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Characterization of interval graphs that are paired 2-disjoint path coverable

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Abstract The performance of a supercomputing system is significantly affected by the network used to interconnect the nodes. One of the key problems associated with parallel processing is finding disjoint paths in the underlying graph of an interconnection network. It is often important to find disjoint paths that collectively pass through all the vertices. A *disjoint path cover* of a graph is a set of vertex-disjoint paths that altogether cover every vertex of the graph. Given disjoint source and sink sets, $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$, in a graph, a *paired k-disjoint path cover* joining S and T is a disjoint path cover $\{P_1, \ldots, P_k\}$ composed of k paths, in which each path P_i runs from s_i to t_i. In this paper, we characterize interval graphs that have a paired 2-disjoint path cover joining S and T for any possible configurations of source and sink sets S and T of size 2 each.

Keywords Disjoint path · Path cover · Disjoint path coverability · Interval graph · r-Scattering number

1 Introduction

One of the central issues in the study of interconnection networks is the detection of parallel paths, which is naturally related to routing among nodes and fault tolerance of the network [9,16]. An interconnection network is frequently modeled as a graph, where vertices and edges represent the nodes and

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communication links of the network, respectively. Parallel paths correspond to pairwise disjoint paths of the graph. Disjoint path is, moreover, a fundamental notion from which many graph properties can be deduced [2,16]. For example, the connectivity of a graph is closely related to the existence of disjoint paths in the graph.

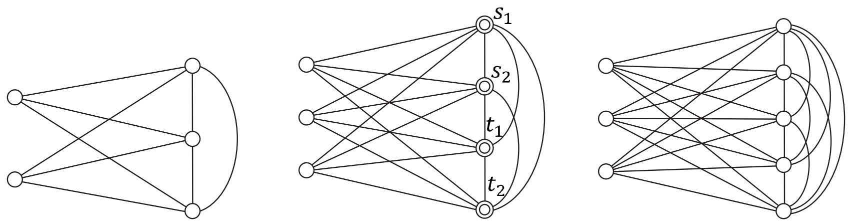
A disjoint path cover of a graph is a set of vertex-disjoint paths that altogether cover every vertex of the graph. The disjoint path cover problems are applicable in many areas such as software testing, database design, and code optimization [1,18]. In addition, the problem is concerned with applications where full utilization of network nodes is important [22]. For instance, basic communication problems for the dissemination of information, such as broadcasting and information gathering, require visiting every node of the network at least once. Since visiting a node more than once results in unnecessary overhead, a disjoint path cover can be employed to avoid this unsatisfactory situation.

Let G be a finite, simple undirected graph whose vertex and edge sets are denoted by $V(G)$ and $E(G)$, respectively. A *path cover* of G is a set of paths in G such that every vertex of G is contained in at least one path. A *disjoint path cover* (DPC for short) of G is a path cover in which every vertex of G is covered by exactly one path. Given disjoint subsets $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ of $V(G)$ for a positive integer k, a *many-to-many k-disjoint path cover* is a DPC composed of k paths, each of which joins a pair of source $s_i \in S$ and sink $t_j \in T$. If each source $s_i \in S$ must be joined to a specific sink $t_i \in T$, the DPC is called *paired*, and it is *unpaired* if no such constraint is imposed.

Definition 1 (Park et al. [23]) A graph G of order $n \geq 2k$ is *paired k-disjoint path coverable* if G has a paired (many-to-many) k-DPC joining S and T for any disjoint source and sink sets, S and T, of size k each.

Analogously, we can define *unpaired k-disjoint path coverable* graphs. Refer to Fig. 1 for examples. The other DPC type is a *one-to-many k-disjoint path cover* for disjoint subsets $S = \{s\}$ and $T = \{t_1, \ldots, t_k\}$, in which each path runs from the common source s to a sink t_j, $j \in \{1, \ldots, k\}$. When $S = \{s\}$ and $T = \{t\}$, a DPC composed of k paths, each of which joins s and t, is named *one-to-one k-disjoint path cover*. Also, we can define one-to-many k-disjoint path coverability and one-to-one k-disjoint path coverability for a graph of order $n \geq k + 1$ similarly.

The disjoint path coverability of a graph is closely related to the Hamiltonian properties (as well as the vertex connectivity) of the graph [25]. The problem of deciding if a general graph is k-disjoint path coverable, whatever its DPC type, is NP-complete for any fixed $k \geq 1$; it is polynomial-time reducible from the Hamiltonian Connectivity problem, which is shown to be NP-complete [5]. Also, the problem of deciding if there exists a k-DPC joining given source and sink sets in a general graph is NP-complete; it is reducible from the 2-Hamiltonian Path problem, which is NP-complete [7]. The disjoint path cover problems have been studied for various classes of graphs, including



(a) $G_{2,3}$ (b) $G_{3,4}$ (c) $G_{3,5}$

Fig. 1: Three complete split graphs [25]. The graphs $G_{2,3}$ and $G_{3,5}$ are both paired 2-disjoint path coverable, whereas $G_{3,4}$ is not even unpaired 2-disjoint path coverable.

recent studies on dense graphs [13], cube of connected graphs [21], balanced hypercubes [14,15], hypercube-like networks [17], directed graphs [4], k-ary n-cubes [10], and torus networks [11].

An *interval graph* is the intersection graph of a family $\mathcal{I}$ of intervals on the real line, where two vertices are connected with an edge if and only if their corresponding intervals intersect. The family $\mathcal{I}$ is usually called an *interval representation* for the graph. A *proper interval graph* is an interval graph with an interval representation in which none of the intervals properly contains another. The interval graphs that are Hamiltonian and that are Hamiltonian-connected, respectively, were characterized by Deogun et al. [6] and by Broersma et al. [3]. Also, the One-to-One k-Disjoint Path Coverability and the One-to-Many k-Disjoint Path Coverability of interval graphs were characterized by Li et al. [12] and by Park et al. [24]. The characterizations are all in terms of the scattering number. For a noncomplete graph G, the *scattering number* $\mathrm{sc}(G)$ of G is defined as

$$\mathrm{sc}(G) = \max\{c(G-X) - |X| : X \subset V(G),\ c(G-X) \geq 2\},$$

where $c(G-X)$ denotes the number of connected components in $G-X$. A vertex cut X of G that fulfills $c(G-X) - |X| = \mathrm{sc}(G)$ is called a *scattering set*. The scattering number of a complete graph K_n is defined to be $3-n$ as in [24].

As far as the authors know, studies on the Paired k-Disjoint Path Coverability of interval graphs cannot be found in the literature; however, a study on proper interval graphs has been known in [19] as follows: A proper interval graph G is paired k-disjoint path coverable for $k \geq 3$ if and only if G is $(2k-1)$-connected; G is paired 2-disjoint path coverable if and only if G is 3-connected and for all i with $1 < i < n-3$, (v_{i-1}, v_{i+3}) or (v_i, v_{i+4}) is an edge of G, where $(v_1, v_2, \ldots, v_n)$ is a consecutive ordering of the vertices of G. For the Unpaired k-Disjoint Path Coverability, it was proved in [20] that an interval graph G of order $n \geq 2k$ for $k \geq 2$ is unpaired k-disjoint path coverable if $\mathrm{sc}(G) \leq -k$. In [25], the authors proposed the notion of r-scattering number, a generalization of the scattering number, and proved that a noncomplete interval graph G of order $n \geq 4$ is unpaired 2-disjoint path coverable if and only if its 4-scattering number $\mathrm{sc}_4(G) \leq -2$.

Definition 2 (Park and Lim [25]) For a noncomplete graph G of order n and a nonnegative integer r with $r \leq n$, the *r-scattering number*, denoted $\mathrm{sc}_r(G)$, of G is the maximum of $c_r(G, X) - |X|$ over all vertex cuts X of G, where $c_r(G, X)$ is the maximum number of connected components in $G - X$ each of which contains no marked vertices when marking exactly r out of n vertices in G, i.e.,

$$c_r(G, X) = \begin{cases} c(G - X) & \text{if } |X| \geq r, \\ c(G - X) - h_{r-|X|}(G - X) & \text{if } |X| < r, \end{cases}$$

where $h_{r-|X|}(G - X)$ is the minimum number of connected components in $G - X$ whose total number of vertices is $r - |X|$ or more. A vertex cut X of G that fulfills $c_r(G, X) - |X| = \mathrm{sc}_r(G)$ is called an *r-scattering set*.

We begin our study on the Paired k-Disjoint Path Coverability of interval graphs by reviewing the two necessary conditions below. Note that a paired k-disjoint path coverable graph is, by definition, unpaired k-disjoint path coverable.

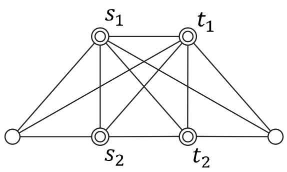
Lemma 1 (Park and Lim [25]) *If a noncomplete graph G of order $n \geq 2k$ for $k \geq 1$ is unpaired k-disjoint path coverable, then $\mathrm{sc}_{2k}(G) \leq -k$.*

Lemma 2 (Park et al. [22]) *If a graph G of order $n \geq 2k$ for $k \geq 1$ is paired k-disjoint path coverable, then G is $(2k - 1)$-connected.*

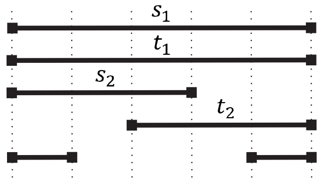
The conjunction of the two conditions '$\mathrm{sc}_{2k}(G) \leq -k$' of Lemma 1 and '$\kappa(G) \geq 2k - 1$' of Lemma 2 produces a necessary condition for G to be paired k-disjoint path coverable. One might guess that the conjunction is sufficient if G is limited to an interval graph, i.e., an interval graph G of order $n \geq 2k$ for $k \geq 2$ is paired k-disjoint path coverable if the conjunction is satisfied. Unfortunately, this is not always the case, as shown in Fig. 2. The interval graph of Fig. 2 is not paired 2-disjoint path coverable, but $\mathrm{sc}_4(G) = -2$ and $\kappa(G) = 3$. In order to tighten the conjunction, we introduce a concept called a snag cut defined below.

Definition 3 (Snag cut) A vertex cut X of size $2k-1$ for $k \geq 2$ in a graph G is a $(2k - 1)$-*snag cut* if $c(G - X) \leq k$ and there exist a connected component H_i of $G - X$ and a vertex $x_a \in X$ such that the subgraph H_i^a induced by $V(H_i) \cup \{x_a\}$ is noncomplete and $\mathrm{sc}(H_i^a) \geq k - c(G - X) + 1$.

In this paper, we prove that a noncomplete interval graph G of order $n \geq 4$ is paired 2-disjoint path coverable if and only if $\mathrm{sc}_4(G) \leq -2$, $\kappa(G) \geq 3$, and G has no 3-snag cut.



(a) an interval graph G



(b) its interval representation

Fig. 2: An interval graph G and its interval representation, where $\mathrm{sc}_4(G) = -2$ and $\kappa(G) = 3$. There is no paired 2-DPC in G composed of s_1–t_1 and s_2–t_2 paths, and there is a 3-snag cut $X = \{s_1, t_1, s_2\}$.

2 Preliminaries

A *path* from u to v, referred to as a u-v path, is a sequence $\langle w_1, \ldots, w_l \rangle$ of distinct vertices of a graph G such that $w_1 = u$, $w_l = v$, and $(w_i, w_{i+1}) \in E(G)$ for all $i \in \{1, \ldots, l-1\}$. If $l \geq 3$ and $(w_l, w_1) \in E(G)$, the sequence is called a *cycle*. A path that visits each vertex exactly once is a *Hamiltonian path*; a cycle that visits each vertex exactly once is a *Hamiltonian cycle*. A graph is *traceable* if a Hamiltonian path exists; a graph is *Hamiltonian* if a Hamiltonian cycle exists; a graph is *Hamiltonian-connected* if every two distinct vertices are joined by a Hamiltonian path.

A graph G' is a *subgraph* of a graph G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. The subgraph G' is a *spanning subgraph* of G if $V(G') = V(G)$; the subgraph G' is an *induced subgraph* of G if G' includes all the edges of G whose endvertices belong to $V(G')$. A spanning subgraph is obtained from G by edge deletions only; an induced subgraph is obtained from G by vertex deletions only. We denote by $N_G(v)$ the open neighborhood of a vertex $v \in V(G)$, i.e. $N_G(v) = \{u \in V(G) : (u, v) \in E(G)\}$.

A *connected component* of G is a maximal connected subgraph of G. A *vertex cut* of G is a set $X \subseteq V(G)$ such that $G - X$ has two or more connected components, where $G - X$ is the subgraph obtained from G by deleting all the vertices of X (or equivalently, $G - X$ is the subgraph of G induced by $V(G) \setminus X$). The *connectivity* of G, denoted $\kappa(G)$, is the minimum number of vertices whose removal results in a disconnected graph or a single vertex. So, $\kappa(G)$ is equal to the size of a minimum vertex cut if G is a noncomplete graph; $\kappa(G) = n - 1$ if G is a complete graph K_n. A graph G is *k-connected* if $\kappa(G) \geq k$. The graph-theoretic terms that are not defined here can be found in [2].

A vertex cut X of an interval graph is called a *guillotine cut* if there exists a point α, called a *cutting point*, on the x-axis such that α is contained in the interval I_v corresponding to v for all $v \in X$. A minimal vertex cut of an interval graph is always a guillotine cut, while the converse is not always true. Interval graphs are a well-studied class of graphs. One of the early characterizations of interval graphs is the following:

Theorem 1 (Gilmore and Hoffman [8]) *A graph G is an interval graph if and only if the maximal cliques of G can be linearly ordered such that, for every vertex v of G, the maximal cliques containing v occur consecutively.*

Let $C_1, \ldots, C_q$ be a linear ordering of the maximal cliques of an interval graph G such that each vertex of G appears in consecutive cliques only. Obviously, the graph G is noncomplete if and only if $q \geq 2$. Since C_1 and C_q are maximal cliques, C_1 and C_q each contains at least one vertex that does not occur in any other maximal clique. Let u_1 be such a vertex in C_1 and let u_n be such a vertex in C_q. The vertices u_1 and u_n, respectively, are referred to as a *left tip* and a *right tip*.

The r-scattering number of a graph G can be seen as bridging the gap between the scattering number and the connectivity of G, in that the 0-scattering number becomes $\mathrm{sc}(G)$ and the n-scattering number becomes $-\kappa(G)$.

Lemma 3 (Park and Lim [25]) *Let G be a noncomplete graph of order $n \geq 2$.*
(a) $\mathrm{sc}_{r+1}(G) \leq \mathrm{sc}_r(G) \leq \mathrm{sc}_{r+1}(G) + 1$ *for all* $r \in \{0, \ldots, n-1\}$.
(b) $-\kappa(G) = \mathrm{sc}_n(G) \leq \mathrm{sc}_{n-1}(G) \leq \cdots \leq \mathrm{sc}_1(G) \leq \mathrm{sc}_0(G) = \mathrm{sc}(G)$.
(c) If G is k-connected, then $\mathrm{sc}_r(G) = \mathrm{sc}(G)$ *for all* $r \in \{0, \ldots, k\}$.

Finally, the following relationship between the scattering number and the connectivity of a noncomplete graph will also be used for our proof:

Lemma 4 (Zhang and Wang [26]) *Let G be a noncomplete graph of order $n \geq 2$. Then,* $\mathrm{sc}(G) \geq 2 - \kappa(G)$.

3 Paired k-disjoint path coverability of general graphs

In addition to the conditions of Lemmas 1 and 2, we can derive a new necessary condition for an undirected graph G to be paired k-disjoint path coverable, as shown in Theorem 2. Roughly speaking, a graph is paired k-disjoint path coverable only if there exists no vertex cut of small size (Lemma 2) and the number of connected components produced by a vertex cut is sufficiently small compared to the size of the cut (Lemma 1). Furthermore, Theorem 2 states that each connected component H_i produced by a vertex cut X necessarily satisfies the following: the subgraph H_i^a induced by H_i and a vertex $x_a \in X$ has no vertex cut producing many connected components compared to the size of the cut.

Theorem 2 *Let G be a graph of order $n \geq 2k$ for $k \geq 2$. If G is paired k-disjoint path coverable, then G has no $(2k-1)$-snag cut.*

Proof Assume G is paired k-disjoint path coverable. Lemmas 1 and 2 lead to $\mathrm{sc}_{2k}(G) \leq -k$ and $\kappa(G) \geq 2k-1$. Suppose, to the contrary, that there exists a $(2k-1)$-snag cut X in G. Then, by the definition of a snag cut, $c(G-X) \leq k$ and the subgraph H_i^a induced by $V(H_i) \cup \{x_a\}$ is noncomplete and $\mathrm{sc}(H_i^a) \geq$

$k - c(G - X) + 1$ for some connected component H_i of $G - X$ and vertex $x_a \in X$. Let Y be a scattering set of H_i^a such that $c(H_i^a - Y) - |Y| = \mathrm{sc}(H_i^a) \geq k - c(G - X) + 1$. The subgraph H_i^a is connected. (Supposing otherwise leads to that H_i^a has two connected components H_i and $\{x_a\}$, meaning $X \setminus \{x_a\}$ would be a vertex cut of G of size $2k - 2$, contradicting the fact $\kappa(G) \geq 2k - 1$.) It follows that $Y \neq \emptyset$ and $Y \neq \{x_a\}$, leading to $Y \cap V(H_i)$ is a nonempty set. Let $X' = X \cup Y$.

Claim $H_i^a - Y$ has a connected component made of only one vertex x_a.

Proof of claim. Suppose for a contradiction that singleton $\{x_a\}$ is not a connected component of $H_i^a - Y$. Then, (i) $x_a \in Y$, or (ii) the connected component of $H_i^a - Y$ that contains x_a is of size 2 or more. If $x_a \in Y$ (i.e., $X \cap Y = \{x_a\}$), then $|Y| \geq 2$ (because $Y \cap V(H_i) \neq \emptyset$) and $|X'| = |X| + |Y| - 1 \geq 2k$. In addition, $G - X'$ has $(c(G - X) - 1) + c(H_i^a - Y)$ connected components. If $x_a \notin Y$ and the connected component of $H_i^a - Y$ that contains x_a is of size 2 or more, then $|X'| = |X| + |Y| \geq 2k$ and $G - X'$ has at least $(c(G - X) - 1) + c(H_i^a - Y)$ connected components. It follows that X' is a vertex cut of G with $|X'| \geq 2k$, so that

$$
\begin{aligned}
\mathrm{sc}_{2k}(G) &\geq c_{2k}(G, X') - |X'| = c(G - X') - |X'| \\
&\geq [(c(G - X) - 1) + c(H_i^a - Y)] - (|X| + |Y|) \\
&= [c(G - X) - |X| - 1] + [c(H_i^a - Y) - |Y|] \\
&\geq [c(G - X) - (2k - 1) - 1] + [k - c(G - X) + 1] \\
&= -k + 1,
\end{aligned}
$$

which contradicts the fact $\mathrm{sc}_{2k}(G) \leq -k$. Thus, the claim is proven. □

We can see from the claim that $X \cap Y = \emptyset$ and the number of connected components in $H_i - Y$ is equal to $c(H_i^a - Y) - 1$ because $H_i - Y$ is the same graph as $(H_i^a - Y) - \{x_a\}$. Thus, there are $(c(G - X) - 1) + (c(H_i^a - Y) - 1)$ connected components in $G - X'$, composed of the connected components of $G - X$ excluding H_i and the connected components of $H_i - Y$. Referring to Fig. 3, consider a paired k-DPC of G, in particular, joining $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ such that $s_1 = x_a$, t_1 is an arbitrary vertex in Y, and $S \cup (T \setminus \{t_1\}) = X$.

Each s_j–t_j path P_j in the DPC for $j \geq 2$ passes through at most $|V(P_j) \cap X'| - 1$ connected components of $G - X'$. We now show that the s_1–t_1 path P_1 is exceptionally via up to $|V(P_1) \cap X'| - 2$ connected components of $G - X'$. Observe that (i) P_1 passes through no connected component H_j of $G - X$ other than H_i (because $t_1 \in V(H_i)$ and every vertex of X is a source or a sink), and (ii) the source s_1 has no neighbor in a connected component of $H_i - Y$. This means that the successor of s_1 in the path P_1, possibly t_1, is contained in $Y \subset X'$; so, P_1 passes through at most $|V(P_1) \cap X'| - 2$ connected components of $G - X'$.

It follows that the k DPC paths collectively pass through at most $(|V(P_1) \cap X'| - 2) + \sum_{j=2}^{k} (|V(P_j) \cap X'| - 1) = |X'| - k - 1 = |Y| + k - 2$

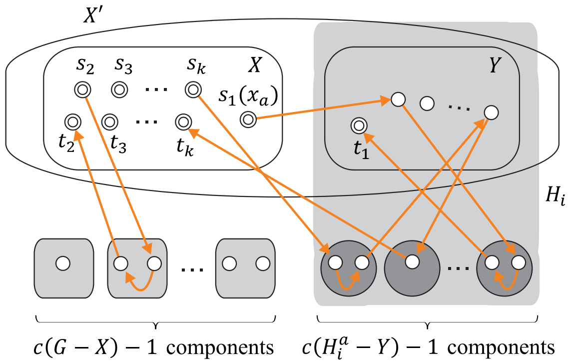


Fig. 3: Illustration of the proof of Theorem 2. The brightly shaded rectangles, including the big one, represent the connected components of $G - X$, and the darkly shaded disks represent the connected components of $H_i - Y$.

connected components in total, where $|X'| = |X| + |Y| = (2k - 1) + |Y|$. However, there are $(c(G - X) - 1) + (c(H_i^a - Y) - 1) \geq |Y| + k - 1$ connected components in $G - X'$, because Y is a scattering set of H_i^a such that $c(H_i^a - Y) - |Y| \geq k - c(G - X) + 1$. Therefore, there exists a connected component of $G - X'$ that does not covered by a DPC path, which contradicts the hypothesis that G is paired k-disjoint path coverable, completing the proof. ■

Restricting our attention to the class of graphs G with $\mathrm{sc}_{2k}(G) \leq -k$ and $\kappa(G) \geq 2k - 1$, some useful properties of a graph in the class are derived in the remaining part of this section.

Lemma 5 *Let G be a noncomplete graph of order $n \geq 2k$ for $k \geq 2$ such that $\mathrm{sc}_{2k}(G) \leq -k$ and $\kappa(G) \geq 2k - 1$. Then, (a) $\mathrm{sc}(G) \leq -k + 1$. (b) Each connected component H_i of $G - X$, where X is a $2k$-scattering set of order $2k - 1$, is a complete graph or $\mathrm{sc}(H_i) \leq 0$.*

Proof Lemma 3 (c) and (a) lead to $\mathrm{sc}(G) = \mathrm{sc}_{2k-1}(G) \leq \mathrm{sc}_{2k}(G) + 1$, proving (a). Next, suppose for a contradiction that H_i is not a complete graph and $\mathrm{sc}(H_i) \geq 1$. Then, there is a scattering set Y of H_i such that $c(H_i - Y) - |Y| = \mathrm{sc}(H_i) \geq 1$. In addition, we have $\mathrm{sc}_{2k}(G) = c_{2k}(G, X) - |X| = (c(G - X) - 1) - |X|$. It follows that $X' := X \cup Y$ would be a vertex cut of G with $|X'| \geq 2k$, leading to that $\mathrm{sc}_{2k}(G) \geq c_{2k}(G, X') - |X'| = c(G - X') - |X'| = (c(G - X) - 1 + c(H_i - Y)) - (|X| + |Y|) = \mathrm{sc}_{2k}(G) + \mathrm{sc}(H_i) \geq \mathrm{sc}_{2k}(G) + 1$, which is a contradiction. This completes the proof. ■

A $(2k-1)$-snag cut X of a graph G is a $2k$-scattering set if $c(G-X) = k$ and $\mathrm{sc}_{2k}(G) \leq -k$. This is because $\mathrm{sc}_{2k}(G) \geq c_{2k}(G, X) - |X| = (k-1) - (2k-1) = -k$, leading to $\mathrm{sc}_{2k}(G) = c_{2k}(G, X) - |X| = -k$ as required.

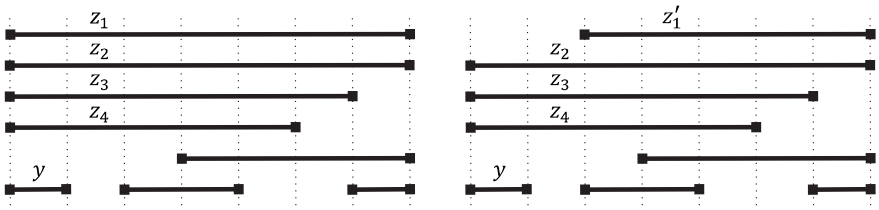
Lemma 6 *Let G be a noncomplete graph of order $n \geq 4$ such that $\mathrm{sc}_4(G) \leq -2$ and $\kappa(G) \geq 3$. Suppose G has a 3-snag cut X, so that $c(G-X)=2$ and the subgraph H_i^a induced by $V(H_i) \cup \{x_a\}$ is noncomplete and $\mathrm{sc}(H_i^a) \geq 1$ for some connected component H_i of $G-X$ and vertex $x_a \in X$. Then, H_i^a has a unique scattering set Y such that $|Y|=1$ and $Y = N_{H_i^a}(x_a)$.*

Proof Toward a contradiction, suppose there is a scattering set Y of H_i^a such that $|Y| \geq 2$ or $Y \neq N_{H_i^a}(x_a)$, and moreover $c(H_i^a - Y) - |Y| = \mathrm{sc}(H_i^a) \geq 1$. Note that H_i^a is connected due to $\kappa(G) \geq 3$, meaning $Y \neq \emptyset$. We first prove a claim: $x_a \notin Y$. Suppose otherwise, $Y' := Y \setminus \{x_a\}$ would be a vertex cut of H_i, leading to $\mathrm{sc}(H_i) \geq c(H_i - Y') - |Y'| = c(H_i^a - Y) - (|Y|-1) = \mathrm{sc}(H_i^a) + 1 \geq 2$, which contradicts the fact $\mathrm{sc}(H_i) \leq 0$ of Lemma 5(b), proving the claim. Then, the set $X' := (X \setminus \{x_a\}) \cup Y$ would be a vertex cut of G such that $c(G-X') = c(H_i^a - Y)$. If $|Y| \geq 2$, then $\mathrm{sc}_4(G) \geq c_4(G, X') - |X'| = c(G-X') - |X'| = c(H_i^a - Y) - (|Y|+2) = \mathrm{sc}(H_i^a) - 2 \geq -1$, which contradicts the hypothesis $\mathrm{sc}_4(G) \leq -2$. So, we have $Y \neq N_{H_i^a}(x_a)$ and $|Y|=1$. It follows that the vertex x_a is contained in a connected component of $H_i^a - Y$ of order at least 2. This implies that Y is also a vertex cut of H_i, leading to that $\mathrm{sc}(H_i) \geq c(H_i - Y) - |Y| \geq 2 - 1 = 1$, contradicting the fact $\mathrm{sc}(H_i) \leq 0$ of Lemma 5(b). Thus, the lemma is proven. ■

4 Paired 2-disjoint path coverability of interval graphs

In this section, we prove that combining the aforementioned three necessities results in a sufficient condition for an interval graph G and $k=2$. Specifically, we prove that a noncomplete interval graph G of order $n \geq 4$ is paired 2-disjoint path coverable if and only if $\mathrm{sc}_4(G) \leq -2$, $\kappa(G) \geq 3$, and G has no 3-snag cut. The sufficiency proof proceeds by induction on n. In order to build a paired 2-DPC of G joining prescribed source and sink sets in a recursive manner, we define several subgraphs of G that admit a paired 2-DPC and/or a Hamiltonian path. The paired 2-DPCs and Hamiltonian paths of the subgraphs are then combined into a required 2-DPC. We assume $n \geq 5$ because G is noncomplete and $\kappa(G) \geq 3$; we further assume $\mathrm{sc}(G) \leq -1$ due to Lemma 5(a). Here, we follow the approach taken in [25] to characterize interval graphs that are unpaired 2-disjoint path coverable.

If $\mathrm{sc}(G) \leq -2$, then $\kappa(G) \geq 2 - \mathrm{sc}(G) \geq 4$ by Lemma 4. In this case, we reduce our problem on G to a problem on a spanning subgraph G' of G such that $\mathrm{sc}_4(G') \leq -2$, $\mathrm{sc}(G') = -1$, $\kappa(G') = 3$, and G' has no 3-snag cut as follows: Let $\mathcal{I}$ denote an interval representation for G, and let $\mathrm{lp}(I_v)$ and $\mathrm{rp}(I_v)$, respectively, denote the left and right endpoints of an interval I_v corresponding to a vertex v. Let $Z = \{z_1, \ldots, z_r\}$ be the leftmost minimal cut of G (i.e., $Z = V(C_1) \cap V(C_2)$ for the linear ordering $C_1, \ldots, C_q$ of the maximal cliques of G such that each vertex of G appears in consecutive cliques only), and assume $\mathrm{rp}(I_{z_1}) \geq \cdots \geq \mathrm{rp}(I_{z_r})$. We define G' to be the intersection graph of a family $\mathcal{I}' := (\mathcal{I} \setminus \{I_{z_1}, \ldots, I_{z_{r-3}}\}) \cup \{I'_{z_1}, \ldots, I'_{z_{r-3}}\}$, where $\mathrm{lp}(I'_{z_i}) =$



(a) an interval representation for G (b) an interval representation for G'

Fig. 4: A noncomplete interval graph G with $\mathrm{sc}_4(G) \leq \mathrm{sc}(G) \leq -2$ and its spanning subgraph G' with $\mathrm{sc}_4(G') \leq -2$ and $\mathrm{sc}(G') = -1$, where $Z = \{z_1, z_2, z_3, z_4\}$ and $V(C_1) = Z \cup \{y\}$ [25].

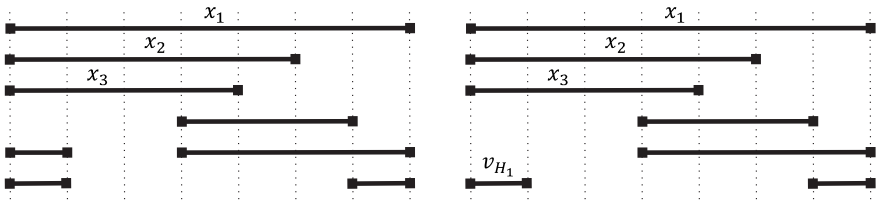
$\min_{v \in V(G) \setminus V(C_1)} \mathrm{lp}(I_v)$ and $\mathrm{rp}(I'_{z_i}) = \mathrm{rp}(I_{z_i})$ for $i \in \{1, \ldots, r-3\}$. Refer to Fig. 4 for an example. The graph G' is obviously a spanning subgraph of G with $\kappa(G') = 3$.

Lemma 7 *For a noncomplete interval graph G with $\mathrm{sc}_4(G) \leq \mathrm{sc}(G) \leq -2$, $\kappa(G) \geq 3$, and no 3-snag cut, let G' be the spanning subgraph of G defined above. Then, (a) $\mathrm{sc}_4(G') \leq -2$ and $\mathrm{sc}(G') = -1$ [25]; (b) G' has no 3-snag cut.*

Proof The subgraph G' has a unique vertex cut of size 3 by the construction of G' because G is 4-connected ($\kappa(G) \geq 2 - \mathrm{sc}(G) \geq 4$). Suppose for a contradiction that G' has a 3-snag cut Y. Then, Y must be equal to $\{z_{r-2}, z_{r-1}, z_r\}$. Let R_1 and R_2 be the connected component of $G' - Y$, where R_1 is to the left of R_2 (i.e., $R_1 = C_1 \setminus C_2$). A vertex $v \in V(R_1)$ is adjacent to every vertex of Y; in addition, there are at least two vertices (including z_1') in R_2 whose left endpoints are the minimum over all vertices of R_2. It follows that Y is not a 3-snag cut by Lemma 6, which is a contradiction. ■

Owing to Lemma 7, we assume hereafter that G is a noncomplete interval graph of order $n \geq 5$ such that $\mathrm{sc}_4(G) \leq -2$, $\mathrm{sc}(G) = -1$, $\kappa(G) \geq 3$, and G has no 3-snag cut. Lemma 3(c) leads to $\kappa(G) = 3$. Let $X = \{x_1, x_2, x_3\}$ be a minimum vertex cut of G, where $\mathrm{rp}(I_{x_3}) \leq \mathrm{rp}(I_{x_2}) \leq \mathrm{rp}(I_{x_1})$. The graph $G - X$ has two connected components, denoted H_1 and H_2, where H_1 is to the left of H_2. The cut X is also a 4-scattering set, as well as a scattering set, of G.

The first interval subgraph we consider is the *contraction* of H_1 in G, denoted by G/H_1, which is defined as a graph whose vertex set is $(V(G) \setminus V(H_1)) \cup \{v_{H_1}\}$, where v_{H_1} is a 'new' vertex not in G, and whose edge set is $\{(u,v) \in E(G) : u, v \in V(G) \setminus V(H_1)\} \cup \{(u, v_{H_1}) : (u,v) \in E(G), u \in V(G) \setminus V(H_1), v \in V(H_1)\}$. In other words, G/H_1 is a graph in which H_1 is replaced with a single vertex v_{H_1} such that v_{H_1} is adjacent to the union of the vertices to which the vertices of H_1 were originally adjacent. See Fig. 5 for an example. The graph G/H_1 is isomorphic to an induced subgraph of G, although G/H' for a connected subgraph H' of G is not in general isomorphic to a subgraph of G.



(a) an interval representation for G (b) an interval representation for G/H_1

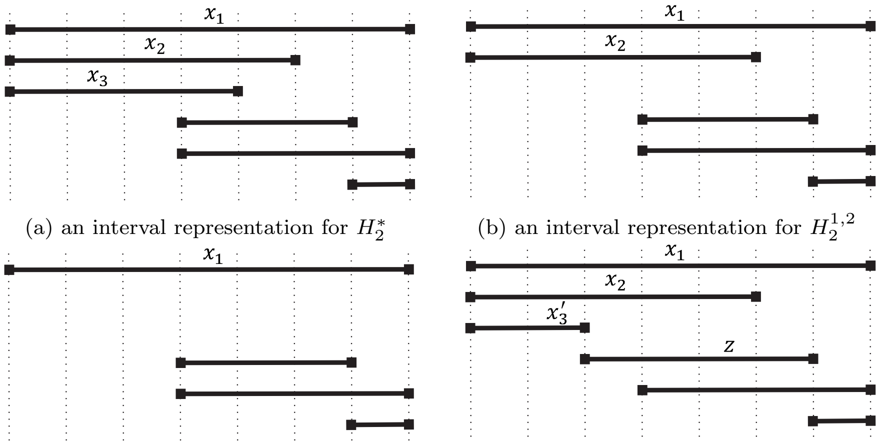
Fig. 5: An interval graph G and the contraction G/H_1 of H_1, where $\mathrm{sc}_4(G) = -2$ and $\mathrm{sc}(G) = -1$.

Lemma 8 *(a)* $\mathrm{sc}_4(G/H_1) \leq -2$ *and* $\kappa(G/H_1) = 3$*; (b)* G/H_1 *has no* 3*-snag cut.*

Proof It is obvious $\kappa(G/H_1) = 3$. Suppose to the contrary $\mathrm{sc}_4(G/H_1) \geq -1$, i.e., there is a 4-scattering set Y of G/H_1 such that $c_4(G/H_1, Y) - |Y| = \mathrm{sc}_4(G/H_1) \geq -1$. Let $Y' = Y \setminus \{v_{H_1}\}$ if $v_{H_1} \in Y$; $Y' = Y$ otherwise. Then, Y' would be a vertex cut of G/H_1 and also a vertex cut of G, leading to $|Y'| \geq 3$ and $c(G/H_1 - Y') = c(G - Y')$, meaning $c_4(G/H_1, Y') = c_4(G, Y')$. Moreover, if $Y' \neq Y$, then $c(G/H_1 - Y) \leq c(G/H_1 - Y')$, hence $c_4(G/H_1, Y) - 1 \leq c_4(G/H_1, Y')$. It follows that $\mathrm{sc}_4(G) \geq c_4(G, Y') - |Y'| = c_4(G/H_1, Y') - |Y'| \geq c_4(G/H_1, Y) - |Y| = \mathrm{sc}_4(G/H_1) \geq -1$, which is a contradiction, proving (a).

Now, suppose for a contradiction that G/H_1 has a 3-snag cut Y. Then, there are exactly two connected components R_1, R_2 in $G/H_1 - Y$, where R_1 is to the left of R_2; also, by Lemma 6, there exist vertices $y \in Y$ and $w \in V(R_i)$ for some R_i with $|V(R_i)| \geq 2$ such that $N_{G/H_1}(y) \cap V(R_i) = \{w\}$. Moreover, Y is a minimal and guillotine cut of G/H_1 because $|Y| = \kappa(G/H_1) = 3$; so, there is a cutting point α of Y. The fact that the vertex v_{H_1} is a left tip of G/H_1 leads to $\alpha > \mathrm{rp}(I_{v_{H_1}})$ and $v_{H_1} \in V(R_1)$. If $w \in V(R_2)$, then Y would be a 3-snag cut of G; if $w \in V(R_1)$, then $|V(R_1)| \geq 2$ and $w \neq v_{H_1}$, hence Y would be also a 3-snag cut of G. Both contradict the assumption that G has no 3-snag cut, completing the proof. ■

In addition to G/H_1, there are some other interval subgraphs on which our subproblems will be defined. Let H_i^* denote the subgraph of G induced by $V(H_i) \cup X$, and let H_i^a and $H_i^{a,b}$, where $a, b \in \{1, 2, 3\}$ and $a \neq b$, be the subgraphs of G induced by $V(H_i) \cup \{x_a\}$ and by $V(H_i) \cup \{x_a, x_b\}$, respectively. Refer to Fig. 6 for examples. The final subgraph, denoted $H_2^\dagger$, is a spanning subgraph of H_2^* defined as follows: Let z be a vertex in $N_{H_2^*}(x_3) \cap V(H_2)$ such that $\mathrm{rp}(I_z) \leq \mathrm{rp}(I_v)$ for all $v \in N_{H_2^*}(x_3) \cap V(H_2)$. We can assume that $\mathrm{lp}(I_z) < \mathrm{lp}(I_v)$ for all $v \in V(H_2) \setminus \{z\}$, because replacing I_z with a new interval I'_z such that $\mathrm{rp}(I'_z) = \mathrm{rp}(I_z)$ and $\max_{v \in V(H_1)} \mathrm{rp}(I_v) < \mathrm{lp}(I'_z) < \min_{v \in V(H_2)} \mathrm{lp}(I_v)$ results in another interval representation for the graph H_2^*. We define $H_2^\dagger$ to be the intersection graph of the family of intervals obtained from the interval representation for H_2^* by replacing the interval I_{x_3} with a new interval I'_{x_3} having $\mathrm{lp}(I'_{x_3}) = \mathrm{lp}(I_{x_3})$ and $\mathrm{rp}(I'_{x_3}) = \mathrm{lp}(I_z)$.



(a) an interval representation for H_2^* (b) an interval representation for $H_2^{1,2}$ (c) an interval representation for H_2^1 (d) an interval representation for $H_2^\dagger$

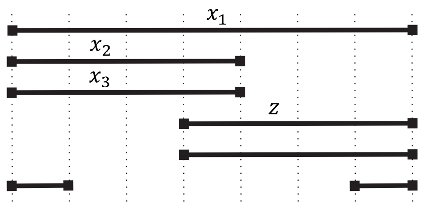
Fig. 6: The interval subgraphs H_2^*, $H_2^{1,2}$, H_2^1, and $H_2^\dagger$ of the graph shown in Fig. 5(a).

Lemma 9 (Park and Lim [25]) *Each of the subgraphs H_i^*, $H_2^{1,2}$, and $H_2^\dagger$ is a complete graph, or its 4-scattering number and connectivity respectively are no more than -2 and no less than 3.*

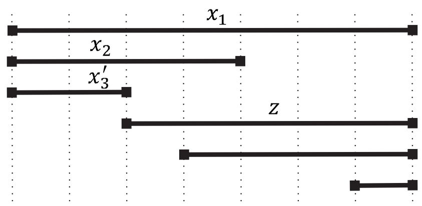
Lemma 10 *(a) H_i^* has no 3-snag cut. (b) $H_2^{1,2}$ has no 3-snag cut.*

Proof To prove (a), it suffices to set $i = 2$. Suppose to the contrary that H_2^* has a 3-snag cut Y. Then, there are two connected components R_1, R_2 in $H_2^* - Y$, where R_1 is to the left of R_2; also, there exist vertices $y \in Y$ and $w \in V(R_i)$ for some R_i with $|V(R_i)| \geq 2$ such that $N_{H_2^*}(y) \cap V(R_i) = \{w\}$. Moreover, Y is a minimal and guillotine cut of H_2^*; so, there is a cutting point α of Y. Supposing $\alpha \leq \mathrm{rp}(I_{x_3})$ leads to $x_1, x_2, x_3 \in Y$, i.e., $Y = \{x_1, x_2, x_3\}$, which contradicts the fact that $H_2 = H_2^* - Y$ is connected. Hence, we have $\alpha > \mathrm{rp}(I_{x_3})$ and $x_3 \in V(R_1)$. Clearly, Y is also a vertex cut of G producing two connected components R_1', R_2', where R_1' is made of R_1 and H_1 and $R_2' = R_2$. Furthermore, $|V(R_i')| \geq 2$ and $N_G(y) \cap V(R_i') = \{w\}$, meaning Y would be a 3-snag cut of G, which is a contradiction.

The proof for (b) is similar to that for (a). Suppose $H_2^{1,2}$ has a 3-snag cut Y. Then, there are two connected components R_1, R_2 in $H_2^{1,2} - Y$, where R_1 is to the left of R_2; also, there exist vertices $y \in Y$ and $w \in V(R_i)$ for some R_i with $|V(R_i)| \geq 2$ such that $N_{H_2^{1,2}}(y) \cap V(R_i) = \{w\}$. There is a cutting point α of Y because Y is a minimum cut of $H_2^{1,2}$. Supposing $\alpha \leq \mathrm{rp}(x_2)$ leads to $x_1, x_2 \in Y$; so, $Y' := Y \cup \{x_3\}$ would be a vertex cut of G producing 3 connected components H_1, R_1 and R_2, which contradicts the assumption $\mathrm{sc}_4(G) \leq -2$. Thus, $\alpha > \mathrm{rp}(x_2)$ and $x_2 \in V(R_1)$. If $w \in V(R_2)$ or if $w \in V(R_1)$ and $w \neq x_2$, then Y would be a 3-snag cut of G, which is a contradiction. Finally, supposing $w \in V(R_1)$ and $w = x_2$ leads to $x_1 \in Y$. So, $Y' := (Y \setminus \{y\}) \cup \{x_2\}$ would be a



(a) an interval representation for G_7



(b) $H_2^\dagger$ with respect to X

Fig. 7: An interval representation for an interval graph of order 7, named G_7, with a vertex cut $X = \{x_1, x_2, x_3\}$ of order 3, where $\mathrm{sc}_4(G) = -2$ and $\mathrm{sc}(G) = -1$. The graph G_7 has no 3-snag cut, but $H_2^\dagger$ has a 3-snag cut $\{x_1, x_2, z\}$.

vertex cut of $H_2^{1,2}$, meaning $Y' \cup \{x_3\}$ would be a vertex cut of G producing at least 3 connected components H_1, $R_1 - \{x_2\}$, and $R_2 + \{y\}$, which contradicts the assumption $\mathrm{sc}_4(G) \leq -2$. This completes the proof. ■

In contrast to the three subgraphs G/H_1, H_i^* and $H_2^{1,2}$ discussed so far, the subgraph $H_2^\dagger$ may have a 3-snag cut even if G does not have, as shown in Fig. 7. Nonetheless, if additional constraints are imposed, as shown in Lemma 11, the nonexistence of a 3-snag cut in $H_2^\dagger$ is maintained.

Lemma 11 *Suppose every vertex cut of size* 3 *in* G *produces a one-vertex connected component and* G *is not isomorphic to the graph* G_7 *shown in Fig. 7(a). Then, the subgraph* $H_2^\dagger$ *has no* 3*-snag cut.*

Proof From the hypothesis of this lemma, we have $|V(H_1)| = 1$ or $|V(H_2)| = 1$. Supposing $|V(H_2)| = 1$ leads to that $H_2^\dagger$ is isomorphic to a complete graph K_4, meaning $H_2^\dagger$ has no 3-snag cut. So, assume $|V(H_1)| = 1$ and $|V(H_2)| \geq 2$ hereafter. Suppose to the contrary that $H_2^\dagger$ has a 3-snag cut Y. Then, there are two connected components R_1, R_2 in $H_2^\dagger - Y$, where R_1 is to the left of R_2. Moreover, $H_2^\dagger$ has 6 or more vertices, meaning $n \geq 7$. Let $n = 7$ first. Then, H_2 is isomorphic to a complete graph K_3 because H_2 is complete or $\mathrm{sc}(H_2) \leq 0$ by Lemma 5(b). The hypothesis that $\kappa(G) = 3$ and G has no 3-snag cut leads to $|N_G(x_1) \cap V(H_2)| = 3$ and $|N_G(x_i) \cap V(H_2)| \geq 2$ for $i \in \{2, 3\}$. If $|N_G(x_2) \cap V(H_2)| = 2$, then G must be isomorphic to the graph G_7; otherwise, $|N_G(x_2) \cap V(H_2)| = 3$ and $H_2^\dagger$ has a unique cut $\{x_1, x_2, z\}$ of size 3, which is obviously not a 3-snag cut, completing the proof for $n = 7$.

The proof for $n \geq 8$ is similar to that of Lemma 10. The snag cut Y is a minimum cut of $H_2^\dagger$; so, there is a cutting point α of Y. Also, there exist vertices $y \in Y$ and $w \in V(R_i)$ for some R_i with $|V(R_i)| \geq 2$ such that $N_{H_2^\dagger}(y) \cap V(R_i) = \{w\}$. Supposing $\alpha \leq \mathrm{rp}(I'_{x_3})$ leads to $Y = \{x_1, x_2, x_3'\}$, which contradicts the fact that Y is a vertex cut of $H_2^\dagger$. Thus, $\alpha > \mathrm{rp}(I'_{x_3})$ and $x_3' \in V(R_1)$. Firstly, suppose $\mathrm{rp}(I'_{x_3}) < \alpha \leq \mathrm{rp}(I_{x_3})$. It follows that $x_1, x_2 \in Y$ and $x_3' \notin Y$. Supposing $z \notin Y$ implies $z \in V(R_1)$, hence $Y' := Y \cup \{x_3\}$ would be a vertex cut of G producing at least 3 connected components H_1, $R_1 - \{x_3'\}$, and R_2, leading to $\mathrm{sc}_4(G) \geq c_4(G, Y') - |Y'| = c(G - Y') - |Y'| \geq -1$, which is a contradiction. So, $Y = \{x_1, x_2, z\}$, meaning $|V(R_1)| = 1$ and $w \in V(R_2)$. If $y \in \{x_1, x_2\}$, then $(Y \setminus \{y\}) \cup \{w\}$ would be a vertex cut of G producing two connected components, $H_1 + \{x_3, y\}$ of size 3 and $R_2 - \{w\}$ of size $n - 6 \geq 2$, which contradicts the hypothesis that every vertex cut of size 3 in G produces a one-vertex connected component. If $y = z$, then $(Y \setminus \{y\}) \cup \{w, x_3\}$ would be a vertex cut of G producing at least 3 connected components, H_1, $\{z\}$, and $R_2 - \{w\}$, which contradicts the assumption $\mathrm{sc}_4(G) \leq -2$.

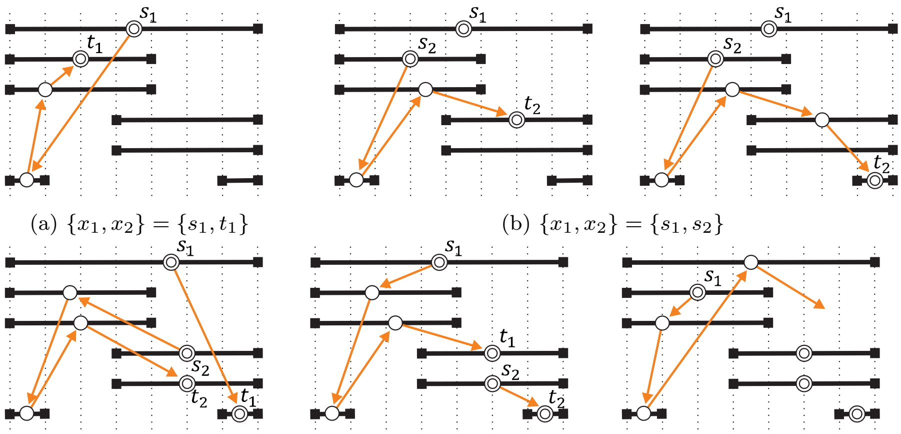
Secondly, suppose $\alpha > \mathrm{rp}(I_{x_3})$. Then, Y would be a vertex cut of G, implying $|V(R_2)| = 1$ and $|V(R_1)| = (n - |V(H_1)|) - (|Y| + |V(R_2)|) = n - 5 \geq 3$ by the hypothesis of the lemma. It follows $w \in V(R_1)$. If $\mathrm{lp}(I_y) \leq \mathrm{rp}(I'_{x_3})$ (i.e., $I_y \cap I'_{x_3} \neq \emptyset$), or if $\mathrm{rp}(I_{x_3}) < \mathrm{lp}(I_y)$ (i.e., $I_y \cap I_{x_3} = \emptyset$), then Y would also be a 3-snag cut of G. Thus, we have $\mathrm{rp}(I'_{x_3}) < \mathrm{lp}(I_y) \leq \mathrm{rp}(I_{x_3})$. We prove a claim: $x_1 \in Y$ and $x_2 \notin Y$. Supposing $x_1 \notin Y$ leads to $x_1, x_2 \in V(R_1)$ and both are neighbors of y because $I_y \cap I_{x_3} \neq \emptyset$, which violates the choice of y. Also, supposing $x_2 \in Y$ leads to $x_1 \in Y$ and thus $Y \cup \{x_3\}$ would be a vertex cut of G that produces at least 3 connected components, which contradicts the assumption $\mathrm{sc}_4(G) \leq -2$, thereby proving the claim. Then, we can see that the vertex w must be x_2 (because $I_y \cap I_{x_2} \neq \emptyset$) and moreover, $I_v \cap I_y = \emptyset$ (i.e., $\mathrm{rp}(I_v) < \mathrm{lp}(I_y)$) for every $v \in V(R_1) \setminus \{x_2, x'_3\}$. Recall $|V(R_1)| \geq 3$. It follows that $(Y \setminus \{y\}) \cup \{x_2, x_3\}$ would be a vertex cut of G producing at least 3 connected components, H_1, $R_1 - \{x_2, x'_3\}$, and $R_2 + \{y\}$, which contradicts the assumption $\mathrm{sc}_4(G) \leq -2$. This completes the entire proof. ∎

Before we go into the proof of our main theorem, we employ the study on the existence of a Hamiltonian path running from x_a in the subgraphs H_2^a and $H_2^{a,b}$. Note that the condition, $|N_G(x_3) \cap V(H_2)| \geq 2$ or $|V(H_2)| = 1$, of Lemma 12 below is always satisfied because G has no 3-snag cut.

Lemma 12 (Park and Lim [25]) *(a) For $a \in \{1, 2\}$, the subgraph H_2^a has a Hamiltonian x_a–y path for every $y \in V(H_2)$. Moreover, if $|N_G(x_3) \cap V(H_2)| \geq 2$ or $|V(H_2)| = 1$, then H_2^3 has a Hamiltonian x_3–y path for every $y \in V(H_2)$. (b) The subgraph $H_2^{a,b}$ has a Hamiltonian x_a–x_b path for distinct $a, b \in \{1, 2, 3\}$.*

Theorem 3 *A noncomplete interval graph G of order $n \geq 4$ is paired 2-disjoint path coverable if and only if $\mathrm{sc}_4(G) \leq -2$, $\kappa(G) \geq 3$, and G has no 3-snag cut.*

Proof The necessity part is due to Lemmas 1, 2 and Theorem 2. The sufficiency proof is by induction on n. Assume that $\mathrm{sc}_4(G) \leq -2$, $\kappa(G) \geq 3$, and G has no 3-snag cut. Lemma 5(a) leads to $\mathrm{sc}(G) \leq -1$. If $\mathrm{sc}(G) \leq -2$, then there is a spanning subgraph G' of G such that $\mathrm{sc}_4(G') \leq -2$, $\mathrm{sc}(G') = -1$, $\kappa(G') = 3$, and G' has no 3-snag cut by Lemma 7. So, we further assume $\mathrm{sc}(G) = -1$ and thus $\kappa(G) = 3$. As before, let $X = \{x_1, x_2, x_3\}$ denote a vertex cut of G with $\mathrm{rp}(I_{x_3}) \leq \mathrm{rp}(I_{x_2}) \leq \mathrm{rp}(I_{x_1})$, and let H_1, H_2 be the connected components of $G - X$, where H_1 is to the left of H_2. Assume w.l.o.g. that $|V(H_1)|, |V(H_2)| \geq 2$,



(a) $\{x_1, x_2\} = \{s_1, t_1\}$ (b) $\{x_1, x_2\} = \{s_1, s_2\}$

(c) $x_1 = s_1$ and $x_2 \notin S \cup T$ (d) $x_2 = s_1$ and $x_1 \notin S \cup T$

Fig. 8: Building a paired 2-DPC in the graph G_7 in case where both vertices in $V(H_1) \cup \{x_3\}$ are nonterminals. Actually, G_7 is paired 2-disjoint path coverable.

or every vertex cut of size 3 in G produces a one-vertex connected component. Given disjoint source and sink sets, $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$ in G, we will build a paired 2-DPC joining them. There are three cases depending on the distribution of terminals, sources and sinks.

Case 1: *There is a connected component, say H_1, of $G - X$ that contains no terminal.* If $|V(H_1)| \geq 2$, then there exists a paired 2-DPC joining S and T in the contraction G/H_1 by the induction hypothesis, because $\mathrm{sc}_4(G/H_1) \leq -2$, $\kappa(G/H_1) \geq 3$, and G/H_1 has no 3-snag cut by Lemma 8. It suffices to replace the subpath $\langle x_a, v_{H_1}, x_b \rangle$, where $a, b \in \{1, 2, 3\}$, of a path in the 2-DPC with a Hamiltonian x_a–x_b path of the subgraph induced by $V(H_1) \cup \{x_a, x_b\}$, which exists by Lemma 12. So, assume $|V(H_1)| = 1$ hereafter in this case, meaning every vertex cut of size 3 in G produces a one-vertex connected component. Firstly, suppose x_3 is a nonterminal. If G is isomorphic to the graph G_7 of Fig. 7(a), we can build a paired 2-DPC joining S and T, as shown in Fig. 8. So, assume G is not isomorphic to G_7. In the subgraph $H_2^{\dagger}$ of G, there exists a paired 2-DPC $\mathcal{P}$ joining S and T by the induction hypothesis and Lemmas 9 and 11. A path in $\mathcal{P}$ that passes through x'_3 as an intermediate vertex visits an edge (x'_3, x_a) for some $a \in \{1, 2\}$ because $N_{H_2^{\dagger}}(x'_3) = \{x_1, x_2, z\}$. Replacing the subpath $\langle x'_3, x_a \rangle$ with a Hamiltonian x_3–x_a path of the subgraph induced by $V(H_1) \cup \{x_3, x_a\}$, which exists by Lemma 12, results in a paired 2-DPC of G joining S and T.

Now, suppose x_3 is a terminal, say source s_1. If some x_a, $a \in \{1, 2\}$, is a nonterminal, it suffices to combine a Hamiltonian x_3–x_a path of the subgraph induced by $V(H_1) \cup \{x_3, x_a\}$ with a paired 2-DPC of $H_2^{1,2}$ composed of x_a–t_1 and s_2–t_2 paths, which exists by the induction hypothesis and Lemmas 9 and 10, into a required 2-DPC of G. (Obviously, the paired 2-DPC of $H_2^{1,2}$ also exists in case when $H_2^{1,2}$ is a complete graph.) Suppose otherwise, i.e., $x_1, x_2 \in S \cup T$. If $x_b = t_1$ for some $b \in \{1,2\}$, it suffices to build two Hamiltonian paths: a Hamiltonian x_3–x_b path in the subgraph induced by $V(H_1) \cup \{x_3, x_b\}$ and a Hamiltonian x_a–y path in H_2^a for some a with $\{a,b\} = \{1,2\}$ and a terminal y contained in H_2. The Hamiltonian x_a–y path exists by Lemma 12. Analogously, if $\{x_1, x_2\} = \{s_2, t_2\}$, then it suffices to build a Hamiltonian x_1–x_2 path in the induced subgraph by $V(H_1) \cup \{x_1, x_2\}$ and a Hamiltonian x_3–y path in H_2^3 for a terminal y in H_2.

Case 2: *There is a connected component, say H_2, of $G - X$ that contains a single terminal, say s_1.* In this case, there exists a nonterminal x_a for some $a \in \{1,2,3\}$; supposing otherwise leads to $V(H_1) \cap (S \cup T) = \emptyset$, reducing to Case 1. It suffices to combine a Hamiltonian s_1–x_a path of H_2^a and a paired 2-DPC of H_1^* composed of x_a–t_1 and s_2–t_2 paths. The paired 2-DPC exists by the induction hypothesis and Lemmas 9 and 10.

Case 3: *Each connected component of $G - X$ contains two terminals.* Assume H_1 contains a source s_1 and a terminal y other than s_1, where y is assumed to be t_1 or s_2. There exists a paired 2-DPC of H_1^* made of s_1–x_1 and y–x_2 paths by the induction hypothesis and Lemmas 9 and 10. If $y = t_1$, it suffices to combine the 2-DPC of H_1^* with a paired 2-DPC of $H_2^{1,2}$ composed of x_1–x_2 and s_2–t_2 paths; if $y = s_2$, it suffices to combine the 2-DPC of H_1^* with a paired 2-DPC of $H_2^{1,2}$ composed of x_1–t_1 and x_2–t_2 paths. This completes the entire proof. ■

5 Concluding remarks

In this paper, we have proved that a noncomplete interval graph G of order $n \geq 4$ is paired 2-disjoint path coverable if and only if $\mathrm{sc}_4(G) \leq -2$, $\kappa(G) \geq 3$, and G has no 3-snag cut. The r-scattering number of an interval graph can be computed in polynomial time [25], so the necessary and sufficient condition can be checked in polynomial time. Also, we can build a paired 2-DPC of an interval graph that is paired 2-disjoint path coverable in polynomial time, according to the proof of Theorem 3. From an algorithmic point of view, we have identified a subclass of interval graphs, in which a graph can be recognized in polynomial time and moreover, a paired 2-DPC joining given source and sink sets in the graph can be built also in polynomial time. Defining a broader graph class that admits an efficient recognition algorithm and an efficient algorithm for constructing a disjoint path cover is a future work. As for the interval graphs, it is open to characterize interval graphs that are paired k-disjoint path coverable for $k \geq 3$. Fig. 9 indicates that it is not enough to simply generalize the condition of Theorem 3 to $\mathrm{sc}_{2k}(G) \leq -k$, $\kappa(G) \geq 2k-1$, and G has no $(2k-1)$-snag cut.

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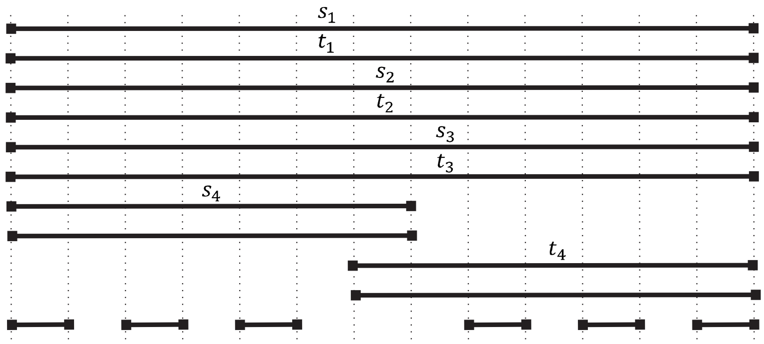

Fig. 9: An interval graph G that is not paired 4-disjoint path coverable, where $\mathrm{sc}_8(G) = \max\{4-8, 6-10\} = -4$, $\kappa(G) = 8$, and G has no 7-snag cut.

Data availability Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Conflict of interest The authors declare that they have no conflict of interest.

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