) holds true. The statement (d) is a direct consequence of the fact that $Q_3 \setminus \{y_1, y_2, y_3\}$ is a union of a single black vertex and a connected component isomorphic to the complete bipartite graph $K_{1,3}$. To prove (e), observe that $Q_3 \setminus \{y_1, y_2\}$ is isomorphic to the graph in Figure 7(a). If $Q_3 \setminus F$ has an isolated vertex, it is isomorphic to the graph in Figure 7(b). Otherwise, it is isomorphic to the graph in Figure 7(c). Since $Q_3 \setminus F \cup \{x_i\}$ and $Q_3 \setminus F \cup \{x'_i\}$ in the two graphs have a path of length five, the claim (e) holds. □

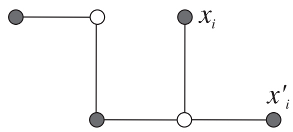
As in the case of restricted HL-graphs, the fault-hamiltonicity of recursive circulant $G(2^m, 4)$, addressed in [14,



(a) $Q_3 \setminus \{y_1, y_2\}$



(b) A vertex isolated



(c) No vertex isolated

Figure 7: Graphs in Lemma 4(e).

17], will play an important role in analyzing its strong matching preclusion properties.

Lemma 5. *Every* $G(2^m, 4)$ *with* $m \geq 3$ *and* $G(2^{m-1}, 4) \times K_2$ *with* $m \geq 4$ *are* $(m-3)$*-fault hamiltonian-connected and* $(m-2)$*-fault hamiltonian.*

Now, we give the last theorem of this article.

Theorem 6. *For every* $m \geq 3$, $smp(G(2^m, 4)) = m$. *Furthermore, (a) for* $m \geq 5$, *each of its minimum SMP sets is trivial, and (b) for* $m = 4$, *each of its minimum SMP sets is either trivial or equivalent to a set consisting of two edges* (v_3, v_4), (v_{11}, v_{12}) *and arbitrary two white vertices in* $W_{3,4}$.

Proof. Due to Theorem 5, it suffices to consider $G(2^m, 4)$ with even $m \geq 4$. Hereafter in this proof, H^m denotes $G(2^m, 4)$ if m is even; otherwise, it denotes $G(2^{m-1}, 4) \times K_2$. Then, $H^m = H_0 \oplus H_1$ for some H_0 and H_1 isomorphic to H^{m-1}. For a technical reason, we will prove that H^m, instead of $G(2^m, 4)$, satisfies the claim of Theorem 6 for every $m \geq 4$. The proof is by induction on m. First of all, $smp(H^m) = m$ by Proposition 3 and Lemma 5. Let F be an arbitrary fault set of H^m with $|F| = m$. The inductive step that given H^m with $m \geq 5$, either $H^m \setminus F$ is matchable or F is its trivial SMP set, can be proven in the same way as in the proof of Theorem 5. Note that both H^m and restricted HL-graph G^m are HL-graphs. Their fault-hamiltonicities (Lemma 1 and Lemma 5) and the common fact that they contain no triangle and that there exist at most two common neighbors for an arbitrary pair of vertices lead to the same proof.

Thus, it remains to prove that, *for any fault set F with* $|F| = 4$, *(i)* $H^4 \setminus F$ *is matchable, (ii) F is a trivial SMP set, or (iii) F is a nontrivial SMP set equivalent to* $\{(v_3, v_4), (v_{11}, v_{12}), v_f, w_f\}$ *with* $v_f, w_f \in W_{3,4}$. Recall that $H^4 = H_0 \oplus H_1$ for the two 3-dimensional hypercubes H_0 and H_1, where $V(H_0) = V(G_0) \cup V(G_1)$ and $V(H_1) = V(G_2) \cup V(G_3)$ (see Figure 6(b) again). $H^4 = G(2^4, 4)$ is 2-fault hamiltonian by Lemma 5, and thus $H^4 \setminus F$ has two disjoint paths that cover all the fault-free vertices. In case that $|F_v|$ is odd, $H^4 \setminus F$ has an almost perfect matching, and hence is matchable. From the fact that every minimum MP set of $G(2^4, 4)$ is trivial [12], F with $|F_v| = 0$ is also a trivial SMP set of H^4.

So, the remaining proof confines to the case of either $|F_v| = 2$ or $|F_v| = 4$. Observe that the eight edges from H_0 to H_1 are classified into four groups of two edges each: black to black, white to white, black to white, and white to black. Let F_i be the set of fault elements in H_i, $i = 0, 1$. We can assume w.l.o.g that *(α) the number of fault edges between* G_3 *and* G_0 *is not less than those of between* G_i *and* G_{i+1} *for* $i = 0, 1, 2$, *(β) there are at least as many white fault vertices as black ones in F, and (γ)* $|F_0| \geq |F_1|$, *and if* $|F_0| = |F_1|$, *then* $|F_0 \cap F_v| \geq |F_1 \cap F_v|$. There are three cases.

Case 1: There exist two fault edges between G_3 and G_0 ($|F_v| = |F_e| = 2$). First, suppose $v_f \in F_0$ and $w_f \in F_1$ for $v_f, w_f \in F_v$. If $c(v_f) \neq c(w_f)$, in which the function c denotes the color of vertex, we may pick up a free edge $(x, \bar{x})$ with $x \in V(H_0)$ such that $c(x) \neq c(v_f)$ and $c(\bar{x}) \neq c(w_f)$. By Lemma 4(a), $H_0 \setminus (F_0 \cup \{x\})$ and $H_1 \setminus (F_1 \cup \{\bar{x}\})$ respectively have perfect matchings. Then, we can build a perfect matching of $H^4 \setminus F$ by merging those two matchings and $\{(x, \bar{x})\}$. Let $c(v_f) = c(w_f) =$ `white` (by the assumption (β), they both may not be `black`). If the two fault edges are (v_3, v_4) and (v_{11}, v_{12}) which join black vertices between G_3 and G_0, then F forms a nontrivial SMP set equivalent to that stated in the statement (iii). Otherwise, one of them is free. Then, similarly as in the case of $c(v_f) \neq c(w_f)$, we can assemble a perfect matching of $H^4 \setminus F$.

Second, let $v_f, w_f \in F_0$. If $c(v_f) \neq c(w_f)$, the union of any two perfect matchings of $H_0 \setminus F_0$ and H_1 is a desired perfect matching of $H^4 \setminus F$. So, let $c(v_f) = c(w_f) =$ `white`. Again, if both fault edges join black vertices of G_3 and G_0, then F is a nontrivial SMP set equivalent to that stated in the statement (iii). Otherwise, select two black vertices x and y in $V(H_0)$ such that the free edges $(x, \bar{x})$ and $(y, \bar{y})$ respectively join a black and a white vertex of H_1. Then, by Lemma 4(c), $H_0 \setminus (F_0 \cup \{x, y\})$ has a perfect matching, and by Lemma 4(a), $H_1 \setminus \{\bar{x}, \bar{y}\}$ has a perfect matching, too. Obviously, those two perfect matchings together with $\{(x, \bar{x}), (y, \bar{y})\}$ form a perfect matching of $H^4 \setminus F$.

Case 2: There exists exactly one fault edge between G_3 and G_0 ($|F_v| = |F_e| = 2$). In this case, we will show that $H^4 \setminus F$ has a perfect matching or F is a trivial SMP set of H^4. First, suppose $v_f \in F_0$ and $w_f \in F_1$ for $v_f, w_f \in F_v$. Regardless of whether $c(v_f) = c(w_f)$ or not, there is always a free edge $(x, \bar{x})$ with $x \in V(H_0)$ such that $c(x) \neq c(v_f)$ and $c(\bar{x}) \neq c(w_f)$. Then, similarly as before, $H^4 \setminus F$ is matchable.

Second, let $v_f, w_f \in F_0$. If $c(v_f) \neq c(w_f)$, merging respective perfect matchings of $H_0 \setminus F_0$ and $H_1 \setminus F_1$ results in a perfect matching of $H^4 \setminus F$. So, let $c(v_f) = c(w_f) =$ `white`. Suppose $H_0 \setminus F_0$ has no isolated vertex. Due to Lemma 4(e) when $|F_0| = 3$ and Lemma 4(c) when $|F_0| = 2$, there always exist a pair of free edges $(x, \bar{x})$ and $(y, \bar{y})$ from black vertices x and y of H_0 such that $H_0 \setminus (F_0 \cup \{x, y\})$ has a perfect matching and $c(\bar{x}) \neq c(\bar{y})$. Then, respective perfect matchings of $H_0 \setminus (F_0 \cup \{x, y\})$ and $H_1 \setminus (F_1 \cup \{\bar{x}, \bar{y}\})$, together with $\{(x, \bar{x}), (y, \bar{y})\}$, form a desired perfect matching. If $H_0 \setminus F_0$ has an isolated vertex x, then $|F_0| = 3$ and x is black colored. If $(x, \bar{x})$ is not free, then $(x, \bar{x})$ is the very fault edge between G_3 and G_0 and thus F is a trivial SMP set. Otherwise, by Lemma 4(e), there is a free edge $(y, \bar{y})$ from a black vertex y of H_0 such that $H_0 \setminus (F_0 \cup \{x, y\})$ has a perfect matching and $c(\bar{x}) \neq c(\bar{y})$. Then, in the same way as just before, we can build a perfect matching of $H^4 \setminus F$.

Case 3: There exists no fault edge between G_3 and G_0. In this case, too, either $H^4 \setminus F$ has a perfect matching or F is a trivial SMP set H^4. Remember that, by the assumption (α), there is no fault edge from G_i to $G_{(i+1) \bmod 4}$ for any i.

Case 3.1: $|F_v| = |F_e| = 2$. Let v_f and w_f be the two fault vertices. Due to the assumption (γ), we only need to consider three subcases. First, let $|F_0| = 2$ and $|F_1| = 2$. If $v_f, w_f \in F_0$ with $c(v_f) \neq c(w_f)$, it suffices to find respective perfect matchings of $H_0 \setminus F_0$ and $H_1 \setminus F_1$ and combine them. If $v_f, w_f \in F_0$ and $c(v_f) = c(w_f) =$ `white`, we first pick up a free edge (x, y) with $x \in V(G_2)$, $y \in V(G_3)$, and $c(x) =$ `white`. Then, $H_1 \setminus (F_1 \cup \{x, y\})$

trivially has a perfect matching made of the tree edges between G_2 and G_3 other than (x, y). By Lemma 4(c), $H_0 \setminus (F_0 \cup \{\bar{x}, \bar{y}\})$ also has a perfect matching since $c(\bar{x}) = c(\bar{y}) = \texttt{black}$. Combining the two matchings with $\{(x, \bar{x}), (y, \bar{y})\}$ produces a perfect matching of $H^4 \setminus F$. If $v_f \in F_0$ and $w_f \in F_1$, choose a free vertex x of H_0 such that $c(x) \neq c(v_f)$ and $c(\bar{x}) \neq c(w_f)$. Then, in spite of each fault edge in H_0 and H_1, both $H_0 \setminus (F_0 \cup \{x\})$ and $H_1 \setminus (F_1 \cup \{\bar{x}\})$ have perfect matchings by Lemma 4(b). So, building a perfect matching of $H^4 \setminus F$ is obvious.

Second, let $|F_0| = 3$ and $|F_1| = 1$. If $v_f, w_f \in F_0$ and $c(v_f) \neq c(w_f)$, $H^4 \setminus F$ is matchable as before. If $v_f, w_f \in F_0$ but $c(v_f) = c(w_f) = \texttt{white}$, Lemma 4(e) implies that there exists a black vertex x in H_0 for which, whether x is isolated in $H_0 \setminus F_0$ or not, we have another black vertex y in H_0 such that $c(\bar{x}) \neq c(\bar{y})$ and $H_0 \setminus (F_0 \cup \{x, y\})$ has a perfect matching. Hence, finding a perfect matching of $H^4 \setminus F$ is possible.

Now, let $v_f \in F_0$ and $w_f \in F_1$. By the assumption (β), we need to consider two subcases. The first one is where $c(v_f) = c(w_f) = \texttt{white}$. If $v_f \in V(G_1)$, then the vertex x in G_0 adjacent to v_f is black colored. Thus, $H_0 \setminus (F_0 \cup \{x\})$ has a perfect matching made of the three edges between G_0 and G_1 other than (x, v_f). Together with $\{(x, \bar{x})\}$ and a perfect matching in $H_1 \setminus (F_1 \cup \{\bar{x}\})$ with black vertex $\bar{x} \in V(G_3)$, it forms a perfect matching of $H^4 \setminus F$. Assume $v_f \in V(G_0)$, and let x be the black vertex in G_1 adjacent to v_f. If $\bar{x} \neq w_f$, then, for an edge (y, z) with $y \in V(G_0)$, $z \in V(G_1)$, and $c(y) = \texttt{black}$, we can trivially find a perfect matching of $H_0 \setminus (F_0 \cup \{x, y, z\})$. Then, together with $\{(x, \bar{x}), (y, \bar{y}), (z, \bar{z})\}$ and a perfect matching in $H_1 \setminus (F_1 \cup \{\bar{x}, \bar{y}, \bar{z}\})$ with $c(\bar{x}) = c(w_f) = \texttt{white}$ and $c(\bar{y}) = c(\bar{z}) = \texttt{black}$, it builds a perfect matching of $H^4 \setminus F$.

If $\bar{x} = w_f$ and there exists a fault-free edge (x, y) in G_1, then for the vertex $z \in V(G_0)$ adjacent to y, we can find a perfect matching in $H_0 \setminus (F_0 \cup \{z\})$ made of the edge (x, y) and two additional edges between G_0 and G_1. Thus, together with $\{(z, \bar{z})\}$ and a perfect matching in $H_1 \setminus (F_1 \cup \{\bar{z}\})$ with $c(\bar{z}) = \texttt{black}$, it builds a perfect matching of $H^4 \setminus F$. If $\bar{x} = w_f$ and both edges in G_1 incident from x are faulty, then F is a trivial SMP set of $H^4 \setminus F$. For the other subcase of $c(v_f) = \texttt{white}$ and $c(w_f) = \texttt{black}$, we can also show that either $H^4 \setminus F$ has a prefect matching or F is a trivial SMP set of H^4 in a way symmetric to the previous subcase.

Third, let $|F_0| = 4$. The case that neither G_0 nor G_1 contain all the fault elements can be reduced to one of the two previous subcases of $|F_0| = 2$ and $|F_0| = 3$, since H_0 can be reformulated as a subgraph induced by either $V(G_3) \cup V(G_0)$ or $V(G_1) \cup V(G_2)$. Suppose G_0 contains all four fault elements. Then, there should be a fault edge e_f incident to some fault vertex. In that case, the graph $H^4 \setminus (F \setminus e_f)$ with $|F \setminus e_f| = 3$ has a perfect matching, and so does $H^4 \setminus F$.

Case 3.2: $|F_v| = 4$. We consider three subcases. First, let $F_0 = \{v_f, w_f\}$ and $F_1 = \{p_f, q_f\}$. If $c(v_f) \neq c(w_f)$ and $c(p_f) \neq c(q_f)$, perfect matchings in $H_0 \setminus F_0$ and $H_1 \setminus F_1$ together build a perfect matching of $H^4 \setminus F$. If not, let $c(v_f) = c(w_f) = \texttt{white}$. Regardless of the colors of p_f and q_f, we can pick up two free edges $(x, \bar{x})$ and $(y, \bar{y})$ with $c(x) = c(y) = \texttt{black}$ such that $c(\bar{x}) \neq c(p_f)$ and $c(\bar{y}) \neq c(q_f)$. Then, together with $\{(x, \bar{x}), (y, \bar{y})\}$, respective perfect matchings of $H_0 \setminus (F_0 \cup \{x, y\})$ and $H_1 \setminus (F_1 \cup \{\bar{x}, \bar{y}\})$ form a perfect matching of $H^4 \setminus F$.

Second, let $F_0 = \{v_f, w_f, p_f\}$ and $F_1 = \{q_f\}$. If $c(v_f) = c(w_f) = c(p_f) = \texttt{white}$, there is an isolated vertex z in $H_0 \setminus F_0$. If $\bar{z} = q_f$, then F is a trivial SMP set. Let $\bar{z} \neq q_f$. By Lemma 4(d), we can pick up two free edges $(x, \bar{x})$ and $(y, \bar{y})$ with $c(x) = c(y) = \texttt{black}$ such that $H_0 \setminus (F_0 \cup \{x, y, z\})$ has a perfect matching and $\{q_f, \bar{z}, \bar{x}, \bar{y}\}$ consists of two black and two white vertices. Then, together with $\{(z, \bar{z}), (x, \bar{x}), (y, \bar{y})\}$, respective perfect matchings of $H_0 \setminus (F_0 \cup \{x, y, z\})$ and $H_1 \setminus (F_1 \cup \{\bar{z}, \bar{x}, \bar{y}\})$ form a perfect matching of $H^4 \setminus F$. If $c(v_f) = c(w_f) = \texttt{white}$ and $c(p_f) = \texttt{black}$, it suffices to pick up a free edge $(x, \bar{x})$ such that $c(x) = \texttt{black}$ and $c(\bar{x}) \neq c(q_f)$ and merge perfect matchings of $H_0 \setminus (F_0 \cup \{x\})$ and $H_1 \setminus (F_1 \cup \{\bar{x}\})$ with it.

Third, let $F_0 = \{v_f, w_f, p_f, q_f\}$. As in the third subcase of ***Case 3.1***, we may assume that G_0 contains all the fault vertices, in which case a perfect matching of $H^4 \setminus F$ can be built from respective perfect matchings of G_i, $i = 1, 2, 3$. This completes the entire proof. □

6. Concluding Remarks

In this paper, we have introduced a more general form of matching preclusion, named strong matching preclusion, and studied its properties against several types of graphs (see Table 1 for comparison of their matching preclusion and strong matching preclusion numbers). As in the matching preclusion problem, the conditional version of strong matching preclusion, demanding deletion of fault elements leaves a graph with no isolated vertices, will be worth investigating.

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Table 1: Comparison of matching preclusion and strong matching preclusion numbers.

Classes of graphs	$smp(G)$	$mp(G)$
Petersen graph	3	3 [4]
Complete graph K_n, $n \geq 2$	$n-1$	$n-1$, even $n \geq 6$ [4] $2n-3$, odd $n \geq 11$ [4]
m-regular bipartite graph, $m \geq 2$	2	m
Restricted HL-graph G^m, $m \geq 3$	m	m [12]
Recursive circulant $G(2^m, 4)$, $m \geq 3$	m	m [12]

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