Algorithms for Finding Disjoint Path Covers in Unit Interval Graphs

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Abstract

A many-to-many *k-disjoint path cover* (k-DPC for short) of a graph G joining the pairwise disjoint vertex sets S and T, each of size k, is a collection of k vertex-disjoint paths between S and T, which altogether cover every vertex of G. This is classified as *paired*, if each vertex of S must be joined to a specific vertex of T, or *unpaired*, if there is no such constraint. In this paper, we develop a linear-time algorithm for the Unpaired DPC problem of finding an unpaired DPC joining S and T given in a unit interval graph, to which the One-to-One DPC and the One-to-Many DPC problems are reduced in linear time. Additionally, we show that the Paired k-DPC problem on a unit interval graph is polynomially solvable for any fixed k.

Keywords: Disjoint path; path cover; path partition; proper interval graph; graph algorithm.

1. Introduction

Let G be a simple undirected graph, whose vertex and edge sets are denoted by $V(G)$ and $E(G)$, respectively. A *path cover* of graph G is a set of paths that altogether cover every vertex of G. Of special interest is the *vertex-disjoint path cover*, or simply called *disjoint path cover*, which has the following additional constraint: every vertex must belong to one and only one path. The disjoint path cover problem finds applications in many areas, such as software testing, database design, and code optimization [2, 28]. In addition, the problem is concerned with applications where full utilization of network nodes is important [32].

The original disjoint path cover problem has no constraints on the positions of terminals or on the lengths of paths. The problem is to determine a disjoint path cover of a graph that uses the minimum number of paths. The minimum number is said to be the *path cover number* of the graph. The path cover (number) problem for a general graph is NP-complete [15], because the path cover number is equal to one if and only if the graph contains a hamiltonian path. Polynomial-time algorithms have been developed for some classes of graphs, for example, interval graphs [1], block graphs and bipartite permutation graphs [39], cographs [25], and distance-hereditary graphs [18].

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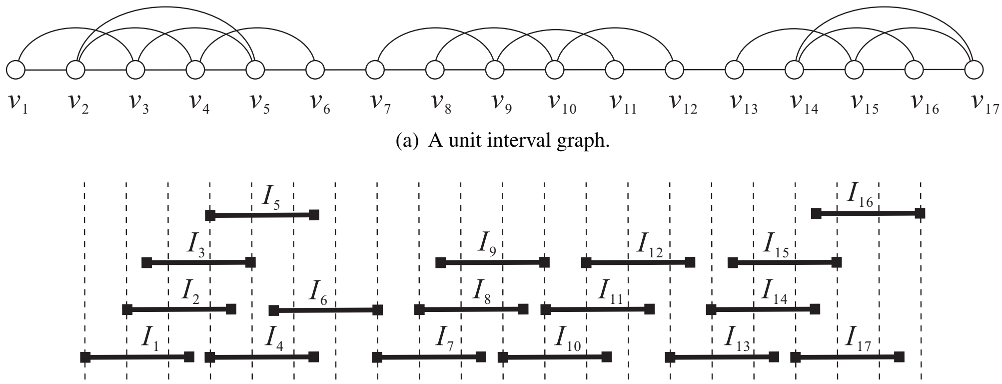


(a) A unit interval graph.

(b) An interval representation of the graph of (a).

Fig. 1: A unit interval graph and its interval representation.

In this paper, we are concerned with the disjoint path cover problem with prescribed sources and sinks, where each path should run from a *source* to a *sink*. The disjoint path cover made of k paths is called the *k-disjoint path cover* (*k-DPC* for short). Given two pairwise disjoint terminal sets, a source set $S = \{s_1, \ldots, s_k\}$ and a sink set $T = \{t_1, \ldots, t_k\}$, of graph G, the *many-to-many k-DPC* is a disjoint path cover, each of whose paths joins a pair of source and sink. The disjoint path cover is *paired* if every source s_i must be matched with a specific sink t_i. On the other hand, it is *unpaired* if any permutation of sinks may be mapped bijectively to sources. There are two simpler variants: the *one-to-many k-DPC* for $S = \{s\}$ and $T = \{t_1, \ldots, t_k\}$, whose paths join the common source to k distinct sinks; and the *one-to-one k-DPC* for $S = \{s\}$ and $T = \{t\}$, whose paths always start from the common source and end up in the common sink. The disjoint path covers of this type have been studied for graphs, such as hypercubes [7, 8, 13, 17, 20], recursive circulants [21, 22], hypercube-like graphs [19, 23, 32, 33], k-ary n-cubes [37, 40], cubes of connected graphs [29, 30], and grid graphs [31].

Some other types of the disjoint path cover problem can also be found in the literature. Given a set of k sources, $S = \{s_1, \ldots, s_k\}$, in graph G, which is associated with k positive integers, $l_1, \ldots, l_k$, such that $\sum_{i=1}^{k} l_i = |V(G)|$, a *prescribed-source-and-length k-DPC* of G is a disjoint path cover composed of k paths, each of whose paths starts at source s_i and contains l_i vertices for $i \in \{1, \ldots, k\}$. For studies on this type of DPCs, refer to [10, 27]. Given a graph G and a subset $\mathcal{T}$ of k vertices of G, a *k-fixed endpoint path cover* of G with respect to $\mathcal{T}$ is a set of vertex-disjoint paths that covers the vertices of G, such that the k vertices of $\mathcal{T}$ are all terminals of the paths in the DPC. For details, refer to [2, 3].

An *interval graph* is the intersection graph of family $\mathcal{I}$ of intervals on the real line, where two vertices are connected with an edge if and only if their corresponding intervals intersect. The family $\mathcal{I}$ is usually called an *interval representation* for the graph. A *unit interval graph* is an interval graph with an interval representation in which all the intervals have unit length. Refer to Fig. 1 for an example of a unit interval graph and its interval representation. In a similar way, a *proper interval graph* is an interval graph with an interval representation in which no interval properly contains another. In 1969, Roberts [34] proved that the classes of unit interval graphs and proper interval graphs coincide.

An ordering, $(v_1, \ldots, v_n)$, of the vertices of a graph of order n is *consecutive* if the vertices contained in a maximal clique are consecutive. A unit interval graph always admits a consecutive ordering because it is evident that the sequence of unit intervals sorted by their left endpoints corresponds to a consecutive order. See Fig. 1 again, where as well as $(v_1, \ldots, v_{17})$, the ordering $(v_1, \ldots, v_{15}, v_{17}, v_{16})$ with v_{16} and v_{17} being switched is also consecutive. A unit interval representation and a consecutive ordering of a unit interval graph can be computed in time linear to the size of the graph [11, 12]. The class of the unit interval graphs is known to admit polynomial solutions for many problems that are NP-complete for general graphs, such as vertex coloring, clique, independent set, etc. [16].

Given a source set $S = \{s_1, \ldots, s_k\}$ and a sink set $T = \{t_1, \ldots, t_k\}$ in a unit interval graph G of order n, we will develop an $O(n)$-time algorithm for determining the existence of an unpaired k-DPC joining S and T and producing an unpaired k-DPC, if it exists, provided that a consecutive ordering of the vertices of G and a unit interval representation for G are both available. We then provide a reduction of the General-Demand k-DPC [24] problem into the Unpaired k-DPC problem in $O(n + k)$ time, so that the One-to-One DPC and the One-to-Many DPC problems are solvable both in $O(n)$ time, provided that the two structures required by the unpaired DPC algorithm are available. Finally, we present an algorithm for the Paired k-DPC problem on a unit interval graph, which runs in time polynomial in n for a fixed k (where k is regarded as a constant).

2. Preliminaries

We begin with a consecutive ordering of the vertices of a unit interval graph G, from which many interesting properties have been deduced. Hereafter, we denote by n the order of G, i.e., $n = |V(G)|$.

Theorem 1 (Roberts [34, 35]). *For a simple graph G, the following statements are equivalent:*
(a) G is a unit interval graph.
(b) G is a proper interval graph.
(c) There is a consecutive ordering, $(v_1, \ldots, v_n)$, of the vertices of G.

Theorem 2 (Looges and Olariu [26]). *A simple graph G of order n is a unit interval graph if and only if there is an ordering, $(v_1, \ldots, v_n)$, of the vertices of G, such that for all $p < q < r$, $(v_p, v_r) \in E(G)$ implies $(v_p, v_q), (v_q, v_r) \in E(G)$.*

Lemma 1. *(See [9, 11], etc.) For an ordering $(v_1, \ldots, v_n)$ of the vertices of a unit interval graph G, the followings are equivalent:*
(a) The ordering is consecutive, i.e., the vertices of a maximal clique are consecutive.
(b) For all $p < q < r$, $(v_p, v_r) \in E(G)$ implies $(v_p, v_q), (v_q, v_r) \in E(G)$.
(c) The vertices in the closed neighbor $N_G(v_i)$ of a vertex v_i are consecutive.

Here, the *closed neighborhood* $N_G[v_i]$ of a vertex v_i refers to the set of vertices adjacent to v_i along with v_i itself, i.e., $N_G[v_i] = \{v_i\} \cup \{v_j \in V(G) : (v_i, v_j) \in E(G)\}$. A graph is k-*connected* if there is no set of fewer than k vertices whose removal results in a trivial graph or a disconnected graph. A *hamiltonian path* is a path that passes through every vertex exactly once, and a *hamiltonian cycle* is a cycle that passes through every vertex exactly once.

Theorem 3 (Chen, Chang, and Chang [9]). *For any positive integer k and any proper interval graph G of $n \geq k+1$ vertices with a consecutive ordering $(v_1, \ldots, v_n)$, G is k-connected if and only if $(v_i, v_j) \in E(G)$ whenever $1 \leq |i-j| \leq k$.*

Theorem 4 (Bertossi [5]). *A unit interval graph G of two or more vertices has a hamiltonian path if and only if G is connected.*

Note that every consecutive ordering $(v_1, \ldots, v_n)$ of a proper interval graph G forms a hamiltonian path if G is connected (or equivalently, 1-connected).

Theorem 5 (Chen, Chang, and Chang [9]). *A unit interval graph G of three or more vertices has a hamiltonian cycle if and only if G is 2-connected.*

Remark 1. If $(v_1, \ldots, v_n)$ is a consecutive ordering of the vertices of a 2-connected unit interval graph G (of order $n \geq 3$), then $(v_1, v_3, \ldots, v_{n-1}, v_n, v_{n-2}, \ldots, v_2)$ will be a hamiltonian cycle of G for even n; $(v_1, v_3, \ldots, v_n, v_{n-1}, v_{n-3}, \ldots, v_2)$ will be a hamiltonian cycle of G for odd n. The hamiltonian cycle passes through both edges (v_1, v_2) and (v_{n-1}, v_n), which implies that graph G has a hamiltonian path joining v_{n-1} and v_n as well as a hamiltonian path joining v_1 and v_2.

All graph-theoretical terms not defined here can be found in [6]. A path in a graph is represented as a sequence of vertices, in which any two consecutive vertices are adjacent. An *s–t path* refers to a path joining vertices s and t, and an *s-path* refers to a path that starts at vertex s. We denote by $V(P)$ and $E(P)$, respectively, the set of vertices and the set of edges of path P.

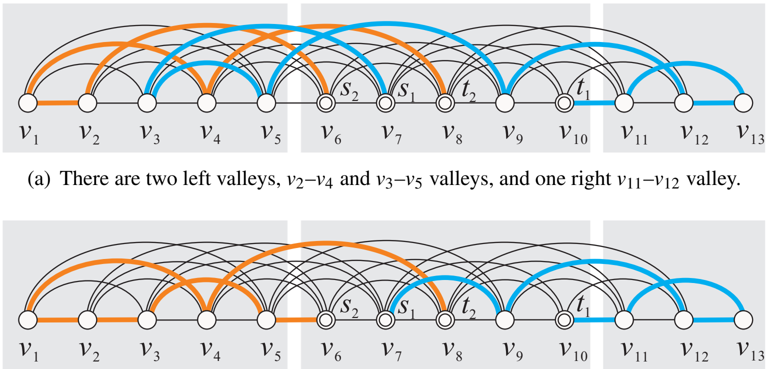
3. Properties of Many-to-Many DPCs

Let $(v_1, \ldots, v_n)$ be a consecutive ordering of the vertices of a unit interval graph G. Vertex v_i is said to be to the left of v_j (or v_j is to the right of v_i) if $i < j$. Let $V_{i,j} = \{v_i, \ldots, v_j\}$ if $i \leq j$ and let $V_{i,j} = \emptyset$ if $i > j$, so that $V_{1,n} = V(G)$. We are given a source set, $S = \{s_1, ..., s_k\}$, and a sink set, $T = \{t_1, ..., t_k\}$, in G such that $S \cap T = \emptyset$. Let $p = \min\{i : v_i \in S \cup T\}$ and $q = \max\{j : v_j \in S \cup T\}$, i.e., v_p is the leftmost terminal and v_q is the rightmost terminal.

For path P that joins a source in S and a sink in T, a *left valley* is a maximal subpath of P, whose vertices are all contained in $V_{1,p-1}$. If we delete all vertices in $V_{p,n}$ from P, then there remain zero or more left valleys that are pairwise disjoint. (See Fig. 2(a).) For a v_a–v_b subpath that is a left valley, called a *left v_a–v_b valley*, there are two distinct vertices, $v_{a'}, v_{b'} \in V_{p,n}$, such that $(v_a, v_{a'}), (v_b, v_{b'}) \in E(P)$. The vertex $v_{a'}$ is called the *mate* of v_a, and $v_{b'}$ is called the *mate* of v_b. (If the v_a–v_b valley is a one-vertex path, i.e., $v_a = v_b$, it does not matter which vertex is the mate of v_a. It is possible that v_a and v_b, respectively, are matched with $v_{b'}$ and $v_{a'}$.) In the same way, a *right valley* is a maximal subpath of P, whose vertices are all contained in $V_{q+1,n}$. The mates, $v_{c'}, v_{d'} \in V_{1,q}$, of the end-vertices of a right v_c–v_d valley is defined similarly.

Lemma 2. *If G has an unpaired (resp. paired) k-DPC joining S and T, then G has an unpaired (resp. paired) k-DPC in which at most one path has a left valley, at most one path has a right valley and all the remaining paths are valley-free.*

Proof. We claim that for every v_a–v_b valley of a path P in the k-DPC, the mate $v_{a'}$ of v_a and the mate $v_{b'}$ of v_b are adjacent. Suppose the valley is a left one first. From the fact that $(v_{a'}, v_a) \in E(G)$ and $a < p \leq a'$, we have $(v_{a'}, v_i) \in E(G)$ for every $p \leq i < a'$ by Lemma 1. Also, we have



(a) There are two left valleys, v_2–v_4 and v_3–v_5 valleys, and one right v_{11}–v_{12} valley.

(b) The two left valleys are merged into one, the left v_4–v_5 valley.

Fig. 2: Illustration of the valley merging process in the proof of Lemma 2, where $p = 6$ and $q = 10$.

$(v_{b'}, v_j) \in E(G)$ for every $p \le j < b'$. It follows that $(v_{a'}, v_{b'}) \in E(G)$ whether $a' < b'$ or $b' < a'$. Symmetrically, we can conclude $(v_{a'}, v_{b'}) \in E(G)$ for a right v_a–v_b valley, proving the claim. This implies that if we remove the v_a–v_b valley from the path P and join $v_{a'}$ and $v_{b'}$ via an edge, we have a new path that joins the same source and sink as P.

Suppose there are two left valleys, v_a–v_b valley and v_c–v_d valley, of some path(s) in the k-DPC. We will show that the two left valleys can be merged into one, resulting in a new k-DPC having one less left valley. (See Fig. 2 for an illustrative example.) We assume w.l.o.g. that $a \le b$, $c \le d$, and $a < c$. Let the mates of v_a, v_b, v_c and v_d be $v_{a'}$, $v_{b'}$, $v_{c'}$ and $v_{d'}$, respectively. Then, $(v_a, v_c), (v_d, v_{a'}) \in E(G)$ because the subgraph induced by $V_{a,a'}$ is a clique by the property of a consecutive ordering. It suffices to (i) remove the v_c–v_d valley of a path and join $v_{c'}$ and $v_{d'}$ via an edge, (ii) insert edge (v_a, v_c) to merge the two valleys into one, a v_b–v_d valley, and (iii) replace the edge $(v_a, v_{a'})$ of the path that contains the old v_a–v_b valley with edge $(v_d, v_{a'})$. Repeating this process, we conclude that there exists a k-DPC having at most one left valley. Similarly, we can repeatedly merge two right valleys into one, resulting in a k-DPC having at most one right valley. This completes the proof. □

From Lemma 2, the left valley of an unpaired (or paired) k-DPC, if it is unique, forms a hamiltonian path of the subgraph $G[V_{1,p-1}]$ induced by $V_{1,p-1}$. Also, the right valley, if any, forms a hamiltonian path in the induced subgraph $G[V_{q+1,n}]$.

Lemma 3. *Let G have an unpaired (resp. paired) k-DPC joining S and T that has at most one left valley and at most one right valley. Then, the following two hold true:*
(a) If $p \ge 2$, there exists an unpaired (resp. paired) k-DPC that has a left v_a–v_b valley, where $a = p-2$ and $b = p-1$ for $p \ge 3$, and $a = b = 1$ for $p = 2$. Moreover, the mate of v_a is to the left of the mate of v_b.
(b) If $q \le n-1$, there exists an unpaired (resp. paired) k-DPC that has a right v_c–v_d valley, where $c = q+1$ and $d = q+2$ for $q \le n-2$, and $c = d = n$ for $q = n-1$. Moreover, the mate of v_c is to the left of the mate of v_d.

Proof. If $p \in \{2, 3\}$, we have only one choice for the left valley: the one-vertex path (v_1) for $p = 2$ and the v_1–v_2 path for $p = 3$. Suppose $p \ge 4$. The subgraph $G[V_{1,p-1}]$ is hamiltonian because the left valley, say v_i–v_j valley, becomes a hamiltonian cycle of $G[V_{1,p-1}]$ if v_i and v_j are joined by an edge $(v_i, v_j) \in E(G)$. It follows, from Theorem 5 and Remark 1, that there exists a hamiltonian v_{p-2}–v_{p-1} path in $G[V_{1,p-1}]$. In the path of the DPC that contains the left v_i–v_j valley, it is possible to switch the left valley with the hamiltonian v_{p-2}–v_{p-1} path (i.e., a new v_{p-2}–v_{p-1} valley) through $\{(v_{p-2}, v_{i'}), (v_{p-1}, v_{j'})\}$ or $\{(v_{p-2}, v_{j'}), (v_{p-1}, v_{i'})\}$ for the mates $v_{i'}$ and $v_{j'}$ of v_i and v_j, respectively. This is because (i) if $i < j$ (where $i \leq p-2$ and $j \leq p-1$), then $(v_{p-2}, v_{i'}), (v_{p-1}, v_{j'}) \in E(G)$, and (ii) if $j < i$ (where $j \leq p-2$ and $i \leq p-1$), then $(v_{p-2}, v_{j'}), (v_{p-1}, v_{i'}) \in E(G)$. Finally, suppose the mate of v_a is to the right of the mate of v_b, where $a \leq b$. It suffices to swap their mates. The proof of (a) is complete. The proof of statement (b) is done similarly. □

Remark 2. The left valley of Lemma 3(a) for $p \geq 4$ can be constructed easily from Remark 1: $(v_{p-1}, v_{p-3}, \ldots, v_2, v_1, v_3, \ldots, v_{p-2})$ for odd p and $(v_{p-1}, v_{p-3}, \ldots, v_1, v_2, v_4, \ldots, v_{p-2})$ for even p. The same follows for the right valley of Lemma 3(b) for $q \leq n-3$.

It follows immediately from Lemma 3 and Theorem 5 that:

Lemma 4. *(a) For $p \geq 4$, G has an unpaired (resp. paired) k-DPC if and only if $G[V_{p-2,n}]$ has an unpaired (resp. paired) k-DPC and $G[V_{1,p-1}]$ is 2-connected.*
(b) For $q \leq n-3$, G has an unpaired (resp. paired) k-DPC if and only if $G[V_{1,q+2}]$ has an unpaired (resp. paired) k-DPC and $G[V_{q+1,n}]$ is 2-connected.

From the discussions of valleys in Lemmas 2 and 3 and from the property of a consecutive ordering, we also have that:

Lemma 5. *(a) For every $1 \leq i < p$, the subgraph $G[V_{i,n}]$ has an unpaired (resp. paired) k-DPC only if $G[V_{i+1,n}]$ has an unpaired (resp. paired) k-DPC.*
(b) For every $q < j \leq n$, the subgraph $G[V_{1,j}]$ has an unpaired (resp. paired) k-DPC only if $G[V_{1,j-1}]$ has an unpaired (resp. paired) k-DPC.

4. Algorithm for the Unpaired DPC Problem

If we restrict our attention to the unpaired many-to-many DPCs of a unit interval graph G, we can discover an additional property on valleys. The left valley, if any, can be associated with the leftmost terminal v_p, and the right valley, if any, can also be associated with the rightmost terminal v_q, as shown below.

Lemma 6. *Let the graph G have an unpaired k-DPC joining S and T, whose valleys are in shape of Lemma 3. Then, there exists an unpaired k-DPC such that*

- *the mate of v_a is v_p if there is a left v_a–v_b valley with $a \leq b$, and*
- *the mate of v_d is v_q if there is a right v_c–v_d valley with $c \leq d$.*

Proof. Suppose that a path P in the k-DPC contains a left v_a–v_b valley with $a \leq b$, such that the mate of v_a is not v_p. The mate of v_b is not v_p, neither. Let $v_{a'}$ and $v_{b'}$, respectively, be the mates of v_a and v_b, where $p < a' < b'$ by Lemma 3. Then, path P can be decomposed into three subpaths: the x–$v_{a'}$ subpath, the v_a–v_b valley and the $v_{b'}$–y subpath for some $x, y \in S \cup T$. Notice that $(v_p, v_a), (v_p, v_b) \in E(G)$ and $(v_b, v_{a'}), (v_b, v_{b'}) \in E(G)$. There are two cases depending on whether $v_p \in \{x, y\}$ or not.

Suppose $v_p \notin \{x, y\}$ first. There exists a path P' (other than P) in the k-DPC, one of whose end-vertices is v_p. Let $v_{p'}$ be the vertex next to v_p on P', i.e., $P' = (v_p, v_{p'}, P_v)$ for some subpath P_v, possibly empty. We decompose path P' into two subpaths: one-vertex path (v_p) and $(v_{p'}, P_v)$. The five subpaths of P and P', in total, will be merged into two paths so as to obtain a new unpaired k-DPC. We have $a', b', p' > p$ and moreover, the subgraph induced by $\{v_{a'}, v_{b'}, v_{p'}\}$ forms a clique of three vertices by Lemma 1. For the simplicity of our discussion, we assume $v_p, x \in S$ and $y \in T$. (The other cases can also be dealt with in the same manner.) If we concatenate the x–$v_{a'}$ subpath and $(v_{p'}, P_v)$ into one, and then concatenate (v_p), the v_a–v_b valley and the $v_{b'}$–y subpath into another, then we can obtain two paths, each of which runs from a source to a sink. If we let the other $k-2$ paths in the k-DPC remain unchanged, we have transformed the original k-DPC successfully into a new unpaired k-DPC that satisfies the first condition of the lemma.

Suppose $v_p \in \{x, y\}$ now. Let $v_{p'}$ be the vertex next to v_p on P. As before, we have $a', b', p' > p$ and moreover, the subgraph induced by $\{v_{a'}, v_{b'}, v_{p'}\}$ forms a clique. Note that the induced graph does not necessarily have three vertices because possibly, $p' = a'$ or $p' = b'$. Assume $x = v_p$. (The other case $y = v_p$ can proved in the same way.) Then, $p' \neq b'$ and $(v_{p'}, v_{b'}) \in E(G)$. We decompose the x–$v_{a'}$ subpath once more into two subpaths: one-vertex path (v_p) and the $v_{p'}$–$v_{a'}$ subpath. If we concatenate the four subpaths (v_p), the v_a–v_b valley, the $v_{a'}$–$v_{p'}$ subpath and $v_{b'}$–y subpath in sequence, then we have another x–y path. If we let the other $k-1$ paths in the k-DPC remain unchanged, we obtain a new unpaired k-DPC that satisfies the first condition of the lemma. In a symmetric way, we can transform the k-DPC that satisfies the first condition into a desired k-DPC, which satisfies the second condition as well as the first condition of the lemma. This completes the proof. $\square$

From the discussions of valleys of an unpaired DPC so far, our problem is reduced to the construction of an unpaired DPC in a unit interval graph whose leftmost and rightmost vertices are both terminals, as summarized below. Note that the class of unit interval graphs is *hereditary*, i.e., every induced subgraph of a graph in the class is contained in the same class.

Lemma 7. *Let $p \geq 2$ or $q \leq n-1$. Then, graph G has an unpaired k-DPC joining S and T if and only if*

- *if $p \geq 2$, there exists a hamiltonian v_p–v_{p-1} path in the subgraph $G[V_{1,p}]$;*
- *if $q \leq n-1$, there exists a hamiltonian v_q–v_{q+1} path in the subgraph $G[V_{q,n}]$; and*
- *the induced subgraph G' has an unpaired k-DPC joining S' and T', where*

$$G' = \begin{cases} G[\{v_{p-1}\} \cup V_{p+1,q-1} \cup \{v_{q+1}\}] & \text{if } p \geq 2 \text{ and } q \leq n-1, \\ G[\{v_{p-1}\} \cup V_{p+1,n}] & \text{if } p \geq 2 \text{ and } q = n, \\ G[V_{1,q-1} \cup \{v_{q+1}\}] & \text{if } p = 1 \text{ and } q \leq n-1; \end{cases}$$

$$S' = \begin{cases} (S \setminus \{v_p, v_q\}) \cup \{v_{p-1}, v_{q+1}\} & \text{if } (p \geq 2 \wedge v_p \in S) \text{ and } (q \leq n-1 \wedge v_q \in S), \\ (S \setminus v_p) \cup \{v_{p-1}\} & \text{if } (p \geq 2 \wedge v_p \in S) \text{ and } \neg(q \leq n-1 \wedge v_q \in S), \\ (S \setminus v_q) \cup \{v_{q+1}\} & \text{if } \neg(p \geq 2 \wedge v_p \in S) \text{ and } (q \leq n-1 \wedge v_q \in S), \\ S & \text{if } \neg(p \geq 2 \wedge v_p \in S) \text{ and } \neg(q \leq n-1 \wedge v_q \in S); \end{cases}$$

$$T' = \begin{cases} (T \setminus \{v_p, v_q\}) \cup \{v_{p-1}, v_{q+1}\} & \text{if } (p \geq 2 \wedge v_p \in T) \text{ and } (q \leq n-1 \wedge v_q \in T), \\ (T \setminus v_p) \cup \{v_{p-1}\} & \text{if } (p \geq 2 \wedge v_p \in T) \text{ and } \neg(q \leq n-1 \wedge v_q \in T), \\ (T \setminus v_q) \cup \{v_{q+1}\} & \text{if } \neg(p \geq 2 \wedge v_p \in T) \text{ and } (q \leq n-1 \wedge v_q \in T), \\ T & \text{if } \neg(p \geq 2 \wedge v_p \in T) \text{ and } \neg(q \leq n-1 \wedge v_q \in T). \end{cases}$$

Remark 3. For $p \geq 2$, the subgraph $G[V_{1,p}]$ has a hamiltonian v_p–v_{p-1} path if and only if

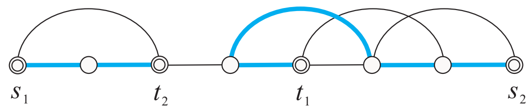


Fig. 3: There exists an unpaired 2-DPC composed of s_1–t_2 path and s_2–t_1 path, but there exists no unpaired 2-DPC whose paths are both monotone.

- if $p = 2$, the subgraph $G[V_{1,p}]$ is connected; and
- if $p \geq 3$, the subgraph $G[V_{1,p}]$ is 2-connected.

For $q \leq n-1$, the subgraph $G[V_{q,n}]$ has a hamiltonian v_q–v_{q+1} path if and only if

- if $q = n-1$, the subgraph $G[V_{q,n}]$ is connected; and
- if $q \leq n-2$, the subgraph $G[V_{q,n}]$ is 2-connected.

Now, we consider the unpaired k-DPC problem in a unit interval graph whose leftmost and rightmost vertices are both terminals, i.e., $v_1, v_n \in S \cup T$. One might expect at a glance that there always exists an unpaired k-DPC in which every path is monotone, where a path $(v_{i_1}, \ldots, v_{i_l})$ is said to be *monotone* if $i_1 < \cdots < i_l$ or $i_1 > \cdots > i_l$. A moment's reflection will convince us that it cannot be so, as shown in Fig. 3. Let us define h_m, $0 \leq m \leq n$, as the number of sources in $V_{1,m}$ minus the number of sinks in $V_{1,m}$, i.e.,

$$h_m := |S \cap V_{1,m}| - |T \cap V_{1,m}|,$$

where $h_0 = h_n = 0$. Similar to Fig. 3, we can construct an instance that admits an unpaired DPC but does not admit an unpaired DPC whose paths are all monotone, if there exists $j < j'$ such that $h_i = 0$ for all $j \leq i \leq j'$.

There are some instances that allow unpaired DPCs whose paths are all monotone. One of them will be the case when $h_m \geq 1$ for all $1 \leq m < n$, which will be clear in Lemma 8. We will devise an algorithm for the special case when $h_m \geq 1$ for all $1 \leq m < n$ ($v_1 \in S$ and $v_n \in T$), and then conquer our Unpaired DPC problem by dividing the input graph into subgraphs of the special case or their slight extensions. Let $X_m \subseteq V_{1,m}$ be the set of h_m non-sink vertices (sources or non-terminals) whose indices are as large as possible. If we pick up h_m non-sink vertices from v_m to v_1 in a decreasing order of their indices, then we have the very set X_m. Let $X_m = \{x_1, \ldots, x_{h_m}\}$ where $x_i = v_{a_i}$ for $1 \leq i \leq h_m$ such that $a_1 < \cdots < a_{h_m}$. Similarly, let $Y_{m+1} \subseteq V_{m+1,n}$ be the set of h_m non-source vertices (sinks or non-terminals) whose indices are as small as possible. Let $Y_{m+1} = \{y_1, \ldots, y_{h_m}\}$ where $y_i = v_{b_i}$ for $1 \leq i \leq h_m$ such that $b_1 < \cdots < b_{h_m}$.

Lemma 8. *Let $h_m \geq 1$ for all $1 \leq m < n$. Then, graph G has an unpaired k-DPC joining S and T if and only if $Z_m \subseteq E(G)$ for every $1 \leq m < n$, where*

$$Z_m := \{(x_1, y_1), \ldots, (x_{h_m}, y_{h_m})\}.$$

Proof. To prove necessity, suppose that G has an unpaired k-DPC. For each $1 \leq m < n$, there exist at least h_m paths, say $P_1, \ldots, P_{h_m}$, in the DPC whose sources are contained in $V_{1,m}$ and whose sinks are contained in $V_{m+1,n}$. If we traverse P_i, $1 \leq i \leq h_m$, starting at its source, we encounter a vertex, z_i, in $V_{1,m}$ and next, a vertex, w_i, in $V_{m+1,n}$, such that $(z_i, w_i) \in E(P_i)$. Consider the set of h_m edges, $Z = \{(z_i, w_i) : 1 \leq i \leq h_m\}$, where z_i's are all non-sinks and w_i's are all non-sources.

If there are edges (z, w) and (z', w') in Z, such that z is to the left of z' and w is to the right of w', then we replace the two edges with (z, w') and (z', w) in $E(G)$. By repeating this process until no such pair exists, we get an edge set $\{(z'_1, w'_1), \ldots, (z'_{h_m}, w'_{h_m})\}$, where z'_1 is to the left of z'_2, z'_2 is to the left of z'_3, and so on, and w'_1 is to the left of w'_2, w'_2 is to the left of w'_3, and so on. Since $(z'_i, w'_i) \in E(G)$ implies $(x_i, y_i) \in E(G)$ for each i, we have $\{(x_1, y_1), \ldots, (x_{h_m}, y_{h_m})\} \subseteq E(G)$. Note that $z'_i = x_i$ or z'_i is to the left of x_i, and $w'_i = y_i$ or w'_i is to the right of y_i.

Sufficiency will be proved by induction on n. For the base case of $n = 2$ (where $v_1 = s_1$ and $v_2 = t_1$), we have $Z_1 = \{(v_1, v_2)\} \subseteq E(G)$, which can be considered as a desired DPC. Suppose $n \geq 3$. Let us denote by v_{l^*} the leftmost non-source vertex of G, i.e., $v_{l^*} \notin S$ and $\{v_1, \ldots, v_{l^*-1}\} \subseteq S$. There are two cases depending on whether v_{l^*} is a non-terminal or a sink.

Case 1: v_{l^*} is a non-terminal, i.e., $v_{l^*} \notin S \cup T$. We claim that the induced subgraph $G[V(G) \setminus v_1]$ has an unpaired k-DPC joining S' and T, where $S' = (S \setminus v_1) \cup \{v_{l^*}\}$ and the vertex v_{l^*} is now a *virtual* source. Similarly, as we define h_m and Z_m for G, we can define h'_m and Z'_m for the induced subgraph where $1 \leq m \leq n$. (That is, $h'_m = |S' \cap V_{2,m}| - |T \cap V_{2,m}|$, and Z'_m is a set of h'_m edges joining h'_m non-sink vertices in $V_{2,m}$ whose indices are as large as possible and h'_m non-source vertices in $V_{m+1,n}$ whose indices are as small as possible.) Then, we have

$$h'_m = \begin{cases} h_m - 1 & \text{if } 1 \leq m < l^*, \\ h_m & \text{if } l^* \leq m \leq n; \end{cases} \quad \text{and} \quad Z'_m = \begin{cases} Z_m \setminus (v_1, v_{l^*}) & \text{if } 2 \leq m < l^*, \\ Z_m & \text{if } l^* \leq m < n. \end{cases}$$

Furthermore, $h'_m \geq 1$ for all $2 \leq m < n$. Thus, there exists an unpaired k-DPC joining S' and T in the induced subgraph $G[V(G) \setminus v_1]$ by the induction hypothesis, thereby proving the claim. An unpaired k-DPC of G joining S and T is obtained from the k-DPC of the induced subgraph by simply replacing the v_{l^*}-path with the concatenation of one-vertex path (v_1) and the v_{l^*}-path.

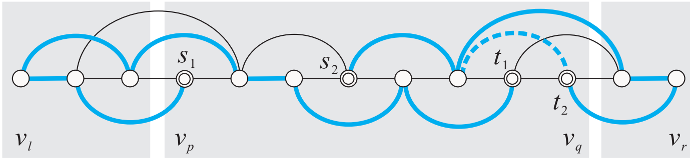
Case 2: v_{l^*} is a sink, i.e., $v_{l^*} \in T$. Since $h_2 \geq 1$, the vertex v_2 in this case should be a source, so that $k \geq 2$ and $l^* \geq 3$. Consider the induced subgraph $G[V(G) \setminus \{v_1, v_{l^*}\}]$. Similar to Case 1, we can see, by the induction hypothesis, that the induced subgraph has an unpaired $(k-1)$-DPC joining S' and T', where $S' = S \setminus v_1$ and $T' = T \setminus v_{l^*}$. Note that h'_m and Z'_m for this induced subgraph are as follows:

$$h'_m = \begin{cases} h_m - 1 & \text{if } 1 \leq m < l^*, \\ h_m & \text{if } l^* < m \leq n; \end{cases} \quad \text{and} \quad Z'_m = \begin{cases} Z_m \setminus (v_1, v_{l^*}) & \text{if } 2 \leq m < l^*, \\ Z_m & \text{if } l^* < m < n. \end{cases}$$

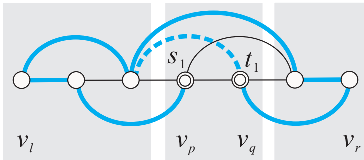
Adding the path (v_1, v_{l^*}) to the $(k-1)$-DPC results in a desired k-DPC of G joining S and T. This completes the proof. □

According to the proof of Lemma 8, we can design a simple $O(n)$-time algorithm for finding an unpaired DPC for the special case where $h_m \geq 1$ for all $1 \leq m < n$. It is worth noting that at all times, from the current leftmost vertex, which is a (virtual) source, we advance to the leftmost non-source vertex. This implies that every path in the DPC is monotone. By virtue of Lemma 7, moreover, the algorithm can be adapted to a slightly more general case where p is not necessarily 1 and q is not necessarily n, i.e., the case when $h_m \geq 1$ for every $p \leq m < q$ (and $h_m = 0$ for $m < p$ or for $m \geq q$).

The algorithm can be summarized as a procedure below, named Find-UDPC-A, that computes an unpaired DPC joining S' and T' in the induced subgraph $G[V_{l,r}]$, in which v_p and v_q, respectively, are the leftmost and rightmost terminals. The procedure is illustrated in Fig. 4. We assume that array $L[1..n]$ is precomputed, where $L[i]$ = nonterminal, source and sink, respectively, if v_i is a non-terminal, source and sink.



(a)



(b)

Fig. 4: The unpaired 2-DPC and 1-DPC produced by procedure Find-UDPC-A.

procedure Find-UDPC-A($S', T', l, r, p, q, \mathcal{I}$) ▷ Assume $h_m \geq 1 \; \forall \; p \leq m < q$.
  if $p = l$ **then** ▷ There exists no left valley.
    Initialize the v_p-path to be a one-vertex path (v_p);
    $i \leftarrow p$; ▷ v_i is the leftmost (virtual) source.
  else ▷ There exists a left valley.
    Initialize the v_p-path to be the extended valley returned by Find-Left-Valley(l, p);
    $i \leftarrow p - 1$; ▷ v_i is the leftmost (virtual) source.
    $L[p] \leftarrow$ discarded; ▷ Hereafter, v_p will be ignored.
  end if
  for all $s \in S'$ other than v_p **do**
    Initialize the s-path to be a one-vertex path (s);
  end for
  $l^* \leftarrow i$;
  repeat ▷ Find the leftmost non-source vertex, v_{l^*}, to the right of v_i.
    $l^* \leftarrow l^* + 1$;
  until $L[l^*] \in$ {nonterminal, sink}
  while $i < q$ **do**
    Append v_{l^*} to the path ending at v_i;
    if $L[l^*]$ = nonterminal **then**
      $L[l^*] \leftarrow$ source; ▷ The vertex v_{l^*} becomes a new *virtual* source.
    else ▷ The vertex v_{l^*} is a sink.
      $L[l^*] \leftarrow$ discarded; ▷ Hereafter, v_{l^*} will be ignored.
    end if
    repeat ▷ Find the next (virtual) source, v_i.
      $i \leftarrow i + 1$;
    until $L[i]$ = source or $i \geq q$
    repeat ▷ Find the leftmost non-source vertex, v_{l^*}, to the right of v_i.
      $l^* \leftarrow l^* + 1$;
    until $L[l^*] \in$ {nonterminal, sink} or $l^* \geq q$

30: **end while**
31: **if** $q < r$ **then** ▷ There exists a right valley.
32: Store the extended valley returned by FIND-RIGHT-VALLEY(q, r);
33: In the path ending at v_q, replace v_q with the extended valley;
34: **end if**
35: **if** every edge of the paths constructed is indeed an edge of the induced subgraph **then**
36: **return** an unpaired DPC composed of the paths constructed;
37: **else**
38: **return** a message saying "there exists no unpaired DPC";
39: **end if**
40: **end procedure**

1: **procedure** FIND-LEFT-VALLEY(l, p) ▷ Return the concatenation of (v_p) and the left valley.
2: **if** $p - l$ is odd **then**
3: **return** $(v_p, v_{p-2}, \ldots, v_{l+1}, v_l, v_{l+2}, \ldots, v_{p-1})$;
4: **else**
5: **return** $(v_p, v_{p-2}, \ldots, v_l, v_{l+1}, v_{l+3}, \ldots, v_{p-1})$;
6: **end if**
7: **end procedure**

1: **procedure** FIND-RIGHT-VALLEY(q, r) ▷ Return the concatenation of the right valley and (v_q).
2: **if** $r - q$ is odd **then**
3: **return** $(v_{q+1}, v_{q+3}, \ldots, v_r, v_{r-1}, v_{r-3}, \ldots, v_q)$;
4: **else**
5: **return** $(v_{q+1}, v_{q+3}, \ldots, v_{r-1}, v_r, v_{r-2}, \ldots, v_q)$;
6: **end if**
7: **end procedure**

Lemma 9. *For the case where $h_m \geq 1$ for every $p \leq m < q$, procedure* FIND-UDPC-A *determines the existence of an unpaired k'-DPC joining S' and T' in the induced subgraph $G[V_{l,r}]$ and produces a k'-DPC, if it exists, in time linear to the number of vertices of the subgraph, where $k' = |S'|$.*

PROOF. From the discussions in Lemmas 7 and 8 and in Remarks 2 and 3, there always exists an unpaired k'-DPC that is exactly the same as that which the procedure produces if the induced subgraph has an unpaired k'-DPC. It follows that the procedure is correct. The valley finding procedures run in time linear to $|V_{l,p-1}| + |V_{q+1,r}|$. Moreover, the total time for finding all the next (virtual) sources v_i (in lines 4, 7, and 24–26) and all the next v_{l^*} (in lines 13–16 and 27–29) is linear to $|V_{p,q}| + 1$. Finally, the query if $(v_j, v_{j'}) \in E(G)$ can be answered in $O(1)$ time from the interval representation $\mathcal{I}$, so the validity check (in lines 35–39) can be done in linear-time to $|V_{l,r}|$. Therefore, the running time of the procedure is linear to $|V_{l,r}|$, i.e., $O(|V_{l,r}|)$. □

Remark 4. Let v_{r^*} be the rightmost non-sink vertex of $V_{p,q}$ if $l = p$, or of $\{v_{p-1}\} \cup V_{p+1,q}$ if $l < p$. Procedure FIND-UDPC-A produces an unpaired k'-DPC, whose right valley, if any, is associated with v_{r^*} as well as v_q, i.e., v_{r^*} and v_q are the two mates of end-vertices of the right valley.

Remark 5. Symmetric to procedure FIND-UDPC-A, we can also develop a procedure, named FIND-UDPC-B, applicable to the case where $h_m \leq -1$ for every $p \leq m < q$. (It suffices to switch the roles of sources and sinks.)

There remains a case where $h_m = 0$ for some $p \leq m < q$. (Note that if $h_{m'} > 0$ and $h_{m''} < 0$ for some $p \leq m', m'' < q$, then there exists such m in the range.) To deal with the general cases of our Unpaired DPC problem, one of the natural approaches we can think of would be to decompose graph G into a number of induced subgraphs to each of which one of the procedures Find-UDPC-A or Find-UDPC-B is applicable. The lemma below suggests that this approach is plausible.

Lemma 10. *Let q' and p' be the indices such that $v_{q'}, v_{p'} \in S \cup T$, $q' < p'$ and $h_m = 0$ for all $q' \leq m < p'$. Suppose the pair, q' and p', is the leftmost one, i.e., $h_i \neq 0$ for all $p \leq i < q'$. Then, graph G has an unpaired k-DPC joining S and T if and only if for some $q' \leq m < p'$, the subgraph, G_1, induced by $V_{1,m}$ has an unpaired k_1-DPC joining S_1 and T_1 and moreover, the subgraph, G_2, induced by $V_{m+1,n}$ has an unpaired k_2-DPC joining S_2 and T_2, where $S_i = S \cap V(G_i)$, $T_i = T \cap V(G_i)$, and $k_i = |S_i|$ for $i \in \{1, 2\}$.*

Proof. The sufficiency proof is straightforward because the union of an unpaired k_1-DPC of G_1 and an unpaired k_2-DPC of G_2 becomes an unpaired k-DPC of G. To prove necessity, suppose graph G has an unpaired k-DPC joining S and T, and let m be $q' \leq m < p'$. If there is a path, P_1, in the DPC joining a source $s \in V(G_1)$ and a sink $t \in V(G_2)$, then there is also a path, P_2, joining a source $s' \in V(G_2)$ and a sink $t' \in V(G_1)$. Let $(x, y) \in E(P_1)$, where $x \in V(G_1)$ and $y \in V(G_2)$, be the first edge encountered when we traverse P_1 starting at s. Also, let $(x', y') \in E(P_2)$, where $x' \in V(G_2)$ and $y' \in V(G_1)$, be the first edge encountered when we traverse P_2 starting at s'. Then, we have $(x, y'), (y, x') \in E(G)$ by Lemma 1. Replacing $\{(x, y), (x', y')\}$ with $\{(x, y'), (x', y)\}$ in P_1 and P_2 results in the s–t' path and s'–t path, which reduces the number of paths in the DPC that join a pair of source and sink, one in G_1 and the other in G_2. If we repeat this process, we can obtain an unpaired DPC in which every path joins a source and a sink, both in G_1 or both in G_2.

The DPC may still have a path that joins a source and a sink both contained in G_2 but passes through some vertices in G_1. For a path P whose source and sink are both in G_2, we refer to a maximal subpath of P whose vertices are all contained in G_1 as a *left valley*, within the scope of this proof. Similarly, the term *right valley* will be used for path P' whose source and sink are both in G_1. In the same way as the proof of Lemma 2, we can transform the DPC into a new DPC with at most one left valley and at most one right valley in total. (We note, for reuse of this lemma for the Paired k-DPC problem in Section 6, that the transformation does not alter the source-sink matching, i.e., if a source s_i is matched with a sink t_j in the old DPC, then s_i is still matched with t_j in the new DPC.) Suppose there are two valleys: one left valley, say v_a–v_b valley with $a \leq b$, and one right valley, say v_c–v_d valley with $c \leq d$. For the mates $v_{a'}$ and $v_{b'}$ of v_a and v_b, respectively, the subgraph induced by $\{v_{a'}, v_{b'}, v_c, v_d\}$ forms a clique because $(v_{a'}, v_m)$, $(v_{b'}, v_m)$, (v_c, v_m), (v_d, v_m) are all edges of G. Similarly, for the mates $v_{c'}$ and $v_{d'}$ of v_c and v_d, respectively, the subgraph $G[\{v_a, v_b, v_{c'}, v_{d'}\}]$ forms a clique. We can swap the two valleys by just replacing the four edges $(v_a, v_{a'})$, $(v_b, v_{b'})$, $(v_c, v_{c'})$, $(v_d, v_{d'})$ with the four new edges, $(v_{a'}, v_c)$, $(v_{b'}, v_d)$, $(v_{c'}, v_a)$, $(v_{d'}, v_b)$. Note that $\{v_a, v_b, v_{a'}, v_{b'}\} \cap \{v_c, v_d, v_{c'}, v_{d'}\} = \emptyset$ because no path contains the left and right valleys simultaneously. Then, the new DPC is valley-free.

Finally, suppose there exists only one valley, say left valley, in the DPC. Let m' be the smallest index such that $V(P) \subseteq V_{1,m'}$ for every path P that joins a source and a sink in G_1. Obviously, we have $q' \leq m' \leq m$ (because we have no right valley). We will transform the DPC into a new one, which is the same as the union of an unpaired k_1-DPC of G_1', the induced subgraph $G[V_{1,m'}]$, and an unpaired k_2-DPC of G_2', the induced subgraph $G[V_{m'+1,n}]$. If there exists no left valley with respect to G'_1 and G'_2, then we are done. Suppose there exists a left valley, say v_a–v_b valley, in the DPC. For the mates $v_{a'}$ and $v_{b'}$ of v_a and v_b, respectively, if we remove the valley from a path that contains it and then join $v_{a'}$ and $v_{b'}$ by an edge, there remains an isolated valley, the v_a–v_b valley. Let $v_{m'} \in V(P)$ for some path P in the DPC, and let $(v_{m'}, v_{m''}) \in E(P)$ for some vertex $v_{m''}$. Note that path P joins a source and a sink in G'_1. Similar to the previous case where there are one left valley and one right valley, we can deduce that $G[\{v_{m'}, v_{m''}, v_a, v_b\}]$ forms a clique. If we delete $(v_{m'}, v_{m''})$ from P and add $(v_{m'}, v_a)$ and $(v_b, v_{m''})$, then the new path P passes through the valley. This completes the entire proof. $\square$

The above lemma indicates that, with respect to some m with $q' \leq m < p'$, our DPC problem can be divided into two: one (special) DPC problem on $G[V_{1,m}]$ and another (possibly general) DPC problem on $G[V_{m+1,n}]$. We are to pick up a specific m with respect to which our problem is divided into two. Lemmas 7 and 8 suggest that if $m \geq q' + 1$, it is necessary for G_1 to have an unpaired k_1-DPC that $(v_{q'+1}, v_{r^*})$ is an edge of G, where v_{r^*} is the vertex defined in Remark 4 when $v_{q'}$ is a sink, i.e., (i) for the case of $v_{q'} \in T$, the vertex v_{r^*} is the rightmost non-sink vertex of $V_{p,q'}$ if $p = l$, or of $\{v_{p-1}\} \cup V_{p+1,q'}$ if $p > l$; (ii) for the case of $v_{q'} \in S$, the vertex v_{r^*} is the rightmost non-source vertex of $V_{p,q'}$ if $p = l$, or of $\{v_{p-1}\} \cup V_{p+1,q'}$ if $p > l$. Define

$$m^* = \begin{cases} q' & \text{if } q' + 1 = p' \text{ or } (v_{q'+1}, v_{r^*}) \notin E(G), \\ q'' & \text{otherwise,} \end{cases} \tag{1}$$

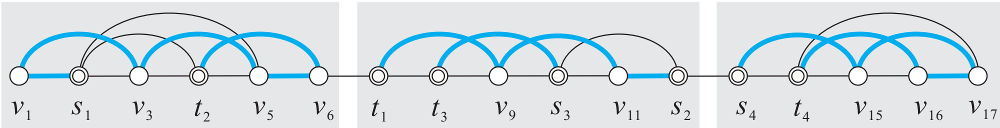
where q'' is the maximum over all j, $q' + 1 \leq j < p'$, subject to that $G[V_{q',j}]$ has a hamiltonian $v_{q'}$–$v_{q'+1}$ path.

Lemma 11. *Let q' and p' be the pair of the leftmost indices of Lemma 10, so that $v_{q'}, v_{p'} \in S \cup T$, $q' < p'$, $h_m = 0$ for all $q' \leq m < p'$, and $h_i \neq 0$ for all $p \leq i < q'$. Then, graph G has an unpaired k-DPC joining S and T if and only if the subgraph, G^*_1, induced by V_{1,m^*}, has an unpaired k_1-DPC joining S_1 and T_1 and moreover, the subgraph, G^*_2, induced by $V_{m^*+1,n}$, has an unpaired k_2-DPC joining S_2 and T_2, where $S_i = S \cap V(G^*_i)$, $T_i = T \cap V(G^*_i)$, and $k_i = |S_i|$ for $i \in \{1, 2\}$.*

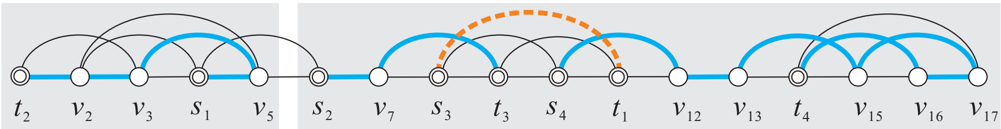
Proof. The sufficiency part of the proof is obvious. To show the necessity part, by Lemma 10, we assume that for some m, where $q' \leq m < p'$, the subgraph, G_1, induced by $V_{1,m}$ has an unpaired k_1-DPC joining S_1 and T_1 and moreover, the subgraph, G_2, induced by $V_{m+1,n}$ has an unpaired k_2-DPC joining S_2 and T_2. We claim $m \leq m^*$. Suppose $m > m^*$ for a contradiction. It follows that $m > q'$, and thus $(v_{q'+1}, v_{r^*}) \in E(G)$. Furthermore, by Lemma 7, $G[V_{q',m}]$ has a hamiltonian $v_{q'}$–$v_{q'+1}$ path. This implies that $m \leq q'' = m^*$, leading to a contradiction. Thus, the claim is proved. Suppose $m < m^*$. (We have nothing to prove if $m = m^*$.) It follows that $m^* > q'$. By the definition of m^* in (1), we have $q' + 1 < p'$, $(v_{q'+1}, v_{r^*}) \in E(G)$, and $G[V_{q',m^*}]$ has a hamiltonian $v_{q'}$–$v_{q'+1}$ path. Thus, G^*_1 has an unpaired k_1-DPC joining S_1 and T_1 by Lemma 7. Also, G^*_2 has an unpaired k_2-DPC joining S_2 and T_2 by Lemma 5(a). Therefore, the lemma is proved. $\square$

Now, we are ready to present an algorithm for finding an unpaired k-DPC of G. It is assumed that we are given a consecutive ordering $(v_1, \ldots, v_n)$ of the vertices of G and a unit interval representation $\mathcal{I}$ for G. Some running examples of the algorithm can be found in Fig. 5.

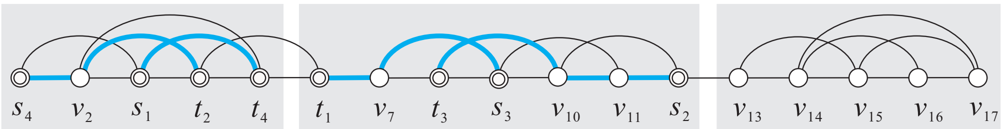
```
procedure Find-Unpaired-DPC(S, T, (v_1, ..., v_n), I)
    P ← ∅; h_0 ← 0;                         ▷ Initialize DPC P to be empty.
```



(a) Find-Unpaired-DPC calls Find-UDPC-A for $(l, r) = (1, 6)$, Find-UDPC-B for $(l, r) = (7, 12)$, and Find-UDPC-A for $(l, r) = (13, 17)$, which collectively lead to an unpaired 4-DPC for this configuration.



(b) Find-UDPC-B for $(l, r) = (1, 5)$ constructs a 1-DPC, and then, however, Find-UDPC-A for $(l, r) = (6, 17)$ reveals that there is no unpaired DPC for this configuration, judging from the fact $(v_8, v_{11}) \notin E(G)$.



(c) Find-UDPC-A for $(l, r) = (1, 5)$ and Find-UDPC-B for $(l, r) = (6, 12)$ are both successful. However, Find-Unpaired-DPC reveals that there is no unpaired DPC, judging from the fact that there are no terminals in $\{v_{13}, \ldots, v_{17}\}$.

Fig. 5: Running examples of procedure Find-Unpaired-DPC for three configurations of sources and sinks on the graph shown in Fig. 1(a).

$l \leftarrow 1$; $p \leftarrow 0$; $S' \leftarrow \emptyset$; $T' \leftarrow \emptyset$; $rn \leftarrow 0$; $m \leftarrow 1$; ▷ Set p and rn to be undefined.
while $m \leq n$ **do**
    if $L[m]$ = source **then**
        $h_m \leftarrow h_{m-1} + 1$; $S' \leftarrow S' \cup \{v_m\}$;
        $rs \leftarrow m$; ▷ $rs \leftarrow$ the index of the rightmost source so far
        if $p = 0$ **then** $p \leftarrow m$; ▷ $p \leftarrow$ the index of the leftmost non-terminal
    else if $L[m]$ = sink **then**
        $h_m \leftarrow h_{m-1} - 1$; $T' \leftarrow T' \cup \{v_m\}$;
        $rt \leftarrow m$; ▷ $rt \leftarrow$ the index of the rightmost sink so far
        if $p = 0$ **then** $p \leftarrow m$; ▷ $p \leftarrow$ the index of the leftmost non-terminal
    else ▷ Here, v_m is a non-terminal.
        $h_m = h_{m-1}$;
        $rn \leftarrow m$; ▷ $rn \leftarrow$ the index of the rightmost non-terminal so far
    end if
    if $h_m = 0$ and $L[m]$ = sink **then** ▷ Procedure Find-UDPC-A is applicable.
        $q' \leftarrow m$; ▷ $q' \leftarrow$ the index of the rightmost non-terminal
        $r^* \leftarrow \max\{rs, rn\}$; ▷ $r^* \leftarrow$ the index of the rightmost non-sink so far
        if $r^* = p$ and $l < p$ **then** $r^* \leftarrow p - 1$; ▷ v_{r^*} is now the vertex of Remark 4.
        if $q' = n$, $L[q'+1] \in$ {source, sink}, or $(v_{q'+1}, v_{r^*}) \notin E(G)$ **then** ▷ Determine m^*.
            $m^* \leftarrow q'$;
        else
            $m^* \leftarrow q' + 1$;

25: **while** $m^* + 1 \le n$, $L[m^* + 1] = \textsf{nonterminal}$, and $(v_{m^*+1}, v_{m^*-1}) \in E(G)$ **do**
26: $m^* \leftarrow m^* + 1$
27: **end while**
28: **end if**
29: $\mathcal{P}' \leftarrow$ Find-UDPC-A(S', T', l, m^*, p, q', $\mathcal{I}$);
30: **if** $\mathcal{P}'$ is a valid unpaired DPC **then**
31: $\mathcal{P} \leftarrow \mathcal{P} \cup \mathcal{P}'$;
32: **else**
33: **return** a message saying "there exists no unpaired DPC";
34: **end if**
35: $h_{m^*} = 0$;
36: $l \leftarrow m^* + 1$; $p \leftarrow 0$; $S' \leftarrow \emptyset$; $T' \leftarrow \emptyset$; $rn \leftarrow 0$; $m \leftarrow m^* + 1$;
37: **else if** $h_m = 0$ and $L[m] = \textsf{source}$ **then** ▹ Procedure Find-UDPC-B is applicable.
38: Similar to lines 18 through 36, an unpaired DPC can be determined and found;
39: **else** ▹ Here, $h_m \neq 0$ or $L[m] = \textsf{nonterminal}$.
40: $m \leftarrow m + 1$;
41: **end if**
42: **end while**
43: **if** $l = n + 1$ **then** ▹ All the vertices v_1 through v_n are processed.
44: **return** $\mathcal{P}$;
45: **else**
46: **return** a message saying "there exists no unpaired DPC";
47: **end if**
48: **end procedure**

Theorem 6. *Procedure* Find-Unpaired-DPC *determines the existence of an unpaired DPC joining S and T in a unit interval graph G and computes the DPC, if any, in $O(n)$ time when we are given a consecutive ordering of the vertices of G and a unit interval representation for G.*

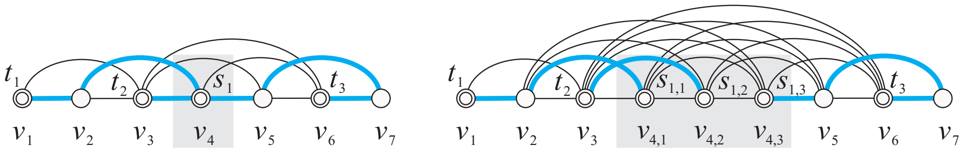
Proof. The correctness of the procedure is due to Lemmas 9 and 11 and Remark 5. To extract a subproblem from the remaining DPC problem on $G[V_{l,n}]$, we find v_p and $v_{q'}$ (along with S' and T') in time linear to $|V_{l,q'}|$ and then determine m^* (in lines 21–28) in time linear to $|V_{q',m^*}|$. Once a subproblem is identified in $O(|V_{l,m^*}|)$ time, it is solved through one of the procedures Find-UDPC-A and Find-UDPC-B also in $O(|V_{l,m^*}|)$ time, totally in time linear to the number of vertices of the subproblem. Thus, the procedure runs in $O(n)$ time. □

Remark 6. The condition $h_m = 0$ in lines 17 and 37 of procedure Find-Unpaired-DPC may be replaced with $|S'| = |T'|$. As a result, all the statements concerning with h_m may be removed.

The algorithm presented in this section is implemented. The source code and some running examples may be downloaded from `http://tcs.catholic.ac.kr/~jhpark/papers/UDPC_in_UIG.zip`.

5. Reduction of General-Demand DPC Problem

The General-Demand DPC problem, suggested in [24], is a generalization of the one-to-one, one-to-many and unpaired many-to-many DPC problem. Given two pairwise disjoint terminal



(a) One-to-many 3-DPC, where $d(v_4) = 3$.

(b) Unpaired 3-DPC, where $S' = \{v_{4,1}, v_{4,2}, v_{4,3}\}$.

Fig. 6: Reduction of One-to-Many 3-DPC problem to Unpaired 3-DPC problem.

sets, a source set $S = \{s_1, \ldots, s_{k'}\}$ and a sink set $T = \{t_1, \ldots, t_{k''}\}$, in which each source and sink is associated with the integer-valued demand $d(\cdot) \geq 1$ such that $\sum_{s_i \in S} d(s_i) = \sum_{t_j \in T} d(t_j) = k$, a *general-demand k-DPC* joining S and T is a k-DPC, in which $d(s_i)$ paths have s_i as their common source for all $s_i \in S$, and $d(t_j)$ paths have t_j as their common sink for all $t_j \in T$. The general-demand k-DPC becomes one-to-one, one-to-many and unpaired many-to-many k-DPCs, respectively, if $k' = k'' = 1$ ($d(s_1) = d(t_1) = k$), if $k' = 1$ and $k'' = k$ ($d(s_1) = k$ and $d(t_j) = 1$ for all $t_j \in T$), and if $k' = k'' = k$ ($d(s_i) = 1$ for all $s_i \in S$ and $d(t_j) = 1$ for all $t_j \in T$).

Consider a General-Demand k-DPC problem on a unit interval graph G with source set S and sink set T, in which we are given a consecutive ordering $(v_1, \ldots, v_n)$ of the vertices of G and a unit interval representation $\mathcal{I}$ for G. Suppose there exists a terminal (source or sink) whose demand is two or greater. Then, the problem can be reduced to a general-demand k-DPC problem on another unit interval graph G', in which the number of terminals whose demand is at least two is one less than that of the original problem on G, as follows: Let the demand of some terminal, say source s_i, be at least two, and let $s_i = v_r$ for some r. If we replace the interval I_r for s_i with $d(s_i)$ copies of I_r, $I_{r,1}, \ldots, I_{r,d(s_i)}$, from $\mathcal{I}$, we obtain an interval representation $\mathcal{I}'$ for G'. Let $S' = (S \setminus s_i) \cup \{s_{i,1}, \ldots, s_{i,d(s_i)}\}$, where $s_{i,j}$ is the vertex $v_{r,j}$ that corresponds to interval $I_{r,j}$, and set $d(s_{i,j}) = 1$ for all $1 \leq j \leq d(s_i)$. Also, $(v_1, \ldots, v_{r-1}, v_{r,1}, \ldots, v_{r,d(s_i)}, v_{r+1}, \ldots, v_n)$ will be a consecutive ordering of the vertices of G'. See Fig. 6 for an illustrative example. It is straightforward to see that G' has a general-demand k-DPC joining S' and T if and only if G has a general-demand k-DPC joining S and T. Furthermore, a general-demand k-DPC of G can be obtained from the general-demand k-DPC of G' by contracting the $d(s_i)$ sources, $s_{i,1}, \ldots, s_{i,d(s_i)}$, into a single source s_i. Repeating this process eventually reduces the General-Demand k-DPC problem to the Unpaired k-DPC problem (without terminals of demand at least two). Therefore, we have:

Theorem 7. *The* General-Demand k-DPC *problem on a unit interval graph G of order n can be solved in $O(n + k)$ time if we are given a consecutive ordering of the vertices of G and a unit interval representation for G.*

Proof. The aforementioned reduction of the General-Demand k-DPC problem to the Unpaired k-DPC problem can be completed in $O(n + k)$ time. Thus, the theorem follows from Theorem 6. □

Remark 7. For graph G to have a general-demand k-DPC joining S and T, it is necessary that $k \leq n^2/4$. If $f(n)$ denotes the maximum k, over all n-vertex graphs along with S and T, such that

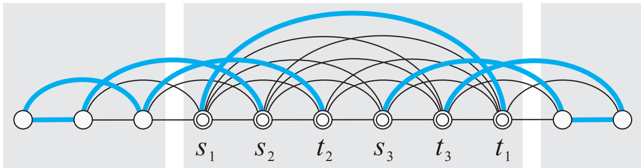


Fig. 7: The left valley is not always associated with the leftmost source, which implies that the counterpart of Lemma 6 for the paired DPC problem is no longer valid. In fact, there exists a unique paired 3-DPC in this instance.

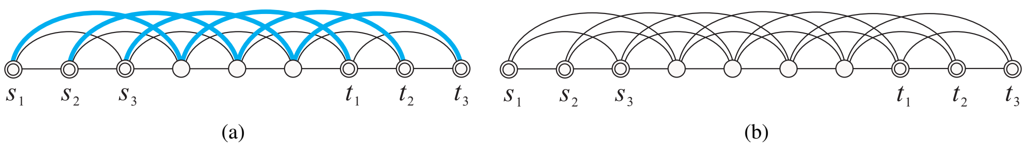


Fig. 8: The instance of (a) has a paired 3-DPC, whereas the instance of (b) does not. They suggest that it is not possible to devise a counterpart of Lemma 8 for the paired DPC problem.

G has a general-demand k-DPC joining S and T, then

$$f(n) = \begin{cases} f(n-1)+1 & \text{if there exists a non-terminal in } G, \\ |S| \cdot |T| \leq n^2/4 & \text{otherwise,} \end{cases}$$

where $f(2) = 1$. Thus, we have $f(n) \leq \max\{f(n-1)+1, n^2/4\} \leq n^2/4$.

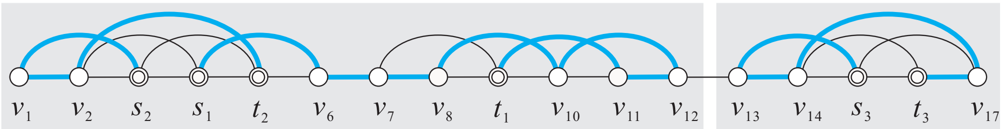
Corollary 1. *The* One-to-One DPC *and* One-to-Many DPC *problems on a unit interval graph G of order n can be solved in $O(n)$ time, if we are given a consecutive ordering of the vertices of G and a unit interval representation for G.*

Proof. There exists a one-to-one k-DPC only if $k \leq n-1$. The same follows for a one-to-many k-DPC. Thus, the corollary follows from Theorem 7. □

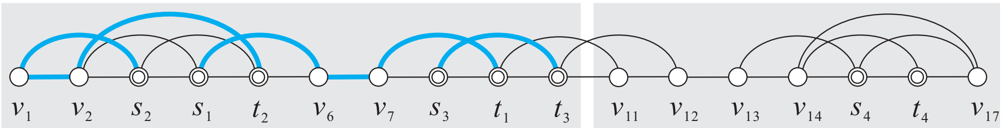
6. Algorithm for the Paired k-DPC Problem

For the Paired k-DPC problem on a unit interval graph G, we assume w.l.o.g. that each source s_i is to the left of its designated sink t_i for $i \in \{1, \ldots, k\}$. (Suppose otherwise, it suffices to exchange s_i and t_i.) By virtue of Lemmas 2 and 3, we can restrict our attention to the paired DPCs having the unique left valley, if any, and the unique right valley, if any, in shape of Lemma 3. The approach taken for the unpaired DPC problem in Section 4 is, however, no longer applicable to the paired DPC problem due to the following: First, the left valley is not always associated with the leftmost terminal v_p, which is in fact a source by our assumption, as shown in Fig. 7. Similarly, the right valley is not always associated with the rightmost terminal v_q; Second, it is hard to devise a counterpart of Lemma 8, as suggested by the two instances shown in Fig. 8.

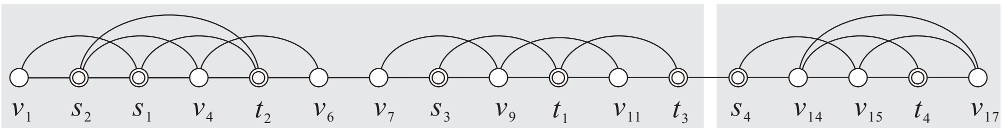
To solve our Paired k-DPC problem, we first divide the problem into smaller subproblems. Let q' and p' be the pair of the leftmost indices of Lemma 10, so that $v_{q'}, v_{p'} \in S \cup T$, $q' < p'$, $h_m = 0$ for all $q' \leq m < p'$, and $h_m \neq 0$ for all $p \leq m < q'$. Recall h_m is defined as



(a) For $q' = 9$ and $p' = 15$, we have $m^* = 12$ because $G[V_{1,12}]$ has a paired 2-DPC but $G[V_{1,13}]$ does not.



(b) For $q' = 10$ and $p' = 15$, we have $m^* = 10$ because $G[V_{1,11}]$ has no paired 3-DPC.



(c) For $q' = 12$ and $p' = 13$, we have $m^* = 12$ because $q' + 1 = p'$.

Fig. 9: Determining m^*, where $q' \leq m^* < p'$, for three configurations of sources and sinks on the graph of Fig. 1(a).

$|S \cap V_{1,m}| - |T \cap V_{1,m}|$. By our assumption that every s_i is to the left of t_i, we have $h_m \geq 0$ for all $0 \leq m \leq n$. Also, we have that for every i, either $s_i, t_i \in V_{p,q'}$ or $s_i, t_i \notin V_{p,q'}$. It is fortunate for us to recycle Lemma 10 (although Lemma 11 is not reusable). More specifically, it holds true the counterpart of Lemma 10 for the Paired k-DPC problem asserting that graph G has a paired k-DPC joining S and T if and only if for some $q' \leq m < p'$, the subgraph, G_1, induced by $V_{1,m}$ has a paired k_1-DPC joining S_1 and T_1 and moreover, the subgraph, G_2, induced by $V_{m+1,n}$ has a paired k_2-DPC joining S_2 and T_2 (where $S_i = S \cap V(G_i)$, $T_i = T \cap V(G_i)$, and $k_i = |S_i|$ for $i \in \{1, 2\}$). Actually, the second and third paragraphs of the proof of Lemma 10 serve as a necessity proof of the counterpart (whereas the sufficiency is obvious). This is because it never occurs the case when there exists a path in the paired DPC of G that joins a source in $V_{1,m}$ and a sink in $V_{m+1,n}$, or joins a source in $V_{m+1,n}$ and a sink in $V_{1,m}$.

Instead of relying on Lemma 11 to pick up a specific m in the range $q' \leq m < p'$ for dividing the problem into two parts, we let

$$m^* = \begin{cases} q' & \text{if } q' + 1 = p' \text{ or } G[V_{1,q'+1}] \text{ has no paired DPC,} \\ q' + 1 & \text{else if } q' + 2 = p' \text{ or } G[V_{1,q'+2}] \text{ has no paired DPC,} \\ q'' & \text{elsewhere,} \end{cases} \tag{2}$$

where q'' is the maximum over all j, $q' + 2 \leq j < p'$, subject to the fact that $G[V_{q'+1,j}]$ has a hamiltonian $v_{q'+1}$–$v_{q'+2}$ path. Refer to Fig. 9 for illustrative examples.

Lemma 12. *Let q' and p' be the pair of the leftmost indices such that $v_{q'}, v_{p'} \in S \cup T$, $q' < p'$, $h_m = 0$ for all $q' \leq m < p'$, and $h_j \geq 1$ for all $p \leq j < q'$. Graph G has a paired k-DPC joining S and T if and only if the subgraph, G_1^*, induced by V_{1,m^*}, has a paired k_1-DPC joining S_1 and T_1 and moreover, the subgraph, G_2^*, induced by $V_{m^*+1,n}$, has a paired k_2-DPC joining S_2 and T_2, where $S_i = S \cap V(G_i^*)$, $T_i = T \cap V(G_i^*)$, and $k_i = |S_i|$ for $i \in \{1, 2\}$.*

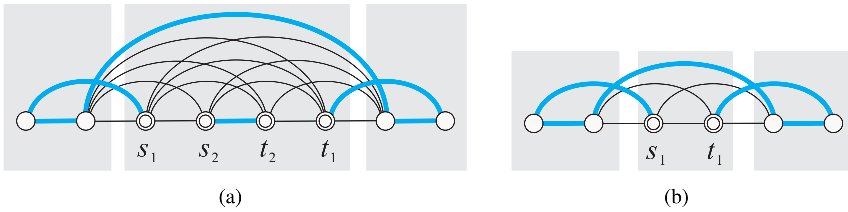


Fig. 10: (a) There exists a unique paired 2-DPC joining $\{s_1, s_2\}$ and $\{t_1, t_2\}$; (b) There is a unique paired (or unpaired) 1-DPC, which is a hamiltonian s_1–t_1 path.

Proof. The sufficiency proof is obvious. To show necessity, we assume (by the counterpart of Lemma 10) that for some m, where $q' \le m < p'$, the subgraph, G_1, induced by $V_{1,m}$, has a paired k_1-DPC joining S_1 and T_1 and moreover, the subgraph, G_2, induced by $V_{m+1,n}$, has a paired k_2-DPC joining S_2 and T_2. By the definition of m^* and Lemma 4(b), we conclude that subgraph G_1^* has a paired k_1-DPC joining S_1 and T_1, and furthermore $m \le m^*$. Also, G_2^* has a paired k_2-DPC joining S_2 and T_2 by Lemma 5(a). Therefore, the proof is complete. □

Remark 8. To determine m^* of Eq. (2), we need to check whether or not there exist paired k_1-DPCs in at most two subgraphs, $G[V_{1,q'+1}]$ and $G[V_{1,q'+2}]$. The maximum j such that $G[V_{q'+1,j}]$ has a hamiltonian $v_{q'+1}$–$v_{q'+2}$ path is computed easily as done in lines 25 through 27 of procedure Find-Unpaired-DPC.

There remains a task to design an algorithm, which is a counterpart of procedure Find-UDPC-A, for determining the existence of a paired DPC in the induced subgraph $G[V_{l,r}]$ as well as building a paired DPC, if any, where v_p and v_q, respectively, are the leftmost source and the rightmost sink, and moreover, $h_j \ge 1$ for all $p \le j < q$. Let H denote the induced subgraph $G[V_{l,r}]$, and let $S_H = S \cap V(H)$, $T_H = T \cap V(H)$, and $k_H = |S_H| = |T_H|$. If it is clear from the context that we are dealing with the subgraph H, we use S, T and k, respectively, in order to denote S_H, T_H and k_H for notational simplicity.

If there are two valleys, one left v_a–v_b valley and one right v_c–v_d valley, such that the mate of an end-vertex, say v_b, of the left valley is an end-vertex, say v_c, of the right valley, then the two valleys may be combined into a single one, named a *wide v_a–v_d valley*, via an edge (v_b, v_c). Here, v_a and v_d are the end-vertices of the wide valley. As illustrated in Fig. 10, a wide valley is not always avoidable when we construct a paired DPC (and also not always avoidable to construct an unpaired 1-DPC, although we think little of the existence of a wide valley when we design an unpaired DPC algorithm in Section 4).

First of all, we study the problem of determining and finding, if any, a paired k-DPC having a wide valley in H. Note that a paired k-DPC with a wide valley exists only if there exists an edge between $V_{l,p-1}$ and $V_{q+1,r}$. Thus, we assume $V_{l,p-1}, V_{q+1,r} \neq \emptyset$ and $(v_{p-1}, v_{q+1}) \in E(H)$. It follows that the induced subgraph $G[V_{p,q}]$ forms a clique by Lemma 1. Let the unique left and right valley, respectively, be v_a–v_b valley and v_c–v_d valley in the shape of Lemma 3, where $a \le b$ and $c \le d$. Then, there are at most four candidates for wide valleys: v_a–v_c, v_a–v_d, v_b–v_c, and v_b–v_d valleys. It can be determined easily whether or not there exists a paired DPC that contains a wide valley as follows:

Lemma 13. *The subgraph H has a paired k-DPC with a wide valley if and only if there exists a wide v_a–v_d valley in H and there exist distinct vertices $x \in N_H(v_a) \cap V_{p,q}$ and $y \in N_H(v_d) \cap V_{p,q}$ such that*

- *at least one of x and y is a non-terminal, or*
- *$\{x, y\} = \{s_i, t_i\}$ for some i and moreover either $k \geq 2$ or $k = 1$ and $V_{p,q} = \{s_i, t_i\}$.*

Proof. Necessity. Suppose that H has a paired k-DPC with a wide α–β valley, where $\alpha \in \{v_a, v_b\}$ and $\beta \in \{v_c, v_d\}$. If $\alpha \neq v_a$, then for the mates $v_{a'}$ and $v_{b'}$ of v_a and v_b, respectively, we have $(v_a, v_{b'}), (v_b, v_{a'}) \in E(H)$ by the existence of a wide valley. Thus, replacing $(v_a, v_{a'})$ and $(v_b, v_{b'})$ with $(v_a, v_{b'}), (v_b, v_{a'})$ from the DPC results in a paired k-DPC with a wide v_a–β valley. Similarly, we can obtain a paired k-DPC with a wide v_a–v_d valley if $\beta \neq v_d$, proving the existence of a wide v_a–v_d valley. Let x and y respectively be the mates of v_a and v_d. Obviously, at least one of x and y is a non-terminal, or $\{x, y\} = \{s_i, t_i\}$ for some i. For the latter case, the s_i–t_i path must be a combination of three paths: one-vertex path (s_i), the wide v_a–v_d valley, and (t_i). Thus, if $k = 1$, there is no vertex in $V_{p,q} \setminus \{s_i, t_i\}$, or equivalently, $V_{p,q} = \{s_i, t_i\}$, proving the necessity.

Sufficiency. We will construct a paired k-DPC with a wide v_a–v_d valley. Recall that the induced subgraph $G[V_{p,q}]$ is a clique. There are three cases. First, suppose that x and y are both non-terminals. Let s_j–t_j path for $j \geq 2$ be a two-vertex path (s_j, t_j). There exists a paired 2-DPC composed of s_1–x path and y–t_1 path in the subgraph induced by $V_{p,q} \setminus \{s_j, t_j : j \geq 2\}$. To obtain a paired k-DPC of H, it suffices to combine the three paths, s_1–x path, the wide v_a–v_d valley, and y–t_1 path, via edges (x, v_a) and (v_d, y) into an s_1–t_1 path. Second, suppose that one of x and y, say y, is a non-terminal. Assume w.l.o.g. $x = s_1$. If we regard y as a *virtual* source, there exists a paired k-DPC composed y–t_1 path and s_j–t_j path for $j \geq 2$ in the subgraph induced by $V_{p,q} \setminus s_1$. It suffices to combine the three paths, one-vertex path (s_1), the wide v_a–v_d valley, and the y–t_1 path, into an s_1–t_1 path. Third, suppose that $\{x, y\} = \{s_i, t_i\}$ for some i and moreover, either $k \geq 2$ or $k = 1$ and $V_{p,q} = \{s_1, t_1\}$. The three paths, one-vertex path (s_i), the wide v_a–v_d valley, and (t_i), are combined into an s_i–t_i path. If $k = 1$ and $V_{p,q} = \{s_1, t_1\}$, we are done; Otherwise, it suffices to find a paired $(k-1)$-DPC in the subgraph induced by $V_{p,q} \setminus \{s_i, t_i\}$, completing the proof. □

Lemma 14. *The problem of finding a paired k-DPC having a wide valley on the subgraph H can be solved in time polynomial to the number of vertices in H.*

Proof. We can determine the existence of a wide v_a–v_d valley and of a pair of distinct vertices x and y of Lemma 13 in time polynomial to $|V(H)|$. Moreover, the paired k-DPC can be constructed, if both exist, in linear-time to $|V(H)|$ according to the sufficiency proof of Lemma 13, and so the lemma follows. □

Hereafter, we then study the problem of determining and finding, if any, a paired k-DPC without a wide valley. We assume w.l.o.g. that there exists no edge between $V_{l,p-1}$ and $V_{q+1,r}$. (For our purpose, every such edge of H may be removed.) To overcome the trouble due to the absence of counterparts of Lemmas 6 and 8, we will reduce our problem to the Paired k-DPC problem on an acyclic digraph. A paired k-DPC in a digraph naturally refers to a disjoint path cover composed of k directed paths, each starting at a source and ending at its designated sink. Let us consider the case when $p = l$ and $q = r$ first, where v_l is a source and v_r is a sink.

Lemma 15. *Suppose that $p = l$, $q = r$, and $h_j \geq 1$ for all $p \leq j < q$. If the graph H has a paired k-DPC joining S and T, then H has a paired k-DPC joining S and T whose paths are all monotone.*

Proof. Suppose there exists a paired k-DPC in which not every path is monotone. We first claim that for each s_i–t_i path, P_i, in the DPC, there exists a monotone s_i–t_i path P'_i such that $V(P'_i) \subseteq V(P_i)$. The path P'_i can be built in a greedy manner, as follows: Let $P_i = (w_1, \ldots, w_m)$, where $w_1 = s_i$ and $w_m = t_i$, and let P'_i be empty initially; For each vertex w_j encountered when we traverse P_i from w_1 to w_m, we pick up w_j and append it to P'_i if and only if (i) w_j is a terminal or (ii) w_j is to the left of t_i and to the right of the latest vertex, $w_{j'}$, appended to P'_i. It remains to show that for each $j \geq 2$, $(w_{j'}, w_j) \in E(H)$ if w_j is picked up and appended to P'_i by the greedy algorithm. Suppose $j' < j-1$ as the case of $j' = j-1$ is obvious. Note that $X := \{w_{j'+1}, \ldots, w_{j-1}\}$ is nonempty and no vertex of X is chosen by the algorithm. It follows that every vertex $x \in X$ is a non-terminal and moreover, it is either to the right of t_i or to the left of $w_{j'}$. If w_{j-1} is to the left of $w_{j'}$, we can conclude $(w_{j'}, w_j) \in E(H)$ by Lemma 1; Otherwise, there exists a vertex $w_{j''} \in X$ to the right of t_i such that $w_{j''-1} = w_{j'}$ or $w_{j''-1}$ is to the left of $w_{j'}$, which implies $(w_{j'}, w_j) \in E(H)$ by Lemma 1 again. Thus, the claim is proved.

Now, we have a set of k pairwise disjoint paths $\{P'_1, \ldots, P'_k\}$, in which each path P'_i joining s_i and t_i is monotone. If there exists a (non-terminal) vertex w not covered by any of the k monotone paths, we pick up a path P'_i such that s_i is to the left of w and t_i is to the right of w. This is possible because $h_j \geq 1$ for every $p \leq j < q$. Then, there is an edge, (x, y), of P'_i such that x is to the left of w and y is to the right of w. If we replace the subpath (x, y) of P'_i with a new subpath (x, w, y), we have a new monotone path P''_i that joins s_i and t_i. Note that $(x, w), (w, y) \in E(H)$ by Lemma 1. Repeating this process eventually produces a paired k-DPC whose paths are all monotone. This completes the proof. □

The above lemma suggests that the existence of a paired DPC, whose paths are all monotone, is linked directly to the paired DPC problem on an acyclic digraph. That is, under the condition that $p = l$, $q = r$, and $h_j \geq 1$ for all $p \leq j < q$, there exists a paired k-DPC in H if and only if there exists a paired k-DPC in an acyclic digraph D

- whose vertex set $V(D) = V(H)$ and
- arc set $E(D) = \{\langle v_i, v_j \rangle : (v_i, v_j) \in E(H) \text{ and } v_i \text{ is to the left of } v_j\}$.

On the other hand, there exists no paired DPC whose paths are all monotone in the remaining case where $p \neq l$ or $q \neq r$, due to the existence of a valley, left or right. However, there exists a paired DPC, if any, whose paths are good from a monotonicity point of view, as discussed in Lemma 16 below.

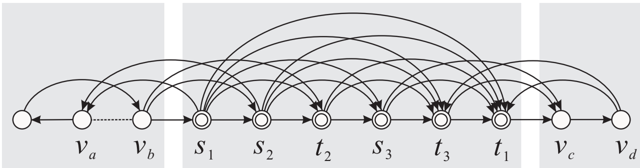
Lemma 16. *Suppose that $l < p$ or $q < r$, and $h_j \geq 1$ for all $p \leq j < q$. Let there exist no edge between $V_{l,p-1}$ and $V_{q+1,r}$. If the graph H has a paired k-DPC whose valley(s) are in the shape of Lemma 3, then H has a paired k-DPC $\mathcal{P}$ such that the valley(s) of $\mathcal{P}$ retain the shape of Lemma 3 and moreover, the set of k paths obtained from $\mathcal{P}$ by the valley removal process below becomes a paired k-DPC of the induced subgraph $G[V_{p,q}]$, whose paths are all monotone:*

- *If a path P_i in the DPC contains a left valley, then we remove the left valley from P_i and add an edge to P_i that joins the two mates of the end-vertices of the left valley;*
- *If a path P_j in the DPC, possibly $j = i$, contains a right valley, then we remove the right valley from P_j and add an edge to P_j that joins the two mates of the end-vertices of the right valley.*

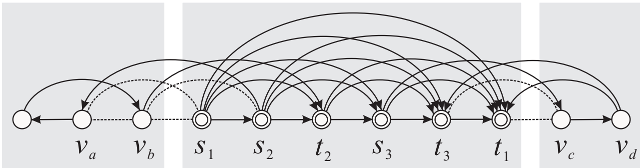
Proof. Suppose there exists a paired k-DPC $\mathcal{P}'$ in H whose valley(s) are in the shape of Lemma 3. If $\mathcal{P}'$ has a left valley, let the valley be a v_a–v_b valley with $a \leq b$ and let $v_{a'}$ and $v_{b'}$, respectively, be the mates of v_a and v_b such that $a' < b'$. Also, if $\mathcal{P}'$ has a right valley, let the valley be a v_c–v_d valley with $c \leq d$ and let $v_{c'}$ and $v_{d'}$, respectively, be the mates of v_c and v_d such that $c' < d'$. If $\mathcal{P}'$ has both valleys, we assume w.l.o.g. that $a' < c'$ or $b' < d'$, so that $a' < d'$; Suppose otherwise (i.e., $c' \leq a'$ and $d' \leq b'$), it suffices to switch the two valleys, i.e., to set $v_{c'}$ and $v_{d'}$, respectively, be the mates of v_a and v_b and set $v_{a'}$ and $v_{b'}$, respectively, be the mates of v_c and v_d. Note that the case $\{a', b'\} = \{c', d'\}$ never occurs; Suppose otherwise, the two valleys and their common mates form a cycle, which contradicts the fact that $\mathcal{P}'$ is a paired DPC. Because the two mates of the end-vertices of a valley, left or right, are adjacent in H, as claimed in the proof of Lemma 2, the set of new paths, Q', after the valley removal process from $\mathcal{P}'$ becomes a paired k-DPC of the subgraph $G[V_{p,q}]$. Suppose not every path in Q' is monotone. (Suppose otherwise, we are done.) By Lemma 15, we can restructure Q' into a new paired k-DPC, Q, of the subgraph so that all paths in Q are monotone. Hereafter, we will build a paired k-DPC, $\mathcal{P}$, of H from Q and the valley(s) of the k-DPC $\mathcal{P}'$ so that $\mathcal{P}$ after the valley removal process becomes Q.

Suppose $\mathcal{P}'$ has a left v_a–v_b valley. Let $v_{a'} \in V(P_i)$ for some s_i–t_i path, P_i, in Q. We claim that P_i and the left valley can be combined into a new s_i–t_i path, P_i', such that P_i' after the valley removal process becomes P_i. If $v_{a'} \neq s_i$, there exists a *predecessor*, α, of $v_{a'}$ in P_i, which is the vertex encountered just before $v_{a'}$ when we traverse P_i from s_i to t_i. If we combine the two subpaths, the s_i–α subpath and the $v_{a'}$–t_i subpath obtained by removing the edge $(\alpha, v_{a'})$ from P_i, with the left valley via edges (v_a, α) and $(v_b, v_{a'})$, we have a new s_i–t_i path that passes through the left valley. If $v_{a'} = s_i$, there exists a *successor*, β, of $v_{a'}$ in P_i, which is the vertex encountered just after $v_{a'}$ when we traverse P_i from s_i to t_i. Then, $\beta = v_{b'}$ or β is to the left of $v_{b'}$. (Suppose to the contrary that $v_{b'}$ is to the left of β. Then, $v_{b'}$ would be a non-terminal that is to the right of $v_{a'}$ and to the left of t_i. Since $(v_{a'}, v_{b'})$ is an edge of the s_i–t_i path of Q', the greedy algorithm for restructuring Q' into Q would pick up $v_{b'}$ and append it to the incomplete s_i-path as a successor of s_i, which contradicts the fact that β is the successor of s_i in P_i.) If we combine the two subpaths, obtained by removing the edge $(v_{a'}, \beta)$ from P_i, and the left valley via edges $(v_a, v_{a'})$ and (v_b, β), we have a new s_i–t_i path that passes through the left valley.

Now, suppose $\mathcal{P}'$ has a right v_c–v_d valley. Let $v_{d'} \in V(P_j)$ for some s_j–t_j path, P_j, in Q, and let the s_i–t_i path P_i', constructed above, pass through the left valley if $\mathcal{P}'$ has a left valley. If $\mathcal{P}'$ has no left valley or $j \neq i$, then we can construct a new s_j–t_j path, P_j', that passes through the right valley in a symmetric way to the construction of P_i'. (That is, if $v_{d'} \neq t_j$, then the s_j–$v_{d'}$ subpath, the β'–t_j subpath and the right valley are combined into a new s_j–t_j path through edges $(v_{d'}, v_c)$ and (β', v_d), where β' is the successor of $v_{d'}$ in P_j; if $v_{d'} = t_j$, then the s_j–α' subpath, one-vertex subpath (t_j) and the right valley are combined into a new s_j–t_j path through edges (α', v_c) and $(v_{d'}, v_d)$, where α' is the predecessor of $v_{d'}$ in P_j.) Let $j = i$ now. We can see that the aforementioned procedure for processing the right valley successfully combines P_i' and the right valley into a final s_i–t_i path that passes through both valleys, if the edge $(v_{d'}, \beta')$ of P_i' (resp. the edge $(\alpha', v_{d'})$ of P_i') is also an edge of the monotone s_i–t_i path P_i of Q for the case of $v_{d'} \neq t_i$ (resp. for the case of $v_{d'} = t_i$). We will show that the precondition is satisfied so as to conclude that P_i' and the right valley are successfully combined. From our assumption of $a' < d'$, the edge $(v_{d'}, \beta')$ of P_i' (for the case of $v_{d'} \neq t_i$) is always an edge of the s_i–t_i path $P_i \in Q$. Also, the edge $(\alpha', v_{d'})$ of P_i' (for the case $v_{d'} = t_i$) is an edge of $P_i \in Q$ if $v_{a'} \neq s_i$, or if $v_{a'} = s_i$ and $\beta \neq v_{d'}$. There remains only one possibility of $v_{a'} = s_i$ and $\beta = v_{d'}$ ($v_{d'} = t_i$). It follows that $P_i \in Q$ is a two-vertex path (s_i, t_i), and that the original s_i–t_i path of $\mathcal{P}'$ contains both valleys and passes through the other two mates, $v_{b'}$ and $v_{c'}$, of the left and right valleys as intermediate vertices. In the process of restructuring Q' into Q, every non-terminal of the s_i–t_i path of Q', including $v_{b'}$ and $v_{c'}$, are not picked up, which implies that every non-terminal of the path is to the left of s_i or



(a) The digraph D, which may not be acyclic.



(b) The digraph $D_{i,j}$ for $i = 5$ and $j = 7$, which is acyclic.

Fig. 11: The digraph D and $D_{i,j}$ built from the graph shown in Fig. 7. The dotted lines indicate the absence of arcs.

to the right of t_i, particularly, $v_{b'}$ is to the right of t_i ($= v_{d'}$), and $v_{c'}$ is to the left of s_i ($= v_{a'}$). Thus, we have $c' < a' < d' < b'$, which contradicts our assumption of $a' < c'$ or $b' < d'$, completing the entire proof. $\square$

Based on the above lemma, we can reduce our Paired k-DPC problem on the graph H into a Paired k-DPC problem on a (not necessarily acyclic) digraph D, whose vertex set $V(D) = V(H)$ and whose arc set $E(D) = C \cup L \cup R$, where C, L and R are defined as follows (see Fig. 11(a)):

- $C := \{\langle v_i, v_j\rangle : (v_i, v_j) \in E(H)$ and v_i is to the left of $v_j\}$;
- $L := \emptyset$ if $l = p$ or there is no hamiltonian v_a–v_b path in $G[V_{l,p-1}]$; Otherwise, $L := L_1 \cup L_2 \cup L_h$, where $L_1 := \{\langle v_i, v_a\rangle : v_i \in N_H(v_a) \cap V_{p,q}\}$, $L_2 := \{\langle v_b, v_{i'}\rangle : v_{i'} \in N_H(v_b) \cap V_{p,q}\}$, and $L_h := \{\langle x_1, x_2\rangle, \ldots, \langle x_{f-1}, x_f\rangle\}$ for a hamiltonian v_a–v_b path, $(x_1, \ldots, x_f)$, of $G[V_{l,p-1}]$.
- $R := \emptyset$ if $r = q$ or there is no hamiltonian v_c–v_d path in $G[V_{q+1,r}]$; Otherwise, $R := R_1 \cup R_2 \cup R_h$, where $R_1 := \{\langle v_j, v_c\rangle : v_j \in N_H(v_c) \cap V_{p,q}\}$, $R_2 := \{\langle v_d, v_{j'}\rangle : v_{j'} \in N_H(v_d) \cap V_{p,q}\}$, and $R_h := \{\langle y_1, y_2\rangle, \ldots, \langle y_{g-1}, y_g\rangle\}$ for a hamiltonian v_c–v_d path, $(y_1, \ldots, y_g)$, of $G[V_{q+1,r}]$.

Lemma 17. *Graph H has a paired k-DPC without a wide valley if and only if the digraph D has a paired k-DPC.*

Proof. The sufficiency part is obvious because D is an orientation of a spanning subgraph of H, i.e., $\langle v_i, v_j\rangle$ is an arc of D only if (v_i, v_j) is an edge of H. The necessity part is due to Lemmas 15 and 16. $\square$

The reduced problem of finding a paired k-DPC in the digraph D, however, is not an easy task because D may have a cycle. To avoid difficulty, we employ a number of *acyclic* digraphs $D_{i,j}$ for $i, j \in \{p, \ldots, q-1\}$ (instead of D), which is defined in the same way as digraph D, except the following four (see Fig. 11(b)):

- $L_1 := \{\langle v_i, v_a\rangle\}$ if $(v_i, v_a) \in E(H)$; $L_1 := \emptyset$ otherwise;
- $L_2 := \{\langle v_b, v_{i'}\rangle : v_{i'} \in N_H(v_b) \cap V_{i+1,q}\}$;

- $R_1 := \{\langle v_j, v_c\rangle\}$ if $(v_j, v_c) \in E(H)$; $R_1 := \emptyset$ otherwise;
- $R_2 := \{\langle v_d, v_{j'}\rangle : v_{j'} \in N_H(v_d) \cap V_{j+1,q}\}$.

Lemma 18. *Graph H has a paired k-DPC without a wide valley if and only if there exists a digraph $D_{i,j}$ that has a paired k-DPC for $i, j \in \{p, \ldots, q-1\}$.*

Proof. The sufficiency part is straightforward by Lemma 17 because every $D_{i,j}$ is a spanning subgraph of D. To prove necessity, suppose H has a paired k-DPC, $\mathcal{P}$, without a wide valley. We assume that the valley(s) of $\mathcal{P}$, if any, are in the shape of Lemma 3. We further assume that every path of $\mathcal{P}$ is monotone if there is no valley (Lemma 15), or every path obtained from $\mathcal{P}$ by the valley removal process is monotone (Lemma 16). If $\mathcal{P}$ has a left v_a–v_b valley, let $i^* = a'$, where $v_{a'}$ is the mate of v_a; otherwise, let i^* be an arbitrary integer in $\{p, \ldots, q-1\}$. Similarly, if $\mathcal{P}$ has a right v_c–v_d valley, let $j^* = c'$, where $v_{c'}$ is the mate of v_c; otherwise, let j^* be an arbitrary integer in $\{p, \ldots, q-1\}$. Then, the digraph D_{i^*,j^*} has a paired k-DPC that corresponds to $\mathcal{P}$. □

An algorithm for the Paired k-DPC problem on an acyclic digraph will be designed by slightly modifying an algorithm for the k-Disjoint Paths or k-Linking problem on an acyclic digraph, where *k-disjoint paths* from $S = \{s_1, \ldots, s_k\}$ to $T = \{t_1, \ldots, t_k\}$ in a digraph D is a set of vertex-disjoint paths $\{P_1, \ldots, P_k\}$, in which each P_i is a directed s_i–t_i path in D. Notice that the k-disjoint paths from S to T, whose paths altogether cover every vertex of the digraph (thus forms a path cover), is the very paired k-DPC joining S and T. It has been known that the 2-Disjoint Paths problem on a general digraph is NP-complete [14] (whereas that problem on a general undirected graph is polynomially solvable for fixed $k \geq 2$ [36, 38]). According to Fortune, Hopcroft, and Wyllie [14], the k-Disjoint Paths problem on an acyclic digraph is polynomially solvable when k is not a part of the input. In fact, they proved that for any fixed directed (pattern) graph G', there is a polynomial-time algorithm for deciding if an acyclic digraph G contains a subgraph homeomorphic to G'. The algorithm works whether or not the node mapping of G' to G is specified. For our purpose, we quote some part from Bang-Jensen and Gutin [4], which describes an algorithm in plain words for the k-Disjoint Paths problem on an acyclic digraph and its correctness as follows:

> *Let $D = (V, A)$ be acyclic and let $x_1, x_2, \ldots, x_k$, $y_1, y_2, \ldots, y_k$ be distinct vertices of D for which we wish to find a k-linking from $(x_1, x_2, \ldots, x_k)$ to $(y_1, y_2, \ldots, y_k)$. We may assume that $d^-_D(x_i) = d^+_D(y_i) = 0$[1] for $i = 1, 2, \ldots, k$, since such arcs play no role in the problem and can therefore be deleted. Form a new digraph $D' = (V', A')$ whose vertex set is the set of all k-tuples of distinct vertices of V. For any such k-tuple $(v_1, v_2, \ldots, v_k)$, there is at least one vertex, say v_r, which cannot be reached by any of the other v_i by a path in D. (Here we used that D is acyclic.) For each out-neighbour w of v_r such that $w \notin \{v_1, v_2, \ldots, v_k\}$, we let A' contain the arc $(v_1, v_2, \ldots, v_{r-1}, v_r, v_{r+1}, \ldots, v_k) \to (v_1, v_2, \ldots, v_{r-1}, w, v_{r+1}, \ldots, v_k)$. Only arcs as those described above are in A'. We claim that D' has a directed path from the vertex $(x_1, x_2, \ldots, x_k)$ to the vertex $(y_1, y_2, \ldots, y_k)$ if and only if D contains disjoint paths $P_1, P_2, \ldots, P_k$ such that P_i is an (x_i, y_i)-path[2] for $i = 1, 2, \ldots, k$. Suppose first that D' has a path P from $(x_1, x_2, \ldots, x_k)$ to $(y_1, y_2, \ldots, y_k)$. By definition, every arc*

[1] Here, $d^-_D(x_i)$ and $d^+_D(y_i)$, respectively, denote the in-degree of x_i and the out-degree of y_i.
[2] An (x_i, y_i)-path denotes a path from x_i to y_i.

of P corresponds to one arc in D. Hence we get a collection of paths $P_1, P_2, \ldots, P_k$ *such that* P_i *is an* (x_i, y_i)*-path by letting* P_i *contain those arcs that correspond to a shift in the ith vertex of a k-tuple,* $i = 1, 2, \ldots, k$.

From the above discussion quoted from [4], we can observe with ease that the number of edges of the path P from $(x_1, \ldots, x_k)$ to $(y_1, \ldots, y_k)$ in D' is equal to the total number of edges of paths in the corresponding k-linking from $(x_1, \ldots, x_k)$ to $(y_1, \ldots, y_k)$ in D. This leads to the fact that there exists a paired k-DPC from $(x_1, \ldots, x_k)$ to $(y_1, \ldots, y_k)$ in D if and only if there exists a path of length $|V(D)| - k$ from $(x_1, \ldots, x_k)$ to $(y_1, \ldots, y_k)$ in D'. The condition is equivalent to that the length of a longest path from $(x_1, \ldots, x_k)$ to $(y_1, \ldots, y_k)$ in D' is $|V(D)| - k$. The longest path in D' is computable in time linear to the size of D' because D' is also an acyclic digraph. Thus, we have the following:

Lemma 19. *The* Paired k-DPC *problem on an acyclic digraph is polynomially solvable for any fixed k.*

Lemmas 14, 18 and 19 lead to the following:

Lemma 20. *The* Paired k-DPC *problem on the graph H is polynomially solvable for any fixed k.*

From the discussions so far in this section, especially in Lemmas 12 and 20 and Remark 8, we conclude that:

Theorem 8. *The* Paired k-DPC *problem on a unit interval graph is polynomially solvable for any fixed k.*

An $O(n^{k+3})$-time and $O(n^{\max\{k,2\}})$-space implementation of the Paired k-DPC algorithm discussed in this section (along with some running examples) may be downloaded from `http://tcs.catholic.ac.kr/~jhpark/papers/PDPC_in_UIG.zip`. The subproblem of determining and building a paired k_1-DPC, if any, in the induced subgraph H of order n_1, where $k_1 = |S \cap V(H)|$, can be solved in $O(n_1)$ time for the DPC with a wide valley and in $O(n_1^{k_1+3})$ time for the DPC without a wide valley. The without-a-wide-valley problem is reduced to $O(n_1^2)$ DPC problems on acyclic digraphs $D_{i,j}$, each of which is reduced again to the longest path problem on some acyclic digraph $D'_{i,j}$ that has an $O(n_1^{k_1+1})$-time solution. If there are r, $r \geq 2$, subproblems on H_1 for (n_1, k_1), …, and on H_r for (n_r, k_r), where $n_1 + \cdots + n_r = n$ and $k_1 + \cdots + k_r = k$, the total running time is $O(n^{k+3})$ since $(n_1 + 1)^{k_1+3} + \cdots + (n_r + 1)^{k_r+3} < n^{k+3}$. (Note that $(n_1+1)^{k_1+3} + (n_2+1)^{k_2+3} < (n_1+n_2)^{k_1+k_2+2} + (n_1+n_2)^{k_1+k_2+2} = 2\cdot(n_1+n_2)^{k_1+k_2+2} < (n_1+n_2)^{k_1+k_2+3}$.) The interested readers may refer to the source code for details.

There seems to be much room for improvement in our Paired k-DPC algorithm. It is open for developing an efficient algorithm for finding a paired k-DPC in a unit interval graph, whose running time is polynomial in both n and k. We hope our naive algorithm could initiate future research.

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