The Journal of Supercomputing manuscript No.
(will be inserted by the editor)

A Sufficient Condition for the Unpaired k-Disjoint Path Coverability of Interval Graphs

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Abstract Given disjoint source and sink sets, $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$, in a graph G, an *unpaired k-disjoint path cover* joining S and T is a set of pairwise vertex-disjoint paths $\{P_1, \ldots, P_k\}$ that altogether cover every vertex of the graph, in which P_i is a path from source s_i to some sink t_j. In this paper, we prove that if the scattering number, $\mathrm{sc}(G)$, of an interval graph G of order $n \geq 2k$ is less than or equal to $-k$, there exists an unpaired k-disjoint path cover joining S and T in G for any possible configurations of source and sink sets S and T of size k each. The bound $\mathrm{sc}(G) \leq -k$ is tight; moreover, the proof directly leads to a quadratic algorithm for building an unpaired k-disjoint path cover.

Keywords Disjoint path · Path cover · Path partition · Scattering number · Interval graph

1 Introduction

A *path cover* of a graph G is a set of paths in G such that every vertex of G is contained in at least one path. A *disjoint path cover* (DPC for short) of G is a set of vertex-disjoint paths that altogether cover every vertex of G. This paper is concerned with a disjoint path cover in which each path runs from a prescribed source to a prescribed sink. Given disjoint subsets $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ of $V(G)$ for a positive integer k, a *many-to-many k-DPC* is a disjoint path cover composed of k paths that collectively join S and T; if each source $s_i \in S$ must be joined to a specific sink $t_i \in T$, the DPC is called *paired*, and it is *unpaired* if no such constraint is imposed.

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The many-to-many k-disjoint path cover has two simpler variants: The *one-to-many k-DPC* for $S = \{s\}$ and $T = \{t_1, \ldots, t_k\}$ is a disjoint path cover made of k paths, each joining a pair of source s and sink t_j for $j \in \{1, \ldots, k\}$; the *one-to-one k-DPC* for $S = \{s\}$ and $T = \{t\}$ is a disjoint path cover each of whose paths joins an identical pair of source s and sink t. The paths in the one-to-many k-DPC or in the one-to-one k-DPC share a source and/or a sink and thus are pairwise internally disjoint.

Definition 1 (Park et al. [27]) A graph G of order $n \geq 2k$ is *unpaired k-disjoint path coverable* if G has an unpaired (many-to-many) k-DPC joining S and T for any disjoint source and sink sets, S and T, of size k each.

Analogously, a graph G of order $n \geq 2k$ is said to be *paired k-disjoint path coverable* if G has a paired (many-to-many) k-DPC joining S and T for any disjoint source and sink sets, S and T, of size k each. A graph G of order $n \geq k + 1$ is *one-to-many k-disjoint path coverable* if for any disjoint subsets S and T of $V(G)$ with $|S| = 1$ and $|T| = k$, there exists a one-to-many k-DPC joining S and T. A graph G of order $n \geq k + 1$ is *one-to-one k-disjoint path coverable* if for any disjoint subsets S and T of $V(G)$ with $|S| = |T| = 1$, there exists a one-to-one k-DPC joining S and T.

The UNPAIRED k-DISJOINT PATH COVERABILITY problem is that of deciding if a given graph contains an unpaired k-DPC for any possible configurations of source and sink sets as follows:

UNPAIRED k-DISJOINT PATH COVERABILITY
INSTANCE: A graph G of order $n \geq 2k$.
QUESTION: Is G unpaired k-disjoint path coverable?

PAIRED, ONE-TO-MANY, and ONE-TO-ONE k-DISJOINT PATH COVERABILITY problems are defined similarly. Meanwhile, the UNPAIRED k-DISJOINT PATH COVER problem refers to that of testing whether there exists an unpaired k-DPC joining given source and sink sets S and T in a graph G. The k-DISJOINT PATH COVER problems for the other types are also defined similarly.

The ONE-TO-ONE k-DISJOINT PATH COVERABILITY of interval graphs was characterized by Li et al. [18] and by Park et al. [28] (Theorem 1 below); also, the ONE-TO-MANY k-DISJOINT PATH COVERABILITY of interval graphs was characterized in [18,28] (Theorem 2 below). Both characterizations can be described in terms of the scattering number $\mathrm{sc}(G)$ of an interval graph G. For a noncomplete graph G, the *scattering number* $\mathrm{sc}(G)$ of G is defined as

$$\mathrm{sc}(G) = \max\{c(G - X) - |X| : X \subset V(G),\ c(G - X) \geq 2\},$$

where $c(G - X)$ denotes the number of connected components in $G - X$. A vertex cut X of G that fulfills $c(G - X) - |X| = \mathrm{sc}(G)$ is called a *scattering set*. For a complete graph K_n of order n, we set $\mathrm{sc}(K_n) = 3 - n$ in this paper, so that the scattering numbers of the n-vertex graphs form a consecutive set $\{3 - n, 4 - n, \ldots, n\}$ [28].

Theorem 1 (Li et al. [18] and Park et al. [28]) *For $k \geq 2$, an interval graph G of order $n \geq k+1$ is one-to-one k-disjoint path coverable if and only if* $\mathrm{sc}(G) \leq 2-k$.

Theorem 2 (Li et al. [18] and Park et al. [28]) *For $k \geq 2$, an interval graph G of order $n \geq k+1$ is one-to-many k-disjoint path coverable if and only if* $\mathrm{sc}(G) \leq 1-k$ *or G is isomorphic to a complete graph K_{k+1}.*

In this paper, we study whether the characterizations of Theorems 1 and 2 can be extended to the Unpaired k-Disjoint Path Coverability of interval graphs, and derive a sufficient condition: An interval graph G of order $n \geq 2k$ for $k \geq 2$ is unpaired k-disjoint path coverable if $\mathrm{sc}(G) \leq -k$. The bound $\mathrm{sc}(G) \leq -k$ is tight in a sense that for any fixed k, there is an interval graph G with $\mathrm{sc}(G) \leq 1-k$ that is not unpaired k-disjoint path coverable.

2 Definitions and previous works

Let G be a finite, simple undirected graph whose vertex and edge sets are denoted by $V(G)$ and $E(G)$, respectively. A *path* from $u \in V(G)$ to $v \in V(G)$, referred to as a *u–v* path, is a sequence $\langle w_1, \ldots, w_l \rangle$ of distinct vertices of G such that $w_1 = u$, $w_l = v$, and $(w_i, w_{i+1}) \in E(G)$ for all $i \in \{1, \ldots, l-1\}$. If $l \geq 3$ and $(w_l, w_1) \in E(G)$, the sequence is called a *cycle*. A cycle that visits each vertex exactly once is a *Hamiltonian cycle*; a path that visits each vertex exactly once is a *Hamiltonian path*. A graph is *Hamiltonian* if a Hamiltonian cycle exists; a graph is *traceable* if a Hamiltonian path exists; a graph is *Hamiltonian-connected* if any pair of vertices are joined by a Hamiltonian path.

A *connected component* of G is a maximal connected subgraph of G. A *vertex cut* of G is a set $X \subseteq V(G)$ such that $G-X$ has two or more connected components, where $G-X$ is the subgraph obtained from G by deleting all the vertices of X or equivalently, the subgraph of G induced by $V(G) \setminus X$. The *connectivity* of G, denoted $\kappa(G)$, is the minimum number of vertices whose removal results in a disconnected graph or a trivial one-vertex graph. So, $\kappa(G)$ is equal to the size of a minimum vertex cut if G is a noncomplete graph; $\kappa(G) = n-1$ if G is a complete graph K_n of order n. A graph G is *k-connected* if $\kappa(G) \geq k$.

An *interval graph* is the intersection graph of a family $\mathcal{I}$ of closed intervals on the real line, where two vertices are connected with an edge if and only if their corresponding intervals intersect. The family $\mathcal{I}$ is usually called an *interval representation* for the graph. A *proper interval graph* is an interval graph with an interval representation in which none of the intervals properly contains another. A *unit interval graph* is an interval graph with an interval representation in which all the intervals have unit length. In 1969, Roberts [29] proved that the classes of proper interval graphs and unit interval graphs coincide.

One of the central issues in the study of interconnection networks is the detection of parallel paths, which is naturally related to routing among nodes and fault tolerance of the network [13, 22]. An interconnection network is frequently modeled as a graph, where vertices and edges represent the nodes and communication links of the network, respectively. Parallel paths correspond to disjoint paths of the underlying graph. Disjoint path is, moreover, a fundamental notion from which many graph properties can be deduced [3, 22].

The disjoint path cover problems are applicable in many areas such as software testing, database design, and code optimization [2, 23]. The problems have been studied for various classes of graphs, including recent studies on dense graphs [20], cube of connected graphs [26], balanced hypercubes [21], hypercubes [30], directed graphs [6], and torus networks [17, 24]. As mentioned earlier, the characterizations of One-to-One and One-to-Many k-Disjoint Path Coverability of interval graphs were established in [18, 28]; moreover, the conditions can be checked in linear time thanks to the linear-time algorithms for finding the scattering number of an interval graph devised by Broersma et al. [5] and by Li et al. [18]. For proper interval graphs, a characterization regarding Unpaired k-Disjoint Path Coverability was derived by Lee et al. [16].

On the other hand, no algorithm for the k-Disjoint Path Cover problem, whatever its type, on interval graphs is reported in the literature. In particular, the 1-Disjoint Path Cover problem is equivalent to the 2-Hamiltonian Path problem of deciding if there exists a Hamiltonian path that joins two given vertices. The 2-Hamiltonian Path problem on a general graph is NP-complete [10]; it remains still open if the problem on an interval graph is polynomially solvable [2, 5, 8], but partial results can be found in [19]. For proper interval graphs, however, a linear-time algorithm for the Unpaired k-Disjoint Path Cover problem was developed by Park et al. [25], to which the One-to-One and One-to-Many k-Disjoint Path Cover problems are reduced in linear time.

The Path Cover problem, also referred to as the Unconstrained Disjoint Path Cover problem, is that of determining a disjoint path cover with the minimum number of paths each of which freely joins a pair of vertices (alleviating the constraint that each path should run from a prescribed source to a prescribed sink). Linear-time algorithms for the Path Cover problem on interval graphs were developed by Arikati et al. [1] and by Hung et al. [14]. Given a vertex u in a graph, the 1-Fixed-Endpoint Path Cover problem is to find a disjoint path cover with the minimum number of paths in which a path contains the vertex u as an endpoint. The 1-Fixed-Endpoint Path Cover problem on interval graphs were solved in polynomial time by Asdre et al. [2] and then in linear time by Li et al. [19].

Interval graphs are a well-studied class of graphs. One of the early characterizations of interval graphs is the following:

Theorem 3 (Gilmore and Hoffman [11]) *A graph G is an interval graph if and only if the maximal cliques of G can be linearly ordered such that, for every vertex v of G, the maximal cliques containing v occur consecutively.*

Let $C_1, \ldots, C_q$ be a linear ordering of the maximal cliques of an interval graph G such that each vertex of G appears in consecutive cliques only. Obviously, the graph G is noncomplete if and only if $q \geq 2$. Since C_1 and C_q are maximal cliques, C_1 and C_q each contains at least one vertex that does not occur in any other clique. Let u_1 be such a vertex in C_1 and let u_n be such a vertex in C_q. The vertices u_1 and u_n, respectively, are referred to as a *left tip* and a *right tip*.

It was proven by Keil [15] that a traceable interval graph (that possesses a Hamiltonian path) always contains a Hamiltonian path that runs from a left tip u_1 to a right tip u_n. A path in an interval graph is *monotone* if every edge (u, v) can be assigned a point from $I_u \cap I_v$, the intersection of the intervals corresponding to u and v, such that the points that are ordered in the appearance of the edges in the path form a nondecreasing sequence.

Lemma 1 (Keil [15]) *If an interval graph contains a Hamiltonian path, then it contains a monotone Hamiltonian path from u_1 to u_n.*

For distinct vertices u_i and u_j of G, there exists a maximal clique C_p, $p \in \{1, \ldots, q\}$, that contains u_i and u_j both if and only if $(u_i, u_j) \in E(G)$. Thus, a vertex u_i of G can be represented by a closed interval $I_{u_i} = [l_i, r_i]$, where $l_i = \min\{j : u_i \in V(C_j)\}$ and $r_i = \max\{j : u_i \in V(C_j)\}$. So, we have $I_{u_1} = [1, 1]$ and $I_{u_n} = [q, q]$; moreover, $(u_1, u_n) \notin E(G)$ if G is noncomplete. Refer to Fig. 1.

Lemma 2 (Broersma et al. [5]) *Let P be a path joining u_1 and u_n of a noncomplete interval graph G, and let Q be a path of G that runs from u_1 or to u_n. If the two paths P and Q are internally disjoint, there exists a path P' joining u_1 and u_n such that $V(P') = V(P) \cup V(Q)$.*

The interval graphs that are Hamiltonian/traceable were characterized by Deogun et al. [9] (Theorem 4 below); also, those that are Hamiltonian-connected were characterized by Broersma et al. [5] (Theorem 5 below). The existence of a one-to-one k-DPC between u_1 and u_n was studied by Broersma et al. [5], as is shown in Theorem 6. A one-to-one k-DPC is also known as a *spanning k-stave* or *k^*-container.*

Theorem 4 (Deogun et al. [9]) *(a) An interval graph G of order $n \geq 3$ is Hamiltonian if and only if $\mathrm{sc}(G) \leq 0$. (b) An interval graph G of order $n \geq 2$ is traceable if and only if $\mathrm{sc}(G) \leq 1$.*

Theorem 5 (Broersma et al. [5]) *An interval graph G of order $n \geq 3$ is Hamiltonian-connected if and only if $\mathrm{sc}(G) \leq -1$ or G is isomorphic to a complete graph K_3.*

Theorem 6 (Broersma et al. [5]) *A noncomplete interval graph G contains a spanning k-stave between u_1 and u_n if and only if $\mathrm{sc}(G) \leq 2 - k$.*

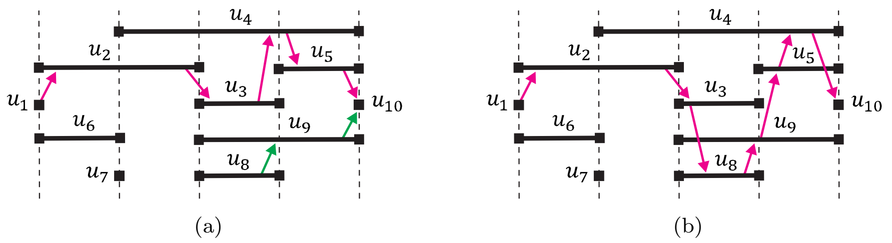


Fig. 1: An interval representation for an interval graph with 10 vertices, in which there are five maximal cliques, $C_1 = \{u_1, u_2, u_6\}$, $C_2 = \{u_2, u_4, u_6, u_7\}$, $\ldots$, $C_5 = \{u_4, u_5, u_9, u_{10}\}$: (a) there are paths, $P = (u_1, u_2, u_3, u_4, u_5, u_{10})$ and $Q = (u_8, u_9, u_{10})$, which are internally disjoint; (b) the two paths can be merged into a new path $P' = (u_1, u_2, u_3, u_8, u_9, u_5, u_4, u_{10})$.

3 Unpaired disjoint path covers

In this section, we will prove by induction on k that an interval graph G of order $n \geq 2k$ for $k \geq 2$ is unpaired k-disjoint path coverable if $\mathrm{sc}(G) \leq -k$. The base step of $k = 2$ and the inductive step of $k \geq 3$, respectively, are addressed in Sections 3.1 and 3.2.

3.1 Unpaired 2-disjoint path covers

We start with a discussion of a Hamiltonian path that runs from a left tip u_1 of an interval graph with $\mathrm{sc}(G) \leq 0$ or of its spanning subgraph that contains u_1. Symmetrically, this discussion is also about a Hamiltonian path that runs to a right tip u_n.

Lemma 3 *In an interval graph G with $\mathrm{sc}(G) \leq 0$, there exists a Hamiltonian u_1–w path for every vertex w of G other than u_1.*

Proof If G is a complete graph, the lemma holds true obviously; so, assume G is noncomplete. There is a one-to-one 2-DPC joining u_1 and u_n by Theorem 6. If $w = u_n$, a Hamiltonian u_1–w path exists by Lemma 1. Otherwise, there is a path in the DPC, represented as $\langle v_1, \ldots, v_l \rangle$ for some $l \geq 3$ with $v_1 = u_1$ and $v_l = u_n$, that passes through w as an intermediate vertex, i.e., $w = v_p$ for some $p \in \{2, \ldots, l-1\}$. It suffices to build a Hamiltonian u_1–u_n path of the subgraph $G - \{v_p, \ldots, v_{l-1}\}$ through Lemma 2 and then combine the Hamiltonian path with the path $\langle v_p, \ldots, v_{l-1} \rangle$ into a Hamiltonian u_1–w path of G. ■

Let G be a noncomplete interval graph with $\mathrm{sc}(G) \leq 0$, in which there exists a one-to-one 2-DPC $\mathcal{P} = \{P_1, P_2\}$ joining a pair of left and right tips, u_1 and u_n, of G. The two paths, P_1 and P_2, are assumed to be monotone due to Lemma 1. (Note that the subgraph $G[V(P_i)]$ induced by $V(P_i)$ is a

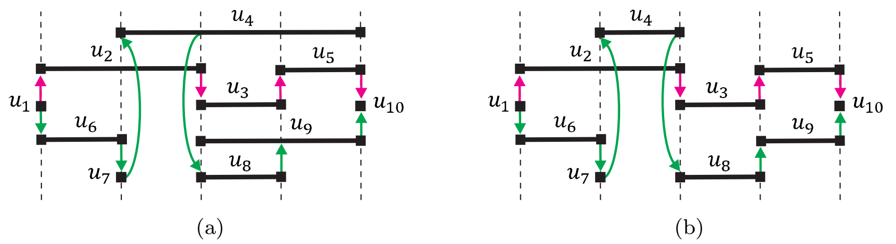


Fig. 2: Interval representations of an interval graph G and its spanning subgraph G': (a) G has a one-to-one 2-DPC $\mathcal{P} = \{(u_1, u_2, u_3, u_5, u_{10}), (u_1, u_6, u_7, u_4, u_8, u_9, u_{10})\}$; (b) $\mathcal{P}$ is also a one-to-one 2-DPC of G', leading to $\mathrm{sc}(G') \leq 0$.

traceable interval graph and moreover, u_1 and u_n are also a pair of left and right tips of the subgraph.) It follows that for a path $P_i = \langle v_1^i, \ldots, v_{l_i}^i \rangle$ in $\mathcal{P}$ where $v_1^i = u_1$ and $v_{l_i}^i = u_n$, there exists a nondecreasing sequence of points $h_1^i, \ldots, h_{l_i-1}^i$ on the real line such that $h_j^i \in I_{v_j^i} \cap I_{v_{j+1}^i}$ for all $j \in \{1, \ldots, l_i - 1\}$. For each $w \in V(G)$, we define a closed interval I'_w (other than the interval I_w corresponding to w) as follows (see Fig. 2):

$$I'_w = \begin{cases} [h_{j-1}^i, h_j^i] & \text{if } w = v_j^i \text{ for some } i \in \{1,2\} \text{ and } j \in \{2, \ldots, l_i - 1\}, \\ I_w & \text{otherwise, i.e., } w \in \{u_1, u_n\}. \end{cases}$$

The intersection graph G' of a family of intervals $\mathcal{I}' = \{I'_w : w \in V(G)\}$ is a spanning subgraph of G because $I'_w \subseteq I_w$ for all w. Nonetheless, u_1 and u_n respectively are left and right tips of G' and moreover, $\mathcal{P}$ is a one-to-one 2-DPC of G' joining u_1 and u_n. Thus, we still have $\mathrm{sc}(G') \leq 0$ by Theorem 6. Let $\mathrm{lp}(I'_w)$ and $\mathrm{rp}(I'_w)$ denote the left and right endpoints of an interval I'_w. An interval I'_w is said to be to the right of I'_v (or I'_v is to the left of I'_w) if $\mathrm{rp}(I'_v) < \mathrm{lp}(I'_w)$.

Lemma 4 *Let $\mathcal{P} = \{P_1 = \langle v_1, \ldots, v_{l_1} \rangle, P_2 = \langle w_1, \ldots, w_{l_2} \rangle\}$ be a one-to-one 2-DPC joining u_1 and u_n of a noncomplete interval graph G' with $v_1 = w_1 = u_1$ and $v_{l_1} = w_{l_2} = u_n$, in which P_1 and P_2 are monotone and moreover, the intersections $I'_{v_i} \cap I'_{v_{i+1}}$ for $1 \leq i < l_1$ and $I'_{w_j} \cap I'_{w_{j+1}}$ for $1 \leq j < l_2$ are all single points. For vertices $v_p \in V(P_1)$ and $w_q \in V(P_2)$ with $q \geq 2$, there exists a Hamiltonian u_1–w_q path in the subgraph H of G' induced by $\{v_i : i \leq p\} \cup \{w_j : j \leq q\}$ if I'_{w_q} is to the right of or intersects with I'_{v_p}.*

Proof If $p = 1$, then $\langle w_1, \ldots, w_q \rangle$ will be a required Hamiltonian path; so, we assume $p \geq 2$. Then, the subgraph H is a traceable graph of order at least 3, in which u_1 remains a left tip. If I'_{w_q} is to the right of I'_{v_p}, then w_q is a right tip of H. So, there exists a Hamiltonian u_1–w_q path in H by Lemma 1. Suppose $I'_{v_p} \cap I'_{w_q} \neq \emptyset$ now. Then H contains a Hamiltonian cycle passing through (v_p, w_q), leading to $\mathrm{sc}(H) \leq 0$ by Theorem 4(a). Thus, there exists a Hamiltonian u_1–w_q path in H by Lemma 3 because $w_q \neq u_1$, completing the proof. ■

Lemmas 5, 6, and 7 below are concerned with the existence of an unpaired 2-DPC joining disjoint source and sink sets S and T of size two each in an interval graph G with $\mathrm{sc}(G) \leq 0$, $\mathrm{sc}(G) \leq -1$, and $\mathrm{sc}(G) \leq -2$, respectively.

Lemma 5 *Let G be a noncomplete interval graph of order $n \geq 4$ with $\mathrm{sc}(G) \leq 0$, in which disjoint terminal sets S and T of size two each are given. If $u_1, u_n \in S \cup T$, then there exists an unpaired 2-DPC joining S and T in G.*

Proof There exists a one-to-one 2-DPC $\mathcal{P} = \{P_1 = \langle v_1, \ldots, v_{l_1} \rangle, P_2 = \langle w_1, \ldots, w_{l_2} \rangle\}$ joining u_1 and u_n in G by Theorem 6, where $v_1 = w_1 = u_1$ and $v_{l_1} = w_{l_2} = u_n$. The paths P_1 and P_2 are assumed to be monotone by Lemma 1. If we define an interval I'_w for each interval I_w corresponding to $w \in V(G)$ in the aforementioned way, the intersection graph G' of a family $\{I'_w : w \in V(G)\}$ is a spanning subgraph of G such that (i) $\mathcal{P}$ is also a one-to-one 2-DPC of G' joining u_1 and u_n, (ii) u_1 and u_n respectively are left and right tips of G', and (iii) the intersections $I'_{v_i} \cap I'_{v_{i+1}}$ for $1 \leq i < l_1$ and $I'_{w_j} \cap I'_{w_{j+1}}$ for $1 \leq j < l_2$ are all single points. For the proof, it suffices to build an unpaired 2-DPC joining S and T in G'. Let x and y be the terminal vertices other than u_1 and u_n, so $S \cup T = \{u_1, u_n, x, y\}$. Assume w.l.o.g. $u_1 \in S$. There are two cases depending on the distribution of x and y.

Case 1: *$x \in V(P_1)$ and $y \in V(P_2)$.* Let $x = v_p$ and $y = w_q$ for some p and q, and assume w.l.o.g. x is a sink (because at least one of x and y are sinks). Suppose $\mathrm{lp}(I'_y) \leq \mathrm{lp}(I'_x)$ first. We will build one DPC path joining u_1 and x in the subgraph H induced by $\{v_i : i \leq p\} \cup \{w_j : j < q\}$ and the other DPC path in $G' - H$. There is a Hamiltonian u_1–x path in H by Lemma 4 because I'_x is to the right of $I'_{w_{q-1}}$ (when $\mathrm{lp}(I'_y) < \mathrm{lp}(I'_x)$) or intersects with $I'_{w_{q-1}}$ (when $\mathrm{lp}(I'_y) = \mathrm{lp}(I'_x)$); also, there is a Hamiltonian u_n–y path in $G' - H$ again by Lemma 4 because I'_y is to the left of $I'_{v_{p+1}}$ (when $\mathrm{rp}(I'_y) < \mathrm{rp}(I'_x)$) or intersects with $I'_{v_{p+1}}$ (when $\mathrm{rp}(I'_y) \geq \mathrm{rp}(I'_x)$). The two Hamiltonian paths form an unpaired 2-DPC of G', as required.

Now, suppose $\mathrm{lp}(I'_x) < \mathrm{lp}(I'_y)$. Let v_r be the rightmost neighbor of w_{q-1} among vertices in $\{v_p, \ldots, v_{l_1-1}\}$, i.e., $r = \max\{j : p \leq j < l_1, I'_{v_j} \cap I'_{w_{q-1}} \neq \emptyset\}$. For the subgraph H induced by $\{v_i : i \leq r\} \cup \{w_j : j < q\}$, two DPC paths will be built: one in H and the other in $G' - H$. The subgraph H has a u_1–x Hamiltonian path by Lemma 3 if $|V(H)| \geq 3$; H has a u_1–x Hamiltonian path $\langle u_1, x \rangle$ otherwise. Also, $G' - H$ has a Hamiltonian u_n–y path $\langle w_{l_2}, w_{l_2-1}, \ldots, w_q \rangle$ if $v_{r+1} = u_n$; otherwise, we have $\mathrm{rp}(I'_{w_{q-1}}) = \mathrm{lp}(I'_y) < \mathrm{lp}(I'_{v_{r+1}})$, so $G' - H$ has a Hamiltonian u_n–y path by Lemma 4 because I'_y is to the left of $I'_{v_{r+1}}$ (when $\mathrm{rp}(I'_y) < \mathrm{lp}(I'_{v_{r+1}})$) or intersects with $I'_{v_{r+1}}$ (when $\mathrm{rp}(I'_y) \geq \mathrm{lp}(I'_{v_{r+1}})$).

Case 2: $x, y \in V(P_2)$. Let $x = w_p$ and $y = w_q$ for some $p < q$. If one of x and y, say x, is a source (i.e. $x \in S$ and $y \in T$), it suffices to set $\langle w_p, \ldots, w_q \rangle$ be a one DPC path and then build a u_1–u_n Hamiltonian path in the subgraph $G' - \{w_p, \ldots, w_q\}$, which exists by Lemma 1. Suppose x and y are both sinks hereafter. The proof is very similar to the second subcase of Case 1. Let v_r be the rightmost neighbor of w_{q-1} among vertices in $\{v_1, \ldots, v_{l_1-1}\}$, and let H be the subgraph induced by $\{v_i : i \le r\} \cup \{w_j : j < q\}$. It follows that $r \ge 2$ and H has a Hamiltonian cycle (passing through (w_{q-1}, v_r)). One DPC path will be built in H and the other in $G' - H$. In H, there is a u_1–x Hamiltonian path by Lemma 3 because u_1 is a left tip of H. Also, $G' - H$ has a Hamiltonian u_n–y path $\langle w_{l_2}, w_{l_2-1}, \ldots, w_q\rangle$ if $v_{r+1} = u_n$; otherwise, $G' - H$ has a Hamiltonian u_n–y path by Lemma 4 because I'_y is to the left of $I'_{v_{r+1}}$ or intersects with $I'_{v_{r+1}}$. This completes the proof. ■

Lemma 6 *Let G be a noncomplete interval graph of order $n \ge 4$ with $\mathrm{sc}(G) \le -1$, in which disjoint terminal sets S and T of size two each are given. If $\{u_1, u_n\} \cap (S \cup T) \ne \emptyset$, then there exists an unpaired 2-DPC joining S and T in G.*

Proof If $u_1, u_n \in S \cup T$, then the lemma follows from Lemma 5, hence assume w.l.o.g. $u_n \notin S \cup T$. By Theorem 6, there exists a one-to-one 3-DPC joining u_1 and u_n in G, in which a path, say P_1, passes though a terminal as an intermediate vertex. Let P_1 be represented as $\langle v_1 = u_1, v_2, \ldots, v_l = u_n\rangle$ for some $l \ge 3$. For the rightmost terminal v_p on P_1 (i.e., $\{v_p, \ldots, v_l\} \cap (S \cup T) = \{v_p\}$), we divide P_1 into two subpaths: $P'_1 = \langle v_1, \ldots, v_{p-1}\rangle$ and $P''_1 = \langle v_p, \ldots, v_l\rangle$. Combining P'_1 with a DPC path P_2 other than P_1 into a new u_1–u_n path through Lemma 2 results in a one-to-one 2-DPC of the subgraph H induced by $V(G) \setminus \{v_p, \ldots, v_{l-1}\}$. Assume w.l.o.g. that v_p is a source. If we regard u_n as a *virtual* source, then H contains an unpaired 2-DPC joining $S' := (S \setminus \{v_p\}) \cup \{u_n\}$ and T by Lemma 5. From concatenating P''_1 and the DPC path that runs from u_n to a sink, we can obtain an unpaired 2-DPC of G joining S and T, completing the proof. ■

Lemma 7 *Let G be an interval graph of order $n \ge 4$. If $\mathrm{sc}(G) \le -2$, then G is unpaired 2-disjoint path coverable.*

Proof A complete graph of order $n \ge 4$ is obviously unpaired 2-disjoint path coverable, so we assume that G is noncomplete. Let S and T be disjoint terminal sets of size two each in G. If $\{u_1, u_n\} \cap (S \cup T) \ne \emptyset$, then there exists an unpaired 2-DPC joining S and T by Lemma 6. So, it is further assumed $u_1, u_n \notin S \cup T$. The remaining part proceeds similar to the proof of Lemma 6. There exists a one-to-one 4-DPC joining u_1 and u_n in G, in which a path, say P_1, passes through a terminal as an intermediate vertex, where P_1 is represented as $\langle v_1 = u_1, v_2, \ldots, v_l = u_n\rangle$. For the rightmost terminal v_p on P_1, we divide P_1 into $P'_1 = \langle v_1, \ldots, v_{p-1}\rangle$ and $P''_1 = \langle v_p, \ldots, v_l\rangle$ subpaths. Combining P'_1 with a DPC path P_2 other than P_1 into a new u_1–u_n path results in a one-to-one 3-DPC of the subgraph H induced by $V(G) \setminus \{v_p, \ldots, v_{l-1}\}$. Assuming w.l.o.g. v_p is a source, we find an unpaired 2-DPC of H joining $S' := (S \setminus \{v_p\}) \cup \{u_n\}$ and T through Lemma 6, from which an unpaired 2-DPC of G joining S and T can be constructed, as required. ■

3.2 Unpaired k-disjoint path covers

Given disjoint terminal sets S and T of size k each in an interval graph G with $\mathrm{sc}(G) \leq -k$ for $k \geq 3$, we will devise a procedure for building an unpaired k-DPC joining S and T in this section. There exists a one-to-one $(k+2)$-DPC $\mathcal{P}$ joining a pair of left and right tips, u_1 and u_n, in G by Theorem 6. As was done in the previous section, $\mathcal{P}$ will be utilized in building an unpaired k-DPC of G. Our problem has an easy solution if there exists a path $P_i \in \mathcal{P}$ that passes through at least one source in S and one sink in T as intermediate vertices, or if at least one of u_1 and u_n are terminal vertices, as shown below.

Lemma 8 *Let S and T be disjoint terminal sets of size $k \geq 3$ each given in a noncomplete interval graph G that has a one-to-one $(k+2)$-DPC $\mathcal{P}$ joining a pair of its left and right tips, u_1 and u_n. If there exists a path $P_i \in \mathcal{P}$ such that $(V(P_i) \setminus \{u_1, u_n\}) \cap S \neq \emptyset$ and $(V(P_i) \setminus \{u_1, u_n\}) \cap T \neq \emptyset$, or if $\{u_1, u_n\} \cap (S \cup T) \neq \emptyset$, then the problem of building an unpaired k-DPC in G is reduced to a problem of building an unpaired $(k-1)$-DPC joining some terminal sets $S' \subset S$ and $T' \subset T$ in an induced subgraph H of G with $\mathrm{sc}(H) \leq -(k-1)$.*

Proof Firstly, suppose $P_i = \langle v_1, \ldots, v_l \rangle$ for some $l \geq 4$ is a path in $\mathcal{P}$ such that $(V(P_i) \setminus \{u_1, u_n\}) \cap S \neq \emptyset$ and $(V(P_i) \setminus \{u_1, u_n\}) \cap T \neq \emptyset$. Then, there exists a subpath $\langle v_p, \ldots, v_q \rangle$, $1 < p < q < l$, of P_i such that $\{v_p, \ldots, v_q\} \cap (S \cup T) = \{v_p, v_q\}$ and either $v_p \in S$ and $v_q \in T$ or $v_p \in T$ and $v_q \in S$. Divide P_i into three subpaths, $P_i^1 = \langle v_1, \ldots, v_{p-1} \rangle$, $P_i^2 = \langle v_p, \ldots, v_q \rangle$, and $P_i^3 = \langle v_{q+1}, \ldots, v_l \rangle$, and let $H = G - V(P_i^2)$. If we combine a DPC path $P_j \in \mathcal{P}$ other than P_i with P_i^1 into a u_1–u_n path P'_j and combine again P'_j with P_i^3 into a u_1–u_n path P''_j through Lemma 2, we obtain a one-to-one $(k+1)$-DPC $\mathcal{P}'$ of the subgraph H, where $\mathcal{P}' = (\mathcal{P} \setminus \{P_i, P_j\}) \cup \{P''_j\}$. It follows that $\mathrm{sc}(H) \leq 2 - (k+1) = -(k-1)$ by Theorem 6. If there exists an unpaired $(k-1)$-DPC $\mathcal{Q}$ joining S' and T' in H, where $S' = S \setminus \{v_p\}$ and $T' = T \setminus \{v_q\}$ if $v_p \in S$ and $v_q \in T$, or $S' = S \setminus \{v_q\}$ and $T' = T \setminus \{v_p\}$ otherwise, then $\mathcal{Q}' = \mathcal{Q} \cup \{P_i^2\}$ will form an unpaired k-DPC of G joining S and T, as required.

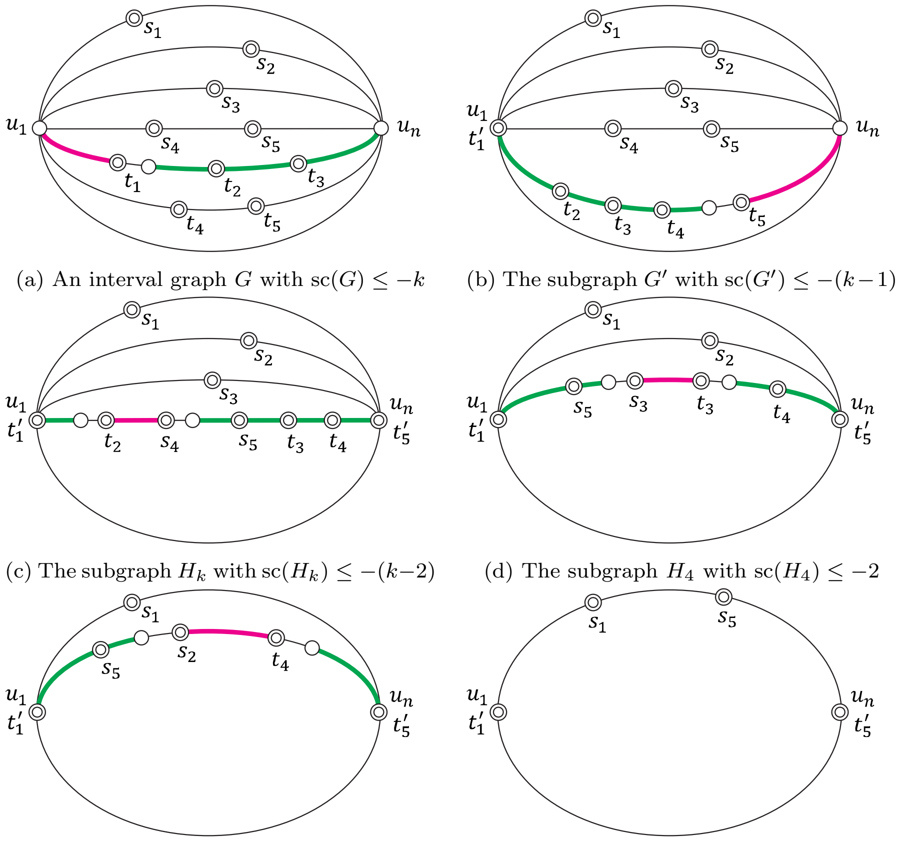
Secondly, suppose $\{u_1, u_n\} \cap (S \cup T) \neq \emptyset$. We further suppose that no path in $\mathcal{P}$ passes through a pair of source and sink as intermediate vertices. Assume w.l.o.g. u_1 is a source. Pick up a path $P_i \in \mathcal{P}$ that passes through a sink as an intermediate vertex, where $P_i = \langle v_1, \ldots, v_l \rangle$ with $v_1 = u_1$ for some $l \geq 3$. For the leftmost sink v_p on P_i (i.e., $\{v_1, \ldots, v_p\} \cap T = \{v_p\}$), we divide P_i into two subpaths, $P_i^1 = \langle v_1, \ldots, v_p \rangle$ and $P_i^2 = \langle v_{p+1}, \ldots, v_l \rangle$, and let $H = G - \{v_2, \ldots, v_p\}$. Combining a path $P_j \in \mathcal{P}$ other than P_i with P_i^2 into a u_1–u_n path P'_j results in a one-to-one $(k+1)$-DPC $\mathcal{P}' = (\mathcal{P} \setminus \{P_i, P_j\}) \cup \{P'_j\}$ of the subgraph H, leading to $\mathrm{sc}(H) \leq 2 - (k+1) = -(k-1)$. If there exists an unpaired $(k-1)$-DPC $\mathcal{Q}$ joining $S' = S \setminus \{u_1\}$ and $T' = T \setminus \{v_p\}$ in H, with u_1 being regarded as a *virtual* nonterminal, an unpaired k-DPC $\mathcal{Q}'$ of G joining S and T can be built as follows: Pick up the unique path $Q_h \in \mathcal{Q}$ that passes through u_1, where Q_h is represented as $\langle w_1, \ldots, w_r \rangle$ with $w_q = u_1$

for some $1 < q < r$. Notice that the predecessor w_{q-1} and the successor w_{q+1} of w_q on Q_h are adjacent in H because w_q $(= u_1)$ is represented as a single-point interval $[1, 1]$. Thus, removing u_1 from Q_h results in a path $Q'_h = \langle w_1, \ldots, w_{q-1}, w_{q+1}, \ldots, w_r\rangle$ that joins the same source-sink pair as Q_h does. Therefore, $\mathcal{Q}' = (\mathcal{Q} \setminus \{Q_h\}) \cup \{Q'_h, P_i^1\}$ will form a required unpaired k-DPC of G, completing the proof. ■

Now, we turn our attention to the remaining case where neither u_1 nor u_n is a terminal and no path in the one-to-one $(k+2)$-DPC $\mathcal{P}$ passes through both a source and a sink. It is assumed w.l.o.g. that among the $k+2$ paths in $\mathcal{P}$, the number of paths that pass through a source is no less than the number of paths that pass through a sink. We will reduce our problem of building an unpaired k-DPC in the graph G having a one-to-one $(k+2)$-DPC joining u_1 and u_n to a problem of building an unpaired k-DPC in an induced subgraph H_k of G having a one-to-one k-DPC joining u_1 and u_n with both u_1 and u_n being *virtual* sinks. For each $r = k$ downto 3, the r-DPC problem on H_r will then be reduced to an $(r-1)$-DPC problem on an induced subgraph H_{r-1} of H_r; eventually, the 2-DPC problem on H_2 will be resolved thanks to Lemma 5.

The one-to-one DPC $\mathcal{P}$ can be partitioned into $\mathcal{P}'$ and $\mathcal{P}''$, where $\mathcal{P}'$ is a set of paths each of which passes through a source or passes through no terminals, and $\mathcal{P}''$ is a set of paths each of which passes through a sink. If there is a path in $\mathcal{P}$ that passes through all sinks, we redefine $\mathcal{P}' := \mathcal{P}' \setminus \{P_i\}$ and $\mathcal{P}'' := \mathcal{P}'' \cup \{P_i\}$ for some P_i that passes through no terminals, so that $|\mathcal{P}'| \geq |\mathcal{P}''| \geq 2$. Let $\mathcal{P}' = \{R_1, \ldots, R_p\}$ and $\mathcal{P}'' = \{P_1, \ldots, P_q\}$, where $p \geq q \geq 2$ and $p + q = k + 2$. Also, let a_i denote the number of sources contained in $R_i \in \mathcal{P}'$, and let b_j denote the number of sinks in $P_j \in \mathcal{P}''$, so $\sum_{i=1}^{p} a_i = \sum_{j=1}^{q} b_j = k$. We assume the sequences $A = (a_1, a_2, \ldots, a_p)$ and $B = (b_1, b_2, \ldots, b_q)$ both are nonincreasing, i.e., $a_1 \geq \cdots \geq a_p$ and $b_1 \geq \cdots \geq b_q$. For the configuration shown in Fig. 3(a), we have $k = 5$, $p = 5$, $q = 2$, $A = (2, 1, 1, 1, 0)$, and $B = (3, 2)$.

Before we go into the details of an algorithm, let us take a look at the basic idea of the algorithm through the example shown in Fig. 3. First, divide $P_1 \in \mathcal{P}''$, represented as $P_1 = \langle v_1, \ldots, v_l\rangle$ for some l, into two subpaths, $P_1^1 = \langle v_1, \ldots, v_g\rangle$ and $P_1^2 = \langle v_{g+1}, \ldots, v_l\rangle$ for the leftmost sink v_g on P_1 with $\{v_1, \ldots, v_g\} \cap T = \{v_g\}$, and then combine $P_2 \in \mathcal{P}''$ with P_1^2 into a new u_1–u_n path P'_2 through Lemma 2, leading to a one-to-one $(k+1)$-DPC of the subgraph $G' := G - \{v_2, \ldots, v_g\}$, as shown in Fig. 3(b). In a similar way, divide the path $P'_2 = \langle w_1, \ldots, w_{l'}\rangle$ into $\langle w_1, \ldots, w_{h-1}\rangle$ and $\langle w_h, \ldots, w_{l'}\rangle$ for the rightmost sink w_h on P'_2 and then combine $R_1 \in \mathcal{P}'$ with $\langle w_1, \ldots, w_{h-1}\rangle$ into a u_1–u_n path R^*, resulting in a one-to-one k-DPC of the subgraph $H_k := G' - \{w_h, \ldots, w_{l'-1}\}$, as shown in Fig. 3(c). If there is an unpaired k-DPC $\mathcal{Q}$ in H_k joining S and $(T \setminus \{v_g, w_h\}) \cup \{u_1, u_n\}$, we can build an unpaired k-DPC of G joining S and T just by extending the path in $\mathcal{Q}$ that runs to u_1 to reach v_g through P_1^1 and extending the path in $\mathcal{Q}$ that runs to u_n to reach w_h through the reverse of the subpath $\langle w_h, \ldots, w_{l'}\rangle$ of P'_2.



(a) An interval graph G with $\mathrm{sc}(G) \leq -k$

(b) The subgraph G' with $\mathrm{sc}(G') \leq -(k-1)$

(c) The subgraph H_k with $\mathrm{sc}(H_k) \leq -(k-2)$

(d) The subgraph H_4 with $\mathrm{sc}(H_4) \leq -2$

(e) The subgraph H_3 with $\mathrm{sc}(H_3) \leq -1$

(f) The subgraph H_2 with $\mathrm{sc}(H_2) \leq 0$

Fig. 3: Illustrations of an algorithm for building an unpaired k-DPC joining S and T in an interval graph G with $\mathrm{sc}(G) \leq -k$, where $k = 5$ in this instance.

As will be demonstrated later, R^* always goes through at least one source and one sink. So, we can extract a subpath from R^* that joins a source and a sink; the remaining two subpaths of R^* are then combined one by one with a carefully chosen path in $\mathcal{P}$ into new R^*. It would be natural to choose a path in $\mathcal{P}'$ that visits as many sources as possible if there are not more sources than sinks in $R^* \setminus \{u_1, u_n\}$, and choose a path in $\mathcal{P}''$ that visits as many sinks as possible otherwise. Repeat this process until $k-2$ paths are extracted (or equivalently, until a one-to-one 2-DPC joining u_1 and u_n appears). Refer to Fig. 3 (c)–(e), where s_4–t_2, s_3–t_3, and s_2–t_4 subpaths are extracted in sequence. Finally, it suffices to build an unpaired 2-DPC in the remaining subgraph H_2, whose existence is guaranteed by Lemma 5.

Given disjoint terminal sets S, T in G and a one-to-one $(k+2)$-DPC $\mathcal{P}$ joining u_1 and u_n such that u_1 and u_n are both nonterminals and no path in $\mathcal{P}$ passes through both a source and a sink, a procedure for building an unpaired

k-DPC of G joining S and T is given in Algorithm 1 below. Steps 1–5 and 24–26 of the algorithm describe the reduction of our problem to a problem of building an unpaired k-DPC in the induced subgraph H_k. An unpaired k-DPC $\{Q_1, \ldots, Q_k\}$ of H_k is built in Steps 6–23. The key point is that on each of the $k-2$ iterations of the for loop in Steps 9–22, R^* should include at least one source and one sink as intermediate vertices. Note that the numbers of sources and sinks in $V(R^*) \setminus \{u_1, u_n\}$ at the beginning of the r-th iteration of the for loop can be represented as $c_r - (r-1)$ and $d_r - (r-1)$, respectively. For the example shown in Fig. 3, $(c_1, d_1) = (2, 3)$, $(c_2 - 1, d_2 - 1) = (2, 2)$, and $(c_3 - 2, d_3 - 2) = (2, 1)$.

Algorithm 1: Constructing an unpaired k-DPC $\mathcal{Q}$ joining S and T

1 $v_g \leftarrow$ the leftmost sink on $P_1 = \langle v_1, \ldots, v_l \rangle$ in $\mathcal{P}''$;
2 $P_2' \leftarrow$ a u_1–u_n path obtained by combining $P_2 \in \mathcal{P}''$ with $\langle v_{g+1}, \ldots, v_l \rangle$;
3 $w_h \leftarrow$ the rightmost sink on $P_2' = \langle w_1, \ldots, w_{l'} \rangle$;
4 $R^* \leftarrow$ a u_1–u_n path made by combining $R_1 \in \mathcal{P}'$ with $\langle w_1, \ldots, w_{h-1} \rangle$;
5 $H_k \leftarrow G - (\{v_2, \ldots, v_g\} \cup \{w_h, \ldots, w_{l'-1}\})$;
6 $\mathcal{Q} \leftarrow \emptyset$;
7 $(c_1, d_1) \leftarrow (a_1, b_1 + b_2 - 2)$;
8 $i \leftarrow 1$; $j \leftarrow 2$;
9 **for** $r \leftarrow 1$ **to** $k-2$ **do**
10   Extract a DPC path, Q_{k-r+1}, that joins a source and a sink from R^*, leaving two subpaths R_1^* and R_2^* of R^*;
11     $\mathcal{Q} \leftarrow \mathcal{Q} \cup \{Q_{k-r+1}\}$;
12     $H_{k-r} \leftarrow H_{k-r+1} - V(Q_{k-r+1})$;
13   **if** $c_r \le d_r$ *or* $d_r = k-2$ **then**
14         $i \leftarrow i+1$;
15         $R^* \leftarrow$ a u_1–u_n path made by combining $R_i \in \mathcal{P}'$, R_1^* and R_2^*;
16         $(c_{r+1}, d_{r+1}) \leftarrow (c_r + a_i, d_r)$;
17   **else**
18         $j \leftarrow j+1$;
19         $R^* \leftarrow$ a u_1–u_n path made by combining $P_j \in \mathcal{P}''$, R_1^* and R_2^*;
20         $(c_{r+1}, d_{r+1}) \leftarrow (c_r, d_r + b_j)$;
21   **end**
22 **end**
23 Build an unpaired 2-DPC $\mathcal{Q}' = \{Q_1, Q_2\}$ in H_2 joining the set of remaining sources and $\{u_1, u_n\}$ according to the proof of Lemma 5;
24 $Q_1' \leftarrow$ concatenation of the path in $\mathcal{Q}'$ that runs to u_1 and $\langle v_1, \ldots, v_g \rangle$;
25 $Q_2' \leftarrow$ concatenation of the path in $\mathcal{Q}'$ that runs to u_n and $\langle w_{l'}, \ldots, w_h \rangle$;
26 $\mathcal{Q} \leftarrow \mathcal{Q} \cup \{Q_1', Q_2'\}$;
27 **return** $\mathcal{Q}$;

Lemma 9 *Given S, T, and $\mathcal{P}$ in G, in which $\{u_1, u_n\} \cap (S \cup T) = \emptyset$ and moreover, $V(P_i) \cap S = \emptyset$ or $V(P_i) \cap T = \emptyset$ for all $P_i \in \mathcal{P}$, Algorithm 1 builds an unpaired k-DPC of G joining S and T.*

Proof We will show the following two in this proof since the rest not covered here is clear from what we have discussed so far: (i) The indices i and j always meet $i \le p$ and $j \le q$ while the algorithm is running; (ii) the path R^* in Step 10 of the algorithm contains at least one source and one sink, from which a DPC path can extracted. We begin with the basics of the nonincreasing sequences $(a_1, \ldots, a_p)$ and $(b_1, \ldots, b_q)$, where $\sum_{i=1}^{p} a_i = \sum_{j=1}^{q} b_j = k$, $p+q = k+2$, and $p \ge q \ge 2$.

Claim 1 (a) $p \ge 3$. (b) $b_1 + b_2 \ge 3$. (c) $a_{p-1} + a_p \le 2$.

Proof of claim. The facts $p \ge q$, $p+q = k+2$ and $k \ge 3$ leads to $p \ge \lceil \frac{k+2}{2} \rceil \ge \lceil \frac{3+2}{2} \rceil = 3$, proving (a). To prove (b), suppose $b_1 + b_2 \le 2$ for a contradiction. It follows that $b_1 = b_2 = 1$ because $k \ge 3$, so $k = \sum_{j=1}^{q} b_j \le \sum_{j=1}^{q} 1 = q$, leading to $p \le 2$, which contradicts the fact $p \ge 3$, proving (b). Supposing $a_{p-1} + a_p \ge 3$ for a contradiction leads to (i) $a_{p-1} \ge 3$ or (ii) $a_{p-1} = 2$ and $a_p \in \{1, 2\}$. It follows that $\sum_{i=1}^{p} a_i \ge 3(p-1)$ or $\sum_{i=1}^{p} a_i \ge 2(p-1)+1$, so $k = \sum_{i=1}^{p} a_i \ge \min\{3(p-1), 2(p-1)+1\} = 2p-1 \ge 2\lceil \frac{k+2}{2} \rceil - 1 \ge k+1$, which is a contradiction, proving (c). □

On each iteration of the for loop (in Steps 9–22) of Algorithm 1, exactly one of i and j is increased by one. So, $i + j = 1 + 2 + (r-1) = r+2$ at the beginning of the r-th iteration and $i + j = k+1 = p+q-1$ after the $k-2$ iterations. In order to ensure $i \le p$ and $j \le q$ while the algorithm is running, we show that:

Claim 2 If the condition, $c_r \le d_r$ or $d_r = k-2$, of Step 13 is satisfied at the r-th iteration of the for loop, then $i < p$; $j < q$ otherwise.

Proof of claim. Note that $c_r = a_1 + \cdots + a_i$ for some $1 \le i \le p$ and $d_r = b_1 + \cdots + b_j - 2$ for some $2 \le j \le q$. It is obvious that $d_r = k-2$ if $j = q$. Also, the converse holds true, because if $b_q \ne 0$, then $b_1 + \cdots + b_{j'} - 2 < k-2$ for every $j' < q$; if $b_q = 0$, then $q = 2$ ($b_1 = k$ and $b_2 = 0$), hence j must be equal to q. That is, the condition $d_r = k-2$ is equivalent to $j = q$. If $d_r = k-2$ ($j = q$), then $i = r+2-j \le (k-2)+2-q = p-2$, as required. Supposing $d_r \ne k-2$ (i.e., $d_r < k-2$) leads to $j < q$, and also leads to $i < p-2$ if $c_r \le d_r$ because $a_1 + \cdots + a_{p-2} \ge k-2$ by Claim 1(c), completing the proof. □

To extract a DPC path from R^* in Step 10 of the algorithm, R^* must have at least one source and one sink as intermediate vertices. The numbers of sources and sinks in $V(R^*) \setminus \{u_1, u_n\}$ at the beginning of the r-th iteration of the for loop are $c_r - (r-1)$ and $d_r - (r-1)$, respectively. So, it remains to verify if $c_r \ge r$ and $d_r \ge r$, i.e., $\min\{c_r, d_r\} \ge r$ for all $r \in \{1, \ldots, k-2\}$. For $r = 1$, we have $c_1 = a_1 \ge 1$ and $d_1 = b_1 + b_2 - 2 \ge 1$ by Claim 1(b). Suppose, to the contrary, that there exists $r \in \{2, \ldots, k-2\}$ such that $\min\{c_r, d_r\} \le r-1$. Let $\hat{r}$ be the minimum such index, so that $\min\{c_{\hat{r}}, d_{\hat{r}}\} \le \hat{r}-1$ and $\min\{c_r, d_r\} \ge r$ for every $r \in \{1, \ldots, \hat{r}-1\}$. It follows that $\hat{r}-1 \le \min\{c_{\hat{r}-1}, d_{\hat{r}-1}\} \le \min\{c_{\hat{r}}, d_{\hat{r}}\} \le \hat{r}-1$, leading to $\min\{c_{\hat{r}-1}, d_{\hat{r}-1}\} = \min\{c_{\hat{r}}, d_{\hat{r}}\} = \hat{r}-1$.

Claim 3 (a) $c_{\hat{r}} \ge \hat{r}$. (b) $c_{\hat{r}-1} = d_{\hat{r}-1} = d_{\hat{r}} = \hat{r}-1$.

Proof of claim. Note that $c_{\hat{r}-1}, d_{\hat{r}-1}, c_{\hat{r}}, d_{\hat{r}} \geq \hat{r} - 1$. Supposing $c_{\hat{r}} \leq \hat{r} - 1$, for a contradiction, leads to that $\hat{r} - 1 \leq c_{\hat{r}-1} \leq c_{\hat{r}} \leq \hat{r} - 1$, meaning $c_{\hat{r}-1} = c_{\hat{r}} = \hat{r}-1$. Since $c_{\hat{r}-1} = \hat{r}-1 \leq d_{\hat{r}-1}$, Steps 14–16 of Algorithm 1 would be executed on the $(\hat{r} - 1)$-th iteration of the for loop, resulting in $c_{\hat{r}} = c_{\hat{r}-1} + a_i$ for some i. This leads to $a_i = 0$ ($a_i = \cdots = a_p = 0$), hence $c_{\hat{r}} = a_1 + \cdots + a_i = k$, which contradicts the fact $c_{\hat{r}} = \hat{r} - 1 \leq k - 3$, proving (a). The fact $c_{\hat{r}} \geq \hat{r}$ implies $d_{\hat{r}} \leq \hat{r} - 1$, meaning $d_{\hat{r}-1} = d_{\hat{r}} = \hat{r} - 1$ because $\hat{r} - 1 \leq d_{\hat{r}-1} \leq d_{\hat{r}} \leq \hat{r} - 1$. It remains to show $c_{\hat{r}-1} = \hat{r} - 1$. Suppose otherwise, i.e., $c_{\hat{r}-1} > \hat{r} - 1 = d_{\hat{r}-1}$, where $d_{\hat{r}-1} \leq k-3$ because $\hat{r} \in \{2, \ldots, k-2\}$. Thus, Steps 18–20 of Algorithm 1 would be executed on the $(\hat{r} - 1)$-th iteration of the for loop, resulting in $d_{\hat{r}} = d_{\hat{r}-1} + b_j$ for some j. This leads to $b_j = 0$ and $d_{\hat{r}} = k - 2$, which contradicts the fact $d_{\hat{r}} = \hat{r} - 1 \leq k - 3$. Hence, the claim is proved. □

Then, $c_{\hat{r}-1}$ and $d_{\hat{r}-1}$ can be represented as $c_{\hat{r}-1} = a_1 + \cdots + a_i = \hat{r} - 1$ and $d_{\hat{r}-1} = b_1 + \cdots + b_j - 2 = \hat{r} - 1$ for some i and j with $i \leq p$, $j \leq q$, and $i + j = \hat{r} + 1$. Moreover, $i < p - 2$ because $c_{\hat{r}-1} = \hat{r} - 1 \leq k - 3$ and $a_1 + \cdots + a_{p-2} \geq k - 2$ by Claim 1(c). The fact that the sequence $(a_1, \ldots, a_p)$ is nonincreasing leads to $a_1 + \cdots + a_i \geq i \cdot \frac{a_1+\cdots+a_{p-2}}{p-2} \geq i \cdot \frac{k-2}{p-2}$, so

$$i \leq \frac{p-2}{k-2}(\hat{r} - 1) = \frac{k-q}{k-2}(\hat{r} - 1). \tag{1}$$

Also, the fact that $(b_1, \ldots, b_q)$ is nonincreasing leads to $b_1 + \cdots + b_j \geq j \cdot \frac{k}{q}$, so

$$j \leq \frac{q}{k}(\hat{r} + 1). \tag{2}$$

Summing up the two inequalities (1) and (2) with $i + j = \hat{r} + 1$ results in

$$\begin{aligned}
& \hat{r} + 1 \leq \frac{k-q}{k-2}(\hat{r} - 1) + \frac{q}{k}(\hat{r} + 1) \\
\Leftrightarrow \quad & \left(\frac{k-q}{k-2} + \frac{q}{k} - 1\right) \cdot \hat{r} \geq \frac{k-q}{k-2} - \frac{q}{k} + 1 \\
\Leftrightarrow \quad & (k(k-q) + (k-2)q - k(k-2)) \cdot \hat{r} \geq k(k-q) - (k-2)q + k(k-2) \\
\Leftrightarrow \quad & 2(k-q) \cdot \hat{r} \geq 2(k-q)(k-1) \\
\Leftrightarrow \quad & \hat{r} \geq k - 1,
\end{aligned}$$

which contradicts the fact $\hat{r} \in \{2, \ldots, k-2\}$, thereby completing the entire proof. ■

What we have discussed so far can be summarized as follows:

Theorem 7 *Let G be an interval graph of order $n \geq 2k$ for $k \geq 2$. If* $\mathrm{sc}(G) \leq -k$*, then G is unpaired k-disjoint path coverable.*

Proof If G is a complete graph, then G is obviously unpaired k-disjoint path coverable; so, assume G is noncomplete. The proof will proceed by induction on k. The base step of $k = 2$ is due to Lemma 7. Let $k \geq 3$ for the inductive step. There exists a one-to-one $(k+2)$-DPC $\mathcal{P}$ joining a pair of left and right tips, u_1 and u_n, in G by Theorem 6. For disjoint terminal sets S and T of size k each in G, we will show that there exists an unpaired k-DPC joining them. In case where there is a path $P_i \in \mathcal{P}$ that passes through at least one source and one sink as intermediate vertices, or where at least one of u_1 and u_n are terminal vertices, our problem is reduced to a problem of building an unpaired $(k-1)$-DPC joining some terminal sets $S' \subset S$ and $T' \subset T$ in an induced subgraph H of G with $\mathrm{sc}(H) \le -(k-1)$ by Lemma 8. An unpaired $(k-1)$-DPC of H joining S' and T' exists by the induction hypothesis, so an unpaired k-DPC of G joining S and T exists. In the remaining case where $\{u_1, u_n\} \cap (S \cup T) = \emptyset$ and moreover, $V(P_i) \cap S = \emptyset$ or $V(P_i) \cap T = \emptyset$ for all $P_i \in \mathcal{P}$, Algorithm 1 builds a required unpaired k-DPC of G by Lemma 9. This completes the proof. ■

The bound $\mathrm{sc}(G) \le -k$ of Theorem 7 is best possible in a sense that there exists an interval graph G with $\mathrm{sc}(G) \le -k+1$ that is not unpaired k-disjoint path coverable for any fixed $k \ge 1$. Think of a *complete split graph* $G_{k+1,2k}$ whose vertex set can be partitioned into an independent set I of size $k+1$ and a clique C of size $2k$ such that every vertex in I is adjacent to every vertex in C. Clearly, the graph $G_{k+1,2k}$ is an interval graph whose scattering number is $(k+1) - 2k = -k+1$. However, the graph is not unpaired k-disjoint path coverable because there is no unpaired k-DPC joining S and T if $S \cup T = C$.

In addition, the sufficient condition "$\mathrm{sc}(G) \le -k$" of Theorem 7 becomes a necessary one for interval graphs that are $2k$-connected, as shown below.

Lemma 10 *If a $2k$-connected graph G is unpaired k-disjoint path coverable for $k \ge 2$, then $\mathrm{sc}(G) \le -k$.*

Proof Let G be a $2k$-connected graph (of order $n \ge 2k+1$) that is unpaired k-disjoint path coverable. If G is a complete graph, then $\mathrm{sc}(G) = 3 - n \le 3 - (2k+1) = -k + (2-k) \le -k$; so, we assume G is noncomplete hereafter. Suppose $\mathrm{sc}(G) \ge 1-k$ for a contradiction. Then, there exists a scattering set X with $|X| \ge 2k$ such that $c(G-X) - |X| = \mathrm{sc}(G) \ge 1-k$. Let $H_1, \ldots, H_p$ be the connected components of $G-X$, where $p = c(G-X) \ge 2$. Suppose S and T are disjoint terminal sets of size k each such that $S \cup T \subseteq X$, as illustrated in Fig. 4. Consider an unpaired k-DPC joining S and T, in which each path P_i passes through at most $|V(P_i) \cap X| - 1$ connected components. It follows that the DPC paths collectively pass through at most $\sum_{i=1}^{k}(|V(P_i) \cap X| - 1) = |X| - k$ connected components in total, leading to $p \le |X| - k$. This contradicts the hypothesis $p - |X| = \mathrm{sc}(G) \ge 1-k$, thereby completing the proof. ■

Theorem 8 *For $k \ge 2$, a $2k$-connected interval graph G is unpaired k-disjoint path coverable if and only if $\mathrm{sc}(G) \le -k$.*

Proof The proof is a direct consequence of Theorem 7 and Lemma 10. ■

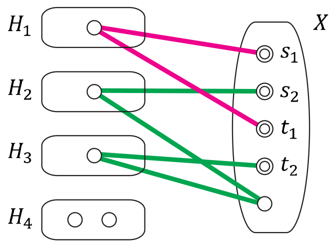


Fig. 4: Illustration of the proof of Lemma 10, where $k = 2$.

3.3 Algorithmic aspects

Interval graphs can be recognized in linear time [4,7,12]. The first linear-time algorithm devised by Booth and Lueker [4] is based on the PQ-tree data structure. If an interval graph is given, the algorithm also finds a linear ordering of the maximal cliques as well as an interval representation for the graph. Some other linear-time recognition algorithms in [7,12] are based on the lexicographic breadth-first-search (LexBFS). Also, the scattering number of an interval graph can be computed in linear time [5,18]; hence, the sufficient condition, $\mathrm{sc}(G) \leq -k$, of Theorem 7 can be checked in linear time. Given u_1 and u_n of an interval graph G and a positive integer k, Damaschke's algorithm [8] builds a one-to-one k-DPC of G joining u_1 and u_n in linear time (if exists); moreover, all paths in the DPC are monotone. The algorithm, in fact, can decide if $\mathrm{sc}(G) \leq -k$ or equivalently, if there exists a one-to-one $(k+2)$-DPC joining u_1 and u_n in G.

The proofs given in this paper suggests an algorithm for building an unpaired k-DPC joining terminal sets S and T of an interval graph G with $\mathrm{sc}(G) \leq -k$. A straightforward implementation leads to an $O(nk+m)$-time algorithm, where n and m respectively are the numbers of vertices and edges in G. This is because (i) a one-to-one $(k+2)$-DPC $\mathcal{P}$ joining u_1 and u_n, composed of all monotone paths, can be built in $O(n+m)$ time; (ii) all the h^i_js can be computed in $O(n)$ time, for example it suffices to let $h^i_1 = 1$ and $h^i_j = \max\{h^i_{j-1}$, the minimum point in $I_{v^i_j} \cap I_{v^i_{j+1}}\}$ for $j \geq 2$; (iii) a path $P_i \in \mathcal{P}$ and a subpath P'_j of a path $P_j \in \mathcal{P}$ other than P_i with $V(P'_j) \cap \{u_1, u_n\} \neq \emptyset$ can be combined into a monotone u_1–u_n path in $O(n)$ time.

4 Concluding remarks

We have proved that an interval graph G of order $n \geq 2k$ for $k \geq 2$ is unpaired k-disjoint path coverable if $\mathrm{sc}(G) \leq -k$. The proofs given for the main theorem are constructive, hence, can be used to design an efficient algorithm for building an unpaired k-DPC in an interval graph G with $\mathrm{sc}(G) \leq -k$. It is open to characterize interval graphs that are unpaired k-disjoint path coverable.

Acknowledgements This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (Grant No. 2018R1D1A1B07045566). This work was also supported by the Catholic University of Korea, Research Fund, 2020.

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