# Characterization of interval graphs that are unpaired 2-disjoint path coverable

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## Abstract

Given disjoint source and sink sets, $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$, in a graph $G$, an *unpaired $k$-disjoint path cover* joining $S$ and $T$ is a set of pairwise vertex-disjoint paths $\{P_1, \ldots, P_k\}$ that altogether cover every vertex of the graph, in which $P_i$ is a path from source $s_i$ to some sink $t_j$. In terms of a generalized scattering number, named an *$r$-scattering number*, we characterize interval graphs that have an unpaired 2-disjoint path cover joining $S$ and $T$ for any possible configurations of source and sink sets $S$ and $T$ of size 2 each. Also, it is shown that the $r$-scattering number of an interval graph can be computed in polynomial time.

*Keywords:* Disjoint path, path cover, path partition, scattering number, $r$-scattering number, interval graphs.

## 1. Introduction

Let $G$ be a finite, simple undirected graph whose vertex and edge sets are denoted by $V(G)$ and $E(G)$, respectively. A *path* from $u \in V(G)$ to $v \in V(G)$, referred to as a $u$-$v$ path, is a sequence $\langle w_1, \ldots, w_l \rangle$ of distinct vertices of $G$ such that $w_1 = u$, $w_l = v$, and $(w_i, w_{i+1}) \in E(G)$ for all $i \in \{1, \ldots, l-1\}$. If $l \geq 3$ and $(w_l, w_1) \in E(G)$, the sequence is called a *cycle*. A *path cover* of a graph $G$ is a set of paths in $G$ such that every vertex of $G$ is contained in at least one path. A *disjoint path cover* (DPC for short) of $G$ is a set of vertex-disjoint paths that altogether cover every vertex of $G$. This paper is concerned with a DPC in which each path runs from a prescribed source to a prescribed sink.

Given disjoint subsets $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ of $V(G)$ for a positive integer $k$, a *many-to-many $k$-disjoint path cover* is a DPC composed of $k$ paths, each of which joins a pair of source $s_i \in S$ and sink $t_j \in T$. If each

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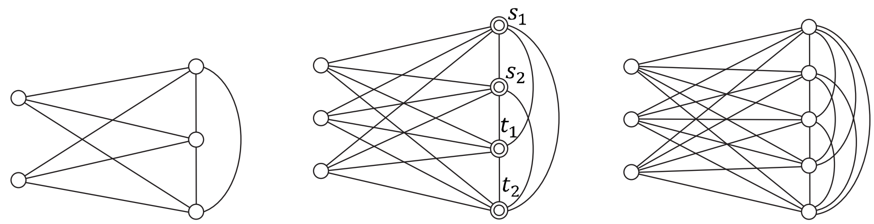



(a) $G_{2,3}$ (b) $G_{3,4}$ (c) $G_{3,5}$

Fig. 1: Three complete split graphs. The graph $G_{3,4}$ is not unpaired 2-disjoint path coverable, whereas $G_{2,3}$ and $G_{3,5}$ both are (un)paired 2-disjoint path coverable.

source $s_i \in S$ must be joined to a specific sink $t_i \in T$, the DPC is called *paired*, and it is *unpaired* if no such constraint is imposed. The other DPC type is a *one-to-many $k$-disjoint path cover* for disjoint subsets $S = \{s\}$ and $T = \{t_1, \ldots, t_k\}$, in which each path runs from the common source $s$ to a sink $t_j$, $j \in \{1, \ldots, k\}$. When $S = \{s\}$ and $T = \{t\}$, a DPC composed of $k$ paths, each of which joins $s$ and $t$, is named *one-to-one $k$-disjoint path cover*.

**Definition 1** (Park et al. [20])**.** A graph $G$ of order $n \geq 2k$ is *unpaired $k$-disjoint path coverable* if $G$ has an unpaired (many-to-many) $k$-DPC joining $S$ and $T$ for any disjoint source and sink sets, $S$ and $T$, of size $k$ each.

Analogously, we can define *paired $k$-disjoint path coverable* graphs. Refer to Fig. 1 for examples. In addition, a graph $G$ of order $n \geq k+1$ is *one-to-many $k$-disjoint path coverable* if for any disjoint subsets $S$ and $T$ of $V(G)$ with $|S| = 1$ and $|T| = k$, there exists a one-to-many $k$-DPC joining $S$ and $T$. A graph $G$ of order $n \geq k+1$ is *one-to-one $k$-disjoint path coverable* if for any disjoint subsets $S$ and $T$ of $V(G)$ with $|S| = |T| = 1$, there exists a one-to-one $k$-DPC joining $S$ and $T$. The disjoint path coverability of a graph is closely related to the Hamiltonian properties (as well as the vertex connectivity) of the graph. A graph of order $n \geq 3$, for instance, is one-to-many 2-disjoint path coverable if and only if it is Hamiltonian-connected. Furthermore, a graph of order $n \geq 3$ is one-to-one 2-disjoint path coverable if and only if it is Hamiltonian.

An *interval graph* is the intersection graph of a family $\mathcal{I}$ of intervals on the real line, where two vertices are connected with an edge if and only if their corresponding intervals intersect. The family $\mathcal{I}$ is usually called an *interval representation* for the graph. A *proper interval graph* is an interval graph with an interval representation in which none of the intervals properly contains another. The interval graphs that are Hamiltonian/traceable were characterized by Deogun et al. [8]; also, those that are Hamiltonian-connected were characterized by Broersma et al. [5] (Theorem 1 below). The characterizations are all in terms of the scattering number. For a noncomplete graph $G$, the *scattering number* $\mathrm{sc}(G)$ of $G$ is defined as

$$\mathrm{sc}(G) = \max\{c(G-X) - |X| : X \subset V(G),\ c(G-X) \geq 2\},$$

where $c(G-X)$ denotes the number of connected components in $G-X$. A vertex cut $X$ of $G$ that fulfills $c(G - X) - |X| = \mathrm{sc}(G)$ is called a *scattering set*. For a complete graph $K_n$ of order $n$, we set $\mathrm{sc}(K_n) = 3 - n$ in this paper, so that the scattering numbers of the $n$-vertex graphs form a consecutive set $\{3 - n, 4 - n, \ldots, n\}$ [21].

**Theorem 1** (Broersma et al. [5])**.** *An interval graph $G$ of order $n \geq 3$ is Hamiltonian-connected if and only if* $\mathrm{sc}(G) \leq -1$ *or $G$ is isomorphic to a complete graph $K_3$.*

Extending Theorem 1, the one-to-one $k$-disjoint path coverability and one-to-many $k$-disjoint path coverability of interval graphs were characterized by Li et al. [13] and by Park et al. [21] (Theorems 2 and 3 below). Both characterizations can be described in terms of the scattering number. In addition, a sufficient condition for an interval graph to be unpaired $k$-disjoint path coverable was established, as shown in Theorem 4.

**Theorem 2** (Li and Wu [13] and Park et al. [21])**.** *For $k \geq 2$, an interval graph $G$ of order $n \geq k + 1$ is one-to-one $k$-disjoint path coverable if and only if* $\mathrm{sc}(G) \leq 2 - k$.

**Theorem 3** (Li and Wu [13] and Park et al. [21])**.** *For $k \geq 2$, an interval graph $G$ of order $n \geq k + 1$ is one-to-many $k$-disjoint path coverable if and only if* $\mathrm{sc}(G) \leq 1 - k$ *or $G$ is isomorphic to a complete graph $K_{k+1}$.*

**Theorem 4** (Park [18])**.** *Let $G$ be an interval graph of order $n \geq 2k$ for $k \geq 2$. If* $\mathrm{sc}(G) \leq -k$, *then $G$ is unpaired $k$-disjoint path coverable.*

The disjoint path cover problems are applicable in many areas such as software testing, database design, and code optimization [1, 16]. The problems have been studied for various classes of graphs, including recent studies on dense graphs [14], cube of connected graphs [19], balanced hypercubes [15], directed graphs [6], and torus networks [12, 17]. The one-to-one and one-to-many $k$-disjoint path coverability of interval graphs can be checked in linear time thanks to the linear-time algorithms for finding the scattering number of an interval graph devised by Broersma et al. [5] and by Li et al. [13]. For proper interval graphs, a characterization regarding the unpaired $k$-disjoint path coverability was derived by Lee et al. [11].

We begin our study on Unpaired $k$-Disjoint Path Coverability of interval graphs by reviewing the sufficient condition of Theorem 4 and the necessary condition of Lemma 1 below derived in terms of the connectivity.

**Lemma 1** (Park et al. [20])**.** *If a graph $G$ of order $n \geq 2k$ for $k \geq 1$ is unpaired $k$-disjoint path coverable, then $G$ is $k$-connected.*

If $G$ is a noncomplete interval graph, the conditions of Theorem 4 and of Lemma 1 respectively can be represented as

$$\mathrm{sc}(G) = \max_X(c(G - X) - |X|) \leq -k, \tag{1}$$

$$-\kappa(G) = \max_X(\quad 0 \quad - |X|) \leq -k, \tag{2}$$

where the maximums are taken over all vertex cuts $X$ of $G$. Comparing the above two in a hope of deriving a necessary and sufficient condition, it can be said that the term "$c(G-X)$" of Eq. (1) is too large whereas the term "0" of Eq. (2) is too small. For the graphs $G_{2,3}$ and $G_{3,4}$ of Fig. 1 and $k=2$, for example, we have $\mathrm{sc}(G_{2,3}) = \text{“2”} - 3 \not\leq -2$ and $-\kappa(G_{3,4}) = \text{“0”} - 4 \leq -2$; also, both graphs have the same scattering number, but one is unpaired 2-disjoint path coverable and the other is not. Accordingly, we introduce a function $c_r(G,X)$, ranging over $\{0,1,\ldots,c(G-X)\}$, which is defined to be the maximum number of connected components in $G-X$ each of which contains no marked vertices when marking exactly $r$ out of $n$ vertices in $G$, where $r$ is an integer between 0 and $n$. In other words,

$$c_r(G,X) = \begin{cases} c(G-X) & \text{if } |X| \geq r, \\ c(G-X) - h_{r-|X|}(G-X) & \text{if } |X| < r, \end{cases}$$

where $h_{r-|X|}(G-X)$ is the minimum number of connected components in $G-X$ whose total number of vertices is $r-|X|$ or more, i.e., $h_{r-|X|}(G-X)$ is equal to the integer $q$ such that $|X| + \sum_{i=1}^{q} |V(H_i)| \geq r > |X| + \sum_{i=1}^{q-1} |V(H_i)|$ for the connected components $H_1,\ldots,H_p$ of $G-X$ with $|V(H_1)| \geq \cdots \geq |V(H_p)|$, where $p = c(G-X)$.

Now, we introduce the notion of $r$-scattering number, a generalization of the scattering number in the sense that the 0-scattering number of a graph is equal to the scattering number of the graph.

**Definition 2** ($r$-Scattering number)**.** For a noncomplete graph $G$ of order $n$ and a nonnegative integer $r$ with $r \leq n$, the *$r$-scattering number*, denoted $\mathrm{sc}_r(G)$, of $G$ is the maximum of $c_r(G,X) - |X|$ over all vertex cuts $X$ of $G$.

A vertex cut $X$ of $G$ that fulfills $c_r(G,X) - |X| = \mathrm{sc}_r(G)$ is called an *$r$-scattering set*. The $r$-scattering number of a complete graph $K_n$ is left undefined. See Fig. 2 for an example of calculating the $r$-scattering numbers. Also, the 4-scattering numbers of the graphs $G_{2,3}$ and $G_{3,4}$ of Fig. 1 are $\mathrm{sc}_4(G_{2,3}) = \text{“1”} - 3 = -2$ (which is smaller than $\mathrm{sc}(G_{2,3}) = \text{“2”} - 3$) and $\mathrm{sc}_4(G_{3,4}) = \text{“3”} - 4 = -1$ (which is larger than $-\kappa(G_{3,4}) = \text{“0”} - 4$). In addition, the $r$-scattering numbers of $G_{3,5}$ are $\mathrm{sc}_8(G_{3,5}) = -5$, $\mathrm{sc}_7(G_{3,5}) = -4$, $\mathrm{sc}_6(G_{3,5}) = -3$, and $\mathrm{sc}_5(G_{3,5}) = \mathrm{sc}_4(G_{3,5}) = \mathrm{sc}_3(G_{3,5}) = \mathrm{sc}_2(G_{3,5}) = \mathrm{sc}_1(G_{3,5}) = \mathrm{sc}_0(G_{3,5}) = -2 = \mathrm{sc}(G_{3,5})$.

In this paper, we derive a necessary condition for a general graph to be unpaired $k$-disjoint path coverable, and then establish a characterization of interval graphs that are unpaired 2-disjoint path coverable. Specifically, we prove that a noncomplete graph $G$ of order $n \geq 2k$ for $k \geq 1$ is unpaired $k$-disjoint path coverable only if $\mathrm{sc}_{2k}(G) \leq -k$, and that a noncomplete interval graph $G$ of order $n \geq 4$ is unpaired 2-disjoint path coverable if and only if $\mathrm{sc}_4(G) \leq -2$. Moreover, we show that the $r$-scattering number of an interval graph can be computed in polynomial time, whereas deciding whether the $r$-scattering number of a general graph is greater than or equal to given $K$ is NP-complete for any fixed $r$.

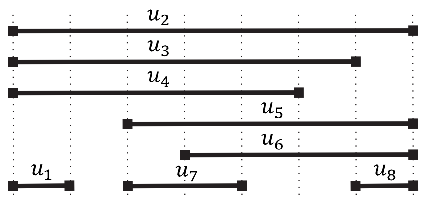



Fig. 2: An interval representation for an interval graph $G$, where $\mathrm{sc}_4(G) = \max\{1 - |\{u_2, u_3, u_4\}|, 2 - |\{u_2, u_3, u_5, u_6\}|, \ldots, 3 - |\{u_2, u_3, u_4, u_5, u_6\}|\} = -2$ and $\mathrm{sc}_6(G) = -2$. The graph $G$ is unpaired 2-disjoint path coverable but has no unpaired 3-DPC joining $S = \{u_2, u_3, u_4\}$ and $T = \{u_5, u_6, u_7\}$.

## 2. Preliminaries

A cycle that visits each vertex exactly once is a *Hamiltonian cycle*; a path that visits each vertex exactly once is a *Hamiltonian path*. A graph is *Hamiltonian* if a Hamiltonian cycle exists; a graph is *traceable* if a Hamiltonian path exists; a graph is *Hamiltonian-connected* if every two distinct vertices are joined by a Hamiltonian path. A *connected component* of $G$ is a maximal connected subgraph of $G$. A *vertex cut* of $G$ is a set $X \subseteq V(G)$ such that $G - X$ has two or more connected components, where $G - X$ is the subgraph obtained from $G$ by deleting all the vertices of $X$ (or equivalently, $G - X$ is the subgraph of $G$ induced by $V(G) \setminus X$). The *connectivity* of $G$, denoted $\kappa(G)$, is the minimum number of vertices whose removal results in a disconnected graph or a single vertex. So, $\kappa(G)$ is equal to the size of a minimum vertex cut if $G$ is a noncomplete graph; $\kappa(G) = n - 1$ if $G$ is a complete graph $K_n$. A graph $G$ is *$k$-connected* if $\kappa(G) \geq k$. In addition, $N_G(v)$, or $N(v)$ if the graph $G$ is clear in the context, represents the open neighborhood of a vertex $v \in V(G)$, i.e. $N_G(v) = \{u \in V(G) : (u, v) \in E(G)\}$, whereas $N_G[v]$, or $N[v]$, denotes the closed neighborhood of $v$, i.e. $N_G[v] = N_G(v) \cup \{v\}$. The graph-theoretic terms that are not defined here can be found in [3].

Interval graphs are a well-studied class of graphs. One of the early characterizations of interval graphs is the following:

**Theorem 5** (Gilmore and Hoffman [9])**.** *A graph $G$ is an interval graph if and only if the maximal cliques of $G$ can be linearly ordered such that, for every vertex $v$ of $G$, the maximal cliques containing $v$ occur consecutively.*

Let $C_1, \ldots, C_q$ be a linear ordering of the maximal cliques of an interval graph $G$ such that each vertex of $G$ appears in consecutive cliques only. Obviously, the graph $G$ is noncomplete if and only if $q \geq 2$. Since $C_1$ and $C_q$ are maximal cliques, $C_1$ and $C_q$ each contains at least one vertex that does not occur in any other maximal clique. Let $u_1$ be such a vertex in $C_1$ and let $u_n$ be such a vertex in $C_q$. The vertices $u_1$ and $u_n$, respectively, are referred to as a *left tip* and a *right tip*.

**Lemma 2** (Park [18])**.** *Let $u_1$ be a left tip of an interval graph $G$ with $\mathrm{sc}(G) \leq 0$. Then, a Hamiltonian $u_1$–$w$ path exists in $G$ for every vertex $w$ other than $u_1$.*

Finally, the following relationship between the scattering number and the connectivity of a noncomplete graph will also be used for our proof:

**Lemma 3** (Zhang and Wang [25]). *Let $G$ be a noncomplete graph of order $n \geq 2$. Then,* $\mathrm{sc}(G) \geq 2 - \kappa(G)$.

## 3. Unpaired $k$-disjoint path coverability of general graphs

In this section, we establish a necessary condition for a noncomplete graph to be unpaired $k$-disjoint path coverable, and then show that the condition is also a sufficient one for two special cases. We begin with some basic properties of the $r$-scattering number. The $r$-scattering number of a graph $G$ can be seen as bridging the gap between the scattering number and the connectivity of $G$, in that the 0-scattering number becomes $\mathrm{sc}(G)$ and the $n$-scattering number becomes $-\kappa(G)$.

**Lemma 4.** *Let $G$ be a noncomplete graph of order $n \geq 2$.*
*(a)* $\mathrm{sc}_{r+1}(G) \leq \mathrm{sc}_r(G) \leq \mathrm{sc}_{r+1}(G) + 1$ *for all* $r \in \{0, \ldots, n-1\}$.
*(b)* $-\kappa(G) = \mathrm{sc}_n(G) \leq \mathrm{sc}_{n-1}(G) \leq \cdots \leq \mathrm{sc}_1(G) \leq \mathrm{sc}_0(G) = \mathrm{sc}(G)$.
*(c) If $G$ is $k$-connected, then* $\mathrm{sc}_r(G) = \mathrm{sc}(G)$ *for all* $r \in \{0, \ldots, k\}$.

*Proof.* For the proof of (a), it suffices to show that $c_{r+1}(G, X) \leq c_r(G, X) \leq c_{r+1}(G, X) + 1$ for every vertex cut $X$ of $G$. If $|X| \geq r + 1$, then $c_r(G, X) = c_{r+1}(G, X) = c(G - X)$; secondly, if $|X| = r$, then $c_r(G, X) = c(G - X)$ and $c_{r+1}(G, X) = c(G-X)-1$; if $|X| < r$ finally, then $h_{r+1-|X|}(G-X) = h_{r-|X|}(G-X)$ or $h_{r-|X|}(G - X) + 1$. This means that $h_{r+1-|X|}(G - X) - 1 \leq h_{r-|X|}(G - X) \leq h_{r+1-|X|}(G-X)$, leading to that $-h_{r+1-|X|}(G-X) \leq -h_{r-|X|}(G-X) \leq -h_{r+1-|X|}(G - X) + 1$. Thus, (a) is proven. The inequalities of (b) is due to (a). In addition, we have $c_n(G, X) = 0$ and $c_0(G, X) = c(G - X)$, respectively leading to $\mathrm{sc}_n(G) = -\kappa(G)$ and $\mathrm{sc}_0(G) = \mathrm{sc}(G)$, proving (b). Finally, if $G$ is $k$-connected and $r \leq k$, then every vertex cut $X$ of $G$ is of size $r$ or more, meaning $c_r(G, X) = c(G - X)$, proving (c). ■

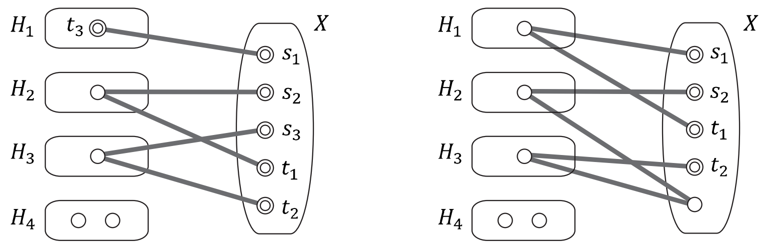
**Lemma 5.** *Let $G$ be a noncomplete graph of order $n \geq 2$. Then,*

$$\mathrm{sc}_r(G) \geq \begin{cases} 4 - n & \text{if } r \leq n-2, \\ 3 - n & \text{if } r = n-1, \\ 2 - n & \text{if } r = n. \end{cases}$$

*Proof.* For a pair of nonadjacent vertices $u$ and $v$ of $G$, let $X = V(G) \setminus \{u, v\}$. It follows that

$$\mathrm{sc}_r(G) \geq c_r(G, X) - |X| = \begin{cases} 2 - (n-2) = 4 - n & \text{if } r \leq n-2, \\ 1 - (n-2) = 3 - n & \text{if } r = n-1, \\ 0 - (n-2) = 2 - n & \text{if } r = n. \end{cases}$$

Thus, the lemma is proven. ■




(a) Case 1, where $k = 3$ (b) Case 2, where $k = 2$

Fig. 3: Illustrations of the proof of Theorem 6.

In terms of the $2k$-scattering number, we can establish a necessary condition for a general graph to be unpaired $k$-disjoint path coverable, as shown in Theorem 6 below. The new condition $\mathrm{sc}_{2k}(G) \leq -k$ of Theorem 6 is stronger than the condition $\kappa(G) \geq k$ of Lemma 1, in the sense that every graph that satisfies $\mathrm{sc}_{2k}(G) \leq -k$ also satisfies $\kappa(G) \geq k$ but the converse is not always true. Note that the hypothesis $\mathrm{sc}_{2k}(G) \leq -k$ implies $\mathrm{sc}_n(G) = -\kappa(G) \leq -k$ by Lemma 4(b), and that the interval graph of Fig. 2, which is not unpaired 3-disjoint path coverable, is 3-connected and $\mathrm{sc}_6(G) = -2 \not\leq -3$.

**Theorem 6.** *If a noncomplete graph $G$ of order $n \geq 2k$ for $k \geq 1$ is unpaired $k$-disjoint path coverable, then $\mathrm{sc}_{2k}(G) \leq -k$.*

*Proof.* Let $G$ be a noncomplete graph of order $n \geq 2k$ that is unpaired $k$-disjoint path coverable. Suppose $\mathrm{sc}_{2k}(G) \geq 1 - k$ for a contradiction. Then, there exists a $2k$-scattering set $X$ such that $c_{2k}(G, X) - |X| = \mathrm{sc}_{2k}(G) \geq 1 - k$; moreover, $|X| \geq k$ because $G$ is $k$-connected by Lemma 1. Let $H_1, \ldots, H_p$ be the connected components of $G - X$ such that $|V(H_1)| \geq \cdots \geq |V(H_p)|$, where $p = c(G - X) \geq 2$. There are two cases according to the size of $X$.

***Case 1:*** $k \leq |X| < 2k$. Let $|X| = k+r$ for some $0 \leq r < k$; let $q$ be an integer (of Definition 2) such that $|X| + \sum_{i=1}^{q} |V(H_i)| \geq 2k > |X| + \sum_{i=1}^{q-1} |V(H_i)|$, so $c_{2k}(G, X) = p - q$. In particular, consider an unpaired $k$-DPC $\{P_1, \ldots, P_k\}$ that joins disjoint terminal sets $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ such that $S \subseteq X$, $X \subseteq S \cup T$, and $T \subseteq X \cup \bigcup_{i=1}^{q} V(H_i)$, as illustrated in Fig. 3(a). Note that $|S \cap X| = k$, $|T \cap X| = r$, and $V(H_i) \cap (S \cup T) = \emptyset$ for all $i \in \{q+1, \ldots, p\}$. Then, there are $c_{2k}(G, X)$ connected components of $G - X$ each of whose intersection with $S \cup T$ is empty, where $c_{2k}(G, X) \geq |X| + 1 - k = r + 1$. A DPC path $P_i$ that runs to a sink in $X$ passes through at most one among the $c_{2k}(G, X)$ components $H_{q+1}, \ldots, H_p$, whereas a DPC path $P_j$ that runs to a sink in $\bigcup_{i=1}^{q} V(H_i)$ passes through no component among $H_{q+1}, \ldots, H_p$. It follows that the $k$ paths in the DPC altogether pass through at most $r$ components among the $c_{2k}(G, X)$ components $H_{q+1}, \ldots, H_p$. This contradicts the fact that $\{P_1, \ldots, P_k\}$ is an unpaired $k$-DPC joining $S$ and $T$, proving Case 1.

***Case 2:*** $|X| \geq 2k$. In this case, we have $c_{2k}(G, X) = p$. Let $S$ and $T$ be disjoint terminal sets of size $k$ each such that $S \cup T \subseteq X$, as illustrated in

Fig. 3(b). Consider an unpaired $k$-DPC joining $S$ and $T$, in which each path $P_i$ passes through at most $|V(P_i) \cap X| - 1$ connected components. It follows that the DPC paths collectively pass through at most $|X| - k$ connected components in total, leading to $|X| - k \geq p$. This contradicts the hypothesis $p - |X| = c_{2k}(G, X) - |X| \geq 1 - k$, thereby completing the proof. ■

Theorem 6 allows us to restrict our attention to the class of graphs whose $2k$-scattering numbers are $-k$ or less. A graph in the class is $k$-connected; its scattering number is considered in the following lemma.

**Lemma 6.** *Let $G$ be a noncomplete graph of order $n \geq 2k$ for $k \geq 1$. If $\mathrm{sc}_{2k}(G) \leq -k$, then $\mathrm{sc}(G) \leq 0$.*

*Proof.* A graph $G$ with $\mathrm{sc}_{2k}(G) \leq -k$ is $k$-connected by Lemma 4(b), leading to $\mathrm{sc}(G) = \mathrm{sc}_k(G)$ by Lemma 4(c). In addition, we have $\mathrm{sc}_k(G) \leq \mathrm{sc}_{k+1}(G) + 1 \leq \cdots \leq \mathrm{sc}_{k+k}(G) + k \leq -k + k = 0$ by Lemma 4(a), completing the proof. ■

In the remaining part of this section, we show that the necessary condition $\mathrm{sc}_{2k}(G) \leq -k$ of Theorem 6 becomes a sufficient one if $n = 2k$, or if $\mathrm{sc}(G) = 0$. Firstly, let $G$ be a noncomplete graph of order $n = 2k$ for $k \geq 2$. An unpaired $k$-DPC of $G$ joining $S$ and $T$ in this case forms a perfect matching in which each edge joins a source and a sink. So, the graph $G$ is unpaired $k$-disjoint path coverable if and only if $G$ is strongly matchable, where a graph $G$ is said to be *strongly matchable* if $G$ has a perfect matching in which each edge joins two vertices, one in $S$ and the other in $T$, for every partition of $V(G)$ into $S$ and $T$ with $|S| = |T| = \frac{n}{2}$. In addition, the condition $\mathrm{sc}_{2k}(G) \leq -k$ is equivalent to that $\kappa(G) \geq k$ because $\mathrm{sc}_{2k}(G) = \mathrm{sc}_n(G) = -\kappa(G)$ by Lemma 4(b).

**Theorem 7.** *Let $G$ be a noncomplete graph of order $n = 2k$ for $k \geq 2$. Then, $G$ is unpaired $k$-disjoint path coverable if and only if $\kappa(G) \geq k$.*

*Proof.* The necessity is due to Theorem 6. For the sufficiency proof, let $\kappa(G) \geq k$. Suppose for a contradiction that $G$ is not unpaired $k$-disjoint path coverable. Then, for some partition $S$ and $T$ of $V(G)$ with $|S| = |T| = k$, there is no perfect matching joining $S$ and $T$. By the well-known Hall's marriage theorem, there exists a subset $W \subseteq S$ such that $|N(W) \cap T| < |W|$, where $N(W)$ denotes the open neighborhood of $W$ defined as $\bigcup_{w \in W} N(w) \setminus W$. It follows that $|W| \geq 1$ and $|N(W) \cap T| < k$, meaning $(S \setminus W) \cup (N(W) \cap T)$ is a vertex cut of $G$ (because $W \neq \emptyset$, $T \setminus (N(W) \cap T) \neq \emptyset$, and there is no edge between the two subsets). However, the size of the cut $|(S \setminus W) \cup (N(W) \cap T)| = |S| - |W| + |N(W) \cap T| < |S| = k$, which contradicts the fact $\kappa(G) \geq k$. Thus, the theorem is proven. ■

Now, we consider the other special case where $\mathrm{sc}(G) = 0$.

**Theorem 8.** *Let $G$ be a noncomplete graph of order $n \geq 2k$ for $k \geq 2$ with $\mathrm{sc}(G) = 0$. Then, $G$ is unpaired $k$-disjoint path coverable if and only if $\mathrm{sc}_{2k}(G) \leq -k$.*

*Proof.* The proof is a direct consequence of Lemmas 7 and 8 below. ■

**Lemma 7** (Park et al. [20]). *A complete bipartite graph $K_{k,k}$, $k \geq 1$, is unpaired $k$-disjoint path coverable.*

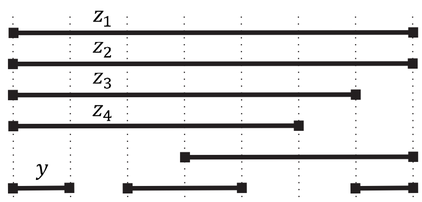
**Lemma 8.** *Let $G$ be a noncomplete graph of order $n \geq 2k$ for $k \geq 2$. If $\mathrm{sc}_{2k}(G) \leq -k$ and $\mathrm{sc}(G) = 0$, then $G$ is isomorphic to a spanning supergraph of a complete bipartite graph $K_{k,k}$ and isomorphic to a spanning subgraph of a complete split graph $G_{k,k}$.*

*Proof.* It holds that $k \leq \kappa(G) < 2k$ by Lemma 4 (b) and (c). There is a scattering set $X$ of $G$ such that $c(G-X) - |X| = \mathrm{sc}(G) = 0$. Then, $|X| \geq k$ and moreover $|X| < 2k$; supposing $|X| \geq 2k$ leads to that $\mathrm{sc}_{2k}(G) \geq c_{2k}(G, X) - |X| = c(G-X) - |X| = 0$, which contradicts the hypothesis $\mathrm{sc}_{2k}(G) \leq -k$. First we prove a claim: $|X| = k$ and every connected component of $G - X$ is a singleton. Suppose to the contrary that $|X| \geq k+1$ or there is a connected component that contains two or more vertices. Then, $h_{2k-|X|}(G-X) \leq 2k - |X| \leq k-1$ if $|X| \geq k+1$; also, $h_{2k-|X|}(G-X) \leq (2k-|X|)-1 \leq k-1$ if there is a connected component of order two or more. It follows that $c_{2k}(G, X) = c(G-X) - h_{2k-|X|}(G-X) \geq c(G-X)-(k-1)$, meaning $\mathrm{sc}_{2k}(G) \geq c_{2k}(G, X) - |X| \geq c(G-X)-(k-1)-|X| = \mathrm{sc}(G)-(k-1) = 1-k$, which is a contradiction. Thus, the claim is proved. In addition, we have $c(G-X) = |X| = k$. It remains to show $(x_i, y_j) \in E(G)$ for every pair $x_i \in X$ and $y_j \in V(G) \setminus X$. Suppose $(x_i, y_j) \notin E(G)$ for some $x_i$ and $y_j$, then $X \setminus \{x_i\}$ would be a vertex cut of $G$, which contradicts the fact $\kappa(G) \geq k$. Therefore, $G$ is isomorphic to a spanning supergraph of $K_{k,k}$ and to a spanning subgraph of $G_{k,k}$, completing the proof. ∎

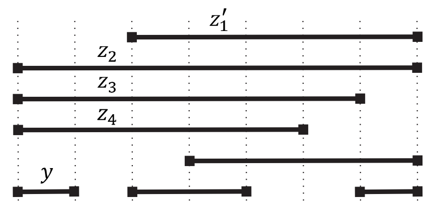
## 4. Unpaired 2-disjoint path coverability of interval graphs

In this section, we prove that the necessity $\mathrm{sc}_{2k}(G) \leq -k$ of Theorem 6 is a sufficient one for an interval graph $G$ and $k = 2$; in other words, we will prove that a noncomplete interval graph $G$ of order $n \geq 4$ is unpaired 2-disjoint path coverable if and only if $\mathrm{sc}_4(G) \leq -2$. The sufficiency proof proceeds by induction on $n$. In order to build an unpaired 2-DPC of $G$ joining prescribed source and sink sets in a recursive manner, we define several subgraphs of $G$ that admit an unpaired 2-DPC and/or a Hamiltoian path. The unpaired 2-DPCs and Hamiltonian paths of the subgraphs are then combined into a required 2-DPC. Thanks to Theorems 7 and 8, we assume $n \geq 5$ and $\mathrm{sc}(G) \leq -1$. (Recall that $\mathrm{sc}(G) \leq 0$ if $\mathrm{sc}_4(G) \leq -2$ by Lemma 6.)

First of all, we reduce our problem on a graph $G$ with $\mathrm{sc}_4(G) \leq -2$ and $\mathrm{sc}(G) \leq -2$ to a problem on a spanning subgraph $G'$ of $G$ with $\mathrm{sc}_4(G') \leq -2$ and $\mathrm{sc}(G') = -1$ as follows: Consider a noncomplete interval graph $G$ with $\mathrm{sc}_4(G) \leq -2$ and $\mathrm{sc}(G) \leq -2$ (i.e., $\mathrm{sc}_4(G) \leq \mathrm{sc}(G) \leq -2$), for which its interval representation is denoted by $\mathcal{I}$. Also, we denote by $\mathrm{lp}(I_v)$ and $\mathrm{rp}(I_v)$, respectively, the left and right endpoints of an interval $I_v$ corresponding to a vertex $v$. Let $Z$ be the leftmost minimal cut of $G$, i.e., $Z = V(C_1) \cap V(C_2)$ for the linear ordering $C_1, \ldots, C_q$ of the maximal cliques of $G$ such that each vertex of $G$ appears in consecutive cliques only. Then, $|Z| \geq 4$ because $\kappa(G) \geq 2 - \mathrm{sc}(G) \geq 4$ by Lemma 3.




(a) an interval representation for $G$




(b) an interval representation for $G'$

Fig. 4: A noncomplete interval graph $G$ with $\mathrm{sc}_4(G) \leq \mathrm{sc}(G) \leq -2$ and its spanning subgraph $G'$ with $\mathrm{sc}_4(G') \leq -2$ and $\mathrm{sc}(G') = -1$, where $Z = \{z_1, z_2, z_3, z_4\}$ and $V(C_1) = Z \cup \{y\}$.

Let $Z = \{z_1, \ldots, z_r\}$ with $\mathrm{rp}(I_{z_1}) \geq \cdots \geq \mathrm{rp}(I_{z_r})$. We define $G'$ to be the intersection graph of a family $\mathcal{I}' := (\mathcal{I} \setminus \{I_{z_1}, \ldots, I_{z_{r-3}}\}) \cup \{I'_{z_1}, \ldots, I'_{z_{r-3}}\}$, where $\mathrm{lp}(I'_{z_i}) = \min_{v \in V(G) \setminus V(C_1)} \mathrm{lp}(I_v)$ and $\mathrm{rp}(I'_{z_i}) = \mathrm{rp}(I_{z_i})$ for $i \in \{1, \ldots, r-3\}$. Refer to Fig. 4 for an example. The graph $G'$ is a spanning subgraph of $G$ because $I'_{z_i} \subsetneq I_{z_i}$ for all $i \in \{1, \ldots, r-3\}$, and moreover $\kappa(G') = 3$.

**Lemma 9.** *Let $G'$ be the spanning subgraph of a noncomplete interval graph $G$ with $\mathrm{sc}_4(G) \leq \mathrm{sc}(G) \leq -2$ defined above. Then, $\mathrm{sc}_4(G') \leq -2$ and $\mathrm{sc}(G') = -1$.*

*Proof.* For the proof of $\mathrm{sc}_4(G') \leq -2$, we suppose to the contrary that there is a 4-scattering set $Y$ of $G'$ such that $c_4(G', Y) - |Y| = \mathrm{sc}_4(G') \geq -1$. Let $Z' = \{z_1, \ldots, z_{r-3}\}$ and $Z'' = \{z_{r-2}, z_{r-1}, z_r\}$, so that $Z' \cup Z'' = Z$ and $\mathrm{rp}(I_{z_i}) \geq \mathrm{rp}(I_{z_j})$ for all $z_i \in Z'$ and $z_j \in Z''$.

*Claim* 1. (a) $Z' \nsubseteq Y$. (b) $Z'' \subseteq Y$. (c) $|Y| \geq 4$.

*Proof of claim.* Suppose $Z' \subseteq Y$, then $Y$ would also be a vertex cut of $G$ and $G - Y$ would have the same connected components as $G' - Y$. This means $c_4(G, Y) = c_4(G', Y)$, leading to that $\mathrm{sc}_4(G) \geq c_4(G, Y) - |Y| = c_4(G', Y) - |Y| = \mathrm{sc}_4(G') \geq -1$, which is a contradiction, proving (a). Suppose $Z'' \nsubseteq Y$, then every vertex $v \in V(C_1) \setminus V(C_2)$ with $v \notin Y$ would be included in the connected component of $G' - Y$ that includes a vertex $z_j \in Z'' \setminus Y$, and thus the vertex sets of the connected components of $G - Y$ would be the same as those of $G' - Y$. This means $c_4(G, Y) = c_4(G', Y)$ again, leading to that $\mathrm{sc}_4(G) \geq c_4(G, Y) - |Y| = c_4(G', Y) - |Y| = \mathrm{sc}_4(G') \geq -1$, which is a contradiction, proving (b). Suppose $|Y| \leq 3$, then $|Y| = 3$, $Y = Z''$, and $G' - Y$ would have only 2 connected components (because $\kappa(G) \geq 4$ and $G - Y$ is connected). This means that $c_4(G', Y) - |Y| = 1 - 3 = -2$, which contradicts the assumption $c_4(G', Y) - |Y| = \mathrm{sc}_4(G') \geq -1$. Thus, the claim is proven. □

Let $H_1, \ldots, H_p$ be the connected components of $G' - Y$, where $H_i$ is assumed to be to the left of $H_{i+1}$ for all $i < p$. Then, $p \geq 3$ because $|Y| \geq 4$ and $c_4(G', Y) - |Y| = p - |Y| \geq -1$. There exists a point $\alpha$ on the real line (between $H_{p-1}$ and $H_p$) such that $\max_{v \in V(H_{p-1})} \mathrm{rp}(I_v) < \alpha < \min_{v \in V(H_p)} \mathrm{lp}(I_v)$. So, the vertex subset $W$ defined as $\{v \in V(G) : \alpha \in I_v\}$ forms a vertex cut of $G'$ satisfying $W \subseteq Y$. The existence of a vertex $z_i \in Z' \setminus Y$ by Claim 1(a) leads to that $z_i \notin W$, $\mathrm{rp}(z_i) < \alpha$, and $Z'' \cap W = \emptyset$. Thus, the set $Y' := Y \setminus Z''$ is a vertex cut of $G'$ and also of $G$. It follows that $|Y'| \geq 4$; moreover, $c(G' - Y') = c(G' - Y)$ if $V(C_1) \setminus V(C_2) \subseteq Y$; $c(G' - Y') = c(G' - Y) - 1$ otherwise. Therefore, $\text{sc}_4(G') \geq c_4(G', Y') - |Y'| = c(G' - Y') - |Y'| \geq (c(G' - Y) - 1) - (|Y| - 3) = c_4(G', Y) - |Y| + 2 = \text{sc}_4(G') + 2$, which is a contradiction, proving $\text{sc}_4(G') \leq -2$.

Now, we show $\text{sc}(G') = -1$. Let $Y$ be an arbitrary vertex cut of $G'$. Then, $Y = Z''$ ($|Y| = 3$) or $|Y| \geq 4$ by the construction of $G'$ and $\kappa(G) \geq 4$. If $Y = Z''$, then $c(G' - Y) - |Y| = 2 - 3 = -1$ because $G - Y$ is connected; if $|Y| \geq 4$, then $c(G' - Y) - |Y| = c_4(G', Y) - |Y| \leq \text{sc}_4(G') \leq -2$. It follows that $\text{sc}(G') = -1$. This completes the proof. ■

Owing to Lemma 9, we further assume $\text{sc}(G) = -1$. One natural way of choosing interval subgraphs of the graph $G$ on which our subproblems are defined would be to associate them with a vertex cut, especially a minimum cut or a 4-scattering set. We observe basics of a minimum vertex cut of $G$ in Lemma 10 below (where the converse of (d) of the lemma is not always true).

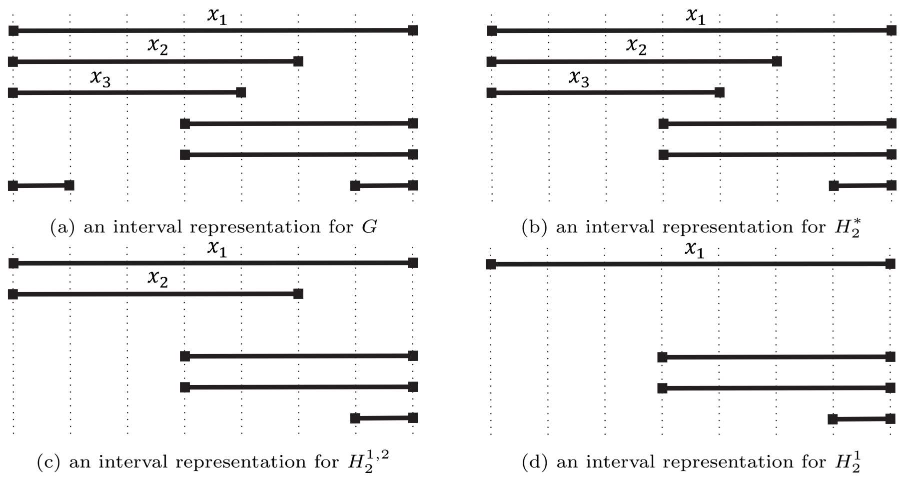
**Lemma 10.** *Let $G$ be a noncomplete interval graph of order $n \geq 5$ with $\text{sc}_4(G) \leq -2$ and $\text{sc}(G) = -1$. Then, (a) $\kappa(G) = 3$; (b) $c(G - X) = 2$ for a vertex cut $X$ of $G$ with $|X| = 3$; (c) $X$ is a scattering set of $G$ if and only if $X$ is a vertex cut with $|X| = 3$; (d) $X$ is a 4-scattering set of $G$ if $X$ is a vertex cut with $|X| = 3$; (e) $\text{sc}_4(G) = -2$.*

*Proof.* The graph $G$ is 3-connected by Lemma 3 because $\text{sc}(G) = -1$; also, supposing that $G$ is 4-connected leads to $\text{sc}_4(G) = \text{sc}(G)$, which contradicts the hypothesis of the lemma. Thus, $\kappa(G) = 3$, proving (a). Supposing $c(G - X) \geq 3$ leads to $\text{sc}(G) \geq c(G - X) - |X| \geq 0$, which is a contradiction, proving (b). To prove (c), it suffices to prove the necessity due to (b). Supposing $X$ is a scattering set with $|X| \geq 4$ leads to $\text{sc}_4(G) \geq c_4(G, X) - |X| = c(G - X) - |X| = \text{sc}(G) = -1$, which is a contradiction, proving (c). If $X$ is a vertex cut with $|X| = 3$, then $\text{sc}_4(G) \geq c_4(G, X) - |X| = c(G - X) - 1 - |X| = -2 \geq \text{sc}_4(G)$, proving (d) and (e). ■

Let $X = \{x_1, x_2, x_3\}$ denote a minimum vertex cut of $G$, and let $H_1, H_2$ be the two connected components of $G - X$. The vertex cut $X$ is also a 4-scattering set, as well as a scattering set, of $G$ by Lemma 10. An interval $I_v$ is said to be to the left of $I_w$ if $\text{rp}(I_v) < \text{lp}(I_w)$. We assume $H_1$ is to the left of $H_2$, i.e., an interval corresponding to a vertex of $H_1$ is to the left of an interval corresponding to a vertex of $H_2$. We further assume w.l.o.g. that $\text{rp}(I_{x_3}) \leq \text{rp}(I_{x_2}) \leq \text{rp}(I_{x_1})$, so that $N_G(x_3) \cap V(H_2) \subseteq N_G(x_2) \cap V(H_2) \subseteq N_G(x_1) \cap V(H_2)$.

**Lemma 11.** *(a) $|N_G(x_3) \cap V(H_2)| \geq 1$. (b) For $a \in \{1, 2\}$, $|N_G(x_a) \cap V(H_2)| \geq 2$ if $|V(H_2)| \geq 2$.*

*Proof.* Supposing $x_3$ has no neighbor in $H_2$ leads to that $\{x_1, x_2\}$ would be a vertex cut of $G$, which contradicts the fact $\kappa(G) = 3$, proving (a). So, $|N_G(x_a) \cap V(H_2)| \geq 1$ for $a \in \{1, 2\}$. It suffices to prove (b) for $a = 2$. Suppose $x_2$ has only one neighbor $v$ in $H_2$ with $|V(H_2)| \geq 2$, then $\{x_1, v\}$ would be a vertex cut of $G$, which also contradicts the fact $\kappa(G) = 3$, proving (b). ■




(a) an interval representation for $G$ (b) an interval representation for $H_2^*$ (c) an interval representation for $H_2^{1,2}$ (d) an interval representation for $H_2^1$

Fig. 5: An interval graph $G$ and its induced subgraphs $H_2^*$, $H_2^{1,2}$, and $H_2^1$, where $\mathrm{sc}_4(G) = -2$ and $\mathrm{sc}(G) = -1$.

The first interval subgraph we consider is a connected component $H_i$ of $G - X$. The component $H_i$ is not always Hamiltonian-connected; for example, the cut $\{z_2, z_3, z_4\}$ of the graph in Fig. 4(b) produces a component isomorphic to $G_{2,2}$ (which is not Hamiltonian-connected). Nonetheless, $H_i$ has a scattering number no more than 0 if it is noncomplete, as shown in the following lemma. So, quite a few vertex pairs in $H_i$ are expected to be joined by a Hamilton path.

**Lemma 12.** *Each connected component $H_i$ of $G - X$ is a complete graph, or* $\mathrm{sc}(H_i) \le 0$.

*Proof.* Suppose $H_i$ is not a complete graph and $\mathrm{sc}(H_i) \ge 1$. Then, there is a scattering set $Y$ of $H_i$ such that $c(H_i - Y) - |Y| = \mathrm{sc}(H_i) \ge 1$. It follows that $X' := X \cup Y$ would be a vertex cut of $G$ with $|X'| \ge 4$. This leads to that $\mathrm{sc}_4(G) \ge c_4(G, X') - |X'| = c(G - X') - |X'| = (c(G - X) - 1 + c(H_i - Y)) - (|X| + |Y|) \ge c(G - X) - |X| = \mathrm{sc}(G) = -1$, which contradicts the fact $\mathrm{sc}_4(G) \le -2$, completing the proof. ■

Hereafter in Lemmas 13 through 17, we discuss several subgraphs induced by a single connected component and a nonempty subset of $X$. The subproblems defined on the subgraphs are then used as building blocks to resolve our problem of constructing an unpaired 2-DPC in $G$. Let $H_i^*$ denote the subgraph of $G$ induced by $V(H_i) \cup X$. In addition, let $H_i^a$ and $H_i^{a,b}$, where $a, b \in \{1, 2, 3\}$ and $a \ne b$, be the induced subgraphs of $G$ by $V(H_i) \cup \{x_a\}$ and by $V(H_i) \cup \{x_a, x_b\}$, respectively. Refer to Fig. 5 for examples.

**Lemma 13.** *The subgraph $H_i^*$ is complete, or* $\mathrm{sc}_4(H_i^*) \le -2$ *and* $\mathrm{sc}(H_i^*) \le -1$.

*Proof.* Let $H_j$ denote the connected component of $G - X$ other than $H_i$, so that $\{i, j\} = \{1, 2\}$. Assume $H_i^*$ is noncomplete. Then, $|V(H_i^*)| \ge 5$ by Lemma 11.

Suppose to the contrary that $\mathrm{sc}_4(H_i^*) \geq -1$, i.e., there is a 4-scattering set $Y$ of $H_i^*$ such that $c_4(H_i^*, Y) - |Y| = \mathrm{sc}_4(H_i^*) \geq -1$. The vertices belonging to $X \setminus Y$, if any, are adjacent to each other and so all belong to the same connected component. Thus, $H_j$ will form a connected component of $G - Y$ if $X \setminus Y = \emptyset$; otherwise, it is included in the connected component that contains $x \in X \setminus Y$. It follows that $Y$ is a vertex cut of $G$; moreover, $\mathrm{sc}_4(G) \geq c_4(G, Y) - |Y| \geq c_4(H_i^*, Y) - |Y| = \mathrm{sc}_4(H_i^*) \geq -1$, which contradicts the fact $\mathrm{sc}_4(G) \leq -2$, proving $\mathrm{sc}_4(H_i^*) \leq -2$. Now, suppose $\mathrm{sc}(H_i^*) \geq 0$ for a contradiction. Then, $\mathrm{sc}(H_i^*) = 0$ by Lemma 6, leading to that $H_i^*$ is isomorphic to $G_{2,2}$ by Lemma 8, which contradicts the fact $|V(H_i^*)| \geq 5$, completing the proof. ■

Lemma 13 allows us to enjoy that $H_i^*$ is unpaired 2-disjoint path coverable (by the induction hypothesis) and also is Hamiltonian-connected (by Theorem 1). The subgraph $H_2^{1,2}$ also has such good properties as $H_i^*$.

**Lemma 14.** *The subgraph $H_2^{1,2}$ is complete, or $\mathrm{sc}_4(H_2^{1,2}) \leq -2$ and $\mathrm{sc}(H_2^{1,2}) \leq -1$.*

*Proof.* Assume $H_2^{1,2}$ is noncomplete. Supposing $|V(H_2)| \leq 2$ leads to that $H_2^{1,2}$ is a complete graph by Lemma 11(b), so we have $|V(H_2)| \geq 3$ and $|V(H_2^{1,2})| \geq 5$. Suppose, for a contradiction, that there is a 4-scattering set $Y$ of $H_2^{1,2}$ such that $c_4(H_2^{1,2}, Y) - |Y| = \mathrm{sc}_4(H_2^{1,2}) \geq -1$. First we prove a claim: There exists $x_a$, $a \in \{1, 2\}$, such that $x_a \notin Y$. Suppose otherwise, i.e., $x_1, x_2 \in Y$. Then, $Y \setminus \{x_1, x_2\}$ would be a vertex cut of $H_2$, meaning $|Y| \geq 4$ by Lemma 12. In addition, $Y' := Y \cup \{x_3\}$ would be a vertex cut of $G$, leading to $c_4(G, Y') = c(G - Y') = c(H_2^{1,2} - Y) + 1 = c_4(H_2^{1,2}, Y) + 1$. It follows that $\mathrm{sc}_4(G) \geq c_4(G, Y') - |Y'| = (c_4(H_2^{1,2}, Y) + 1) - (|Y| + 1) = \mathrm{sc}_4(H_2^{1,2}) \geq -1$, which is a contradiction, thereby proving the claim. In addition, the set $Y$ would be a vertex cut of $H_2^*$, because the vertex $x_3$ of $H_2^* - Y$ must be included in the connected component that contains $x_a$. (Recall that $N_G(x_3) \cap V(H_2) \subseteq N_G(x_a) \cap V(H_2)$.) It follows that $c_4(H_2^*, Y) \geq c_4(H_2^{1,2}, Y)$, implying $\mathrm{sc}_4(H_2^*) \geq c_4(H_2^*, Y) - |Y| \geq c_4(H_2^{1,2}, Y) - |Y| \geq -1$, which is a contradiction. Thus, $\mathrm{sc}_4(H_2^{1,2}) \leq -2$. Finally, supposing $\mathrm{sc}(H_2^{1,2}) \geq 0$ leads to that $\mathrm{sc}(H_2^{1,2}) = 0$ and $H_2^{1,2}$ is isomorphic to $G_{2,2}$ by Lemmas 6 and 8, contradicting the fact $|V(H_2^{1,2})| \geq 5$. Thus, $\mathrm{sc}(H_2^{1,2}) \leq -1$. This completes the proof. ■

Except for the subgraphs $H_2^*$ and $H_2^{1,2}$ (and symmetrical graphs with them), it is not easy to find other induced subgraphs with such good properties of the two. Instead, we study a Hamiltonian property of the subgraphs induced by $V(H_2)$ and a subset of $X$. Specifically, the existence of a Hamiltonian path running from a vertex of $X$ is dealt with in Lemmas 15 through 17 below.

**Lemma 15.** *For $a \in \{1, 2\}$, the subgraph $H_2^a$ is complete or $\mathrm{sc}(H_2^a) \leq 0$. Moreover, if $|N_G(x_3) \cap V(H_2)| \geq 2$ or $|V(H_2)| = 1$, then the subgraph $H_2^3$ is complete or $\mathrm{sc}(H_2^3) \leq 0$.*

*Proof.* It suffices to prove that for an arbitrary $b \in \{1, 2, 3\}$, the subgraph $H_2^b$ is complete or $\mathrm{sc}(H_2^b) \leq 0$ under the condition $|N_G(x_b) \cap V(H_2)| \geq 2$

or $|V(H_2)| = 1$. This is because the condition holds true for $b \in \{1, 2\}$ by Lemma 11. Suppose to the contrary that $H_2^b$ is noncomplete and $\mathrm{sc}(H_2^b) \geq 1$, i.e., there is a scattering set $Y$ of $H_2^b$ such that $c(H_2^b - Y) - |Y| = \mathrm{sc}(H_2^b) \geq 1$. Then, $H_2^b$ is a 2-connected graph, because $|N_G(x_b) \cap V(H_2)| \geq 2$ and $H_2$ is complete or $\mathrm{sc}(H_2) \leq 0$ by Lemma 12, leading to $|Y| \geq 2$. Moreover, the connected components of $H_2^b - Y$ coincide with those of $H_2^* - Y'$ where $Y' = Y \cup (X \setminus \{x_b\})$. It follows that $c_4(H_2^*, Y') = c(H_2^* - Y') = c(H_2^b - Y) \geq 2$, leading to that $\mathrm{sc}_4(H_2^*) \geq c_4(H_2^*, Y') - |Y'| = c(H_2^b - Y) - (|Y| + 2) = \mathrm{sc}(H_2^b) - 2 \geq -1$, which is a contradiction. Thus, $H_2^b$ is complete or $\mathrm{sc}(H_2^b) \leq 0$. ■

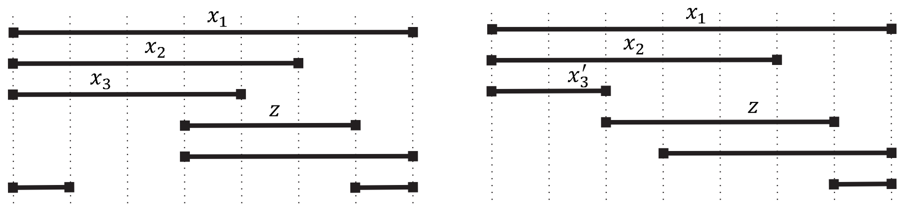
**Lemma 16.** *For $a \in \{1, 2\}$, the subgraph $H_2^a$ has a Hamiltonian $x_a$–$y$ path for every $y \in V(H_2)$. Moreover, if $|N_G(x_3) \cap V(H_2)| \geq 2$ or $|V(H_2)| = 1$, then $H_2^3$ has a Hamiltonian $x_3$–$y$ path for every $y \in V(H_2)$.*

*Proof.* The lemma holds true obviously if $|V(H_2)| = 1$ by Lemma 11; so, assume $|V(H_2)| \geq 2$. Since $|N_G(x_1) \cap V(H_2)|, |N_G(x_2) \cap V(H_2)| \geq 2$ again by Lemma 11, it suffices to prove that the subgraph $H_2^b$, $b \in \{1, 2, 3\}$, has a Hamiltonian $x_b$–$y$ path for every $y \in V(H_2)$ provided $|N_G(x_b) \cap V(H_2)| \geq 2$. If $H_2$ is a complete graph, such a Hamiltonian $x_b$–$y$ path exists obviously; so, assume $H_2$ is noncomplete. Also, $H_2^b$ is noncomplete, and $\mathrm{sc}(H_2^b) \leq 0$ by Lemma 15. Let $w$ be a vertex of $H_2^b$ such that $\mathrm{rp}(I_w) \leq \mathrm{rp}(I_v)$ for all $v \in V(H_2^b)$, so that $w$ is a left tip of $H_2^b$. If $x_b = w$, then a required Hamiltonian $x_b$–$y$ path exists in $H_2^b$ by Lemma 2. Let $x_b \neq w$ now. Then, $x_b$ and $w$ are adjacent each other because $I_w \subseteq I_{x_b}$. Moreover, $w$ is a left tip of $H_2$, so there exists a Hamiltonian $w$–$y$ path in $H_2$ for every $y \in V(H_2)$ other than $w$. (Recall $\mathrm{sc}(H_2) \leq 0$ by Lemma 12.) If $y \neq w$, a required Hamiltonian $x_b$–$y$ path is obtained by combining one-vertex path $\langle x_b \rangle$ with a Hamiltonian $w$–$y$ path of $H_2$; if $y = w$, for a neighbor $z$ of $x_b$ other than $w$, combining $\langle x_b \rangle$ and a Hamiltonian $z$–$w$ path of $H_2$ results in a required Hamiltonian $x_b$–$y$ path. The neighbor $z$ of $x_b$ exists by the hypothesis $|N_G(x_b) \cap V(H_2)| \geq 2$. Therefore, the lemma is proved. ■

**Lemma 17.** *The subgraph $H_2^{a,b}$ has a Hamiltonian $x_a$–$x_b$ path for distinct $a, b \in \{1, 2, 3\}$.*

*Proof.* It suffices to show that $H_2^{2,3}$ has a Hamiltonian $x_2$–$x_3$ path. For a neighbor $z \in V(H_2)$ of $x_3$, a Hamiltonian $x_2$–$z$ path of $H_2^2$ exists by Lemma 16. Combining the Hamiltonian path with one-vertex path $\langle x_3 \rangle$ results in a required Hamiltonian path. ■

The final subgraph on which our subproblems are defined is a spanning subgraph of $H^*$, possibly not an induced subgraph but an interval subgraph of $G$. Let $z$ be a vertex in $N_{H_2^*}(x_3) \cap V(H_2)$ such that $\mathrm{rp}(I_z) \leq \mathrm{rp}(I_v)$ for all $v \in N_{H_2^*}(x_3) \cap V(H_2)$. We can assume that $\mathrm{lp}(I_z) < \mathrm{lp}(I_v)$ for all $v \in V(H_2) \setminus \{z\}$, because replacing $I_z$ with a new interval $I'_z$ such that $\mathrm{rp}(I'_z) = \mathrm{rp}(I_z)$ and $\max_{v \in V(H_1)} \mathrm{rp}(I_v) < \mathrm{lp}(I'_z) < \min_{v \in V(H_2)} \mathrm{lp}(I_v)$ results in another interval representation for the graph $H_2^*$. (Supposing $I'_z \cap I_w \neq \emptyset$ and $I_z \cap I_w = \emptyset$ for some $w \in V(H_2^*)$ leads to $\mathrm{rp}(I_w) < \mathrm{lp}(I_z) \leq \mathrm{rp}(I_{x_3})$, meaning




(a) an interval representation for $G$ (b) an interval representation for $H_2^\dagger$

Fig. 6: An interval graph $G$ and its subgraph $H_2^\dagger$, where $\mathrm{sc}_4(G) = -2$ and $\mathrm{sc}(G) = -1$.

$w \in V(H_2)$ and $I_w$ intersects $I_{x_3}$, which contradicts the choice of $z$.) We define $H_2^\dagger$ to be the intersection graph of the family of intervals obtained from the interval representation for $H_2^*$ by replacing the interval $I_{x_3}$ with a new interval $I'_{x_3}$ having $\mathrm{lp}(I'_{x_3}) = \mathrm{lp}(I_{x_3})$ and $\mathrm{rp}(I'_{x_3}) = \mathrm{lp}(I_z)$. Refer to Fig. 6 for an example. The graph $H_2^\dagger$ is obviously a spanning subgraph of $H_2^*$.

**Lemma 18.** *The subgraph $H_2^\dagger$ is complete, or $\mathrm{sc}_4(H_2^\dagger) \leq -2$ and $\mathrm{sc}(H_2^\dagger) \leq -1$.*

*Proof.* Assume $H_2^\dagger$ is noncomplete. It follows that $|V(H_2)| \geq 2$ and $|V(H_2^\dagger)| \geq 5$. Suppose to the contrary $\mathrm{sc}_4(H_2^\dagger) \geq -1$, i.e., there is a 4-scattering set $Y$ of $H_2^\dagger$ such that $c_4(H_2^\dagger, Y) - |Y| = \mathrm{sc}_4(H_2^\dagger) \geq -1$.

*Claim* 2. (a) $x_3 \notin Y$. (b) $N_{H_2^\dagger}(x_3) = \{x_1, x_2, z\} \subseteq Y$. (c) $|Y| \geq 4$.

*Proof of claim.* Supposing $x_3 \in Y$ leads to that $Y$ would also be a vertex cut of $H_2^*$ and moreover, $H_2^* - Y$ and $H_2^\dagger - Y$ coincide, meaning $sc_4(H_2^*) \geq c_4(H_2^*, Y) - |Y| = c_4(H_2^\dagger, Y) - |Y| = \mathrm{sc}_4(H_2^\dagger) \geq -1$, which is a contradiction, proving (a). Suppose $N_{H_2^\dagger}(x_3) \nsubseteq Y$ for a contradiction, i.e., the connected component, $R_1$, of $H_2^\dagger - Y$ that contains $x_3$ is of size two or more. Then, $R_1 - \{x_3\}$ is connected. (This is because $x_a \in V(R_1)$ and $N_{R_1}(x_3) \subseteq N_{R_1}(x_a)$ if $x_a \notin Y$ for some $a \in \{1, 2\}$; $|N_{R_1}(x_3)| = 1$ otherwise.) Moreover, $(H_2^\dagger - Y) - \{x_3\}$ and $H_2^{1,2} - Y$ coincide. These lead to that $c(H_2^{1,2} - Y) = c(H_2^\dagger - Y)$ and $Y$ is a vertex cut of $H_2^{1,2}$, where $|Y| \geq 3$ by Lemmas 3 and 14. Thus, we have $\mathrm{sc}_4(H_2^{1,2}) \geq c_4(H_2^{1,2}, Y) - |Y| = c_4(H_2^\dagger, Y) - |Y| \geq -1$, which is a contradiction, completing the proof of (b). Finally, suppose $|Y| \leq 3$ for a contradiction. Then, $Y = \{x_1, x_2, z\}$ by Claim 2(b). So, $c(H_2^\dagger - Y) = c_4(H_2^\dagger, Y) + 1 = |Y| + \mathrm{sc}_4(H_2^\dagger) + 1 \geq |Y| = 3$, implying that $Y' := Y \setminus \{x_1, x_2\} = \{z\}$ would be a vertex cut of $H_2$, which contradicts the fact that $H_2$ is complete or $\mathrm{sc}(H_2) \leq 0$. Therefore, the claim is proved. □

Claim 2 leads to that $c(H_2^\dagger - Y) = c_4(H_2^\dagger, Y) = |Y| + \mathrm{sc}_4(H_2^\dagger) \geq 3$. Thus, $Z := Y \setminus \{x_1, x_2\}$ is a vertex cut of $H_2$, where $c(H_2 - Z) = c(H_2^\dagger - Y) - 1 \geq 2$.

*Claim* 3. (a) $c(H_2 - Z) - |Z| = \mathrm{sc}(H_2) = 0$. (b) $N_{H_2^*}(x_3) \cap V(H_2) \subseteq Z$.

*Proof of claim.* The proof of (a) is a direct consequence of the following two inequalities: $\mathrm{sc}(H_2) \leq 0$ and $\mathrm{sc}(H_2) \geq c(H_2 - Z) - |Z| = (c(H_2^\dagger - Y) - 1) - (|Y| -$

2) $= \mathrm{sc}_4(H_2^\dagger) + 1 \geq 0$. Now, suppose for a contradiction that there is a vertex $y \in N_{H_2^*}(x_3) \cap V(H_2)$ such that $y \notin Z$. Assume w.l.o.g. that $\mathrm{lp}(I_y) \leq \mathrm{lp}(I_v)$ for all $v \in N_{H_2^*}(x_3) \cap V(H_2)$ with $v \notin Z$. So, there is no interval $I_u$ with $u \in V(H_2) \setminus Z$ to the left of $I_y$. Moreover, $\mathrm{lp}(I_z) < \mathrm{lp}(I_y)$ and $\mathrm{rp}(I_z) \leq \mathrm{rp}(I_y)$ by the choice of $z$. Then, $Z' := Z \setminus \{z\}$ would be a vertex cut of $H_2$, because $z$ belongs to the connected component of $H_2 - Z'$ that includes $y$, or forms a one-vertex connected component, leading to $c(H_2 - Z') \geq c(H_2 - Z)$. It follows that $\mathrm{sc}(H_2) \geq c(H_2 - Z') - |Z'| \geq c(H_2 - Z) - (|Z| - 1) = 1$, which is a contradiction, thereby proving (b). □

Claims 2(b) and 3(b) lead to $N_{H_2^*}(x_3) \subseteq Y$. Thus, the connected components of $H_2^* - Y$ are the same as those of $H_2^\dagger - Y$, meaning $Y$ is also a vertex cut of $H_2^*$. It follows that $\mathrm{sc}_4(H_2^*) \geq c_4(H_2^*, Y) - |Y| = c_4(H_2^\dagger, Y) - |Y| \geq -1$, which is a contradiction. Thus, $\mathrm{sc}_4(H_2^\dagger) \leq -2$. Finally, supposing $\mathrm{sc}(H_2^\dagger) \geq 0$ leads to that $\mathrm{sc}(H_2^\dagger) = 0$ and $H_2^\dagger$ is isomorphic to $G_{2,2}$ by Lemmas 6 and 8, contradicting the fact $|V(H_2^\dagger)| \geq 5$. Thus, $\mathrm{sc}(H_2^\dagger) \leq -1$. This completes the entire proof. ■

Now, we are ready to prove our main theorem.

**Theorem 9.** *A noncomplete interval graph $G$ of order $n \geq 4$ is unpaired* 2-*disjoint path coverable if and only if* $\mathrm{sc}_4(G) \leq -2$.

*Proof.* The necessity part is due to Theorem 6. The sufficiency proof proceeds by induction on $n$. Assume $\mathrm{sc}_4(G) \leq -2$. Then, $\mathrm{sc}(G) \leq 0$ by Lemma 6. The graph $G$ is unpaired 2-disjoint path coverable if $n = 4$ by Theorem 7; the same follows if $\mathrm{sc}(G) = 0$ by Theorem 8. In addition, due to Lemma 9, there is a spanning subgraph $G'$ of $G$ with $\mathrm{sc}_4(G') \leq -2$ and $\mathrm{sc}(G') = -1$. So, we further assume $n \geq 5$ and $\mathrm{sc}(G) = -1$. As before, let $X = \{x_1, x_2, x_3\}$ denote a minimum vertex cut of $G$ with $\mathrm{rp}(I_{x_3}) \leq \mathrm{rp}(I_{x_2}) \leq \mathrm{rp}(I_{x_1})$, and let $H_1, H_2$ be the connected components of $G - X$, where $H_1$ is to the left of $H_2$. Given disjoint source and sink sets, $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$ in $G$, we will build an unpaired 2-DPC joining them. There are three cases depending on the distribution of the four terminals in $S \cup T$.

***Case 1:*** *There is a connected component, say $H_1$, that contains no terminal.* Firstly, suppose $x_3$ is a nonterminal. In the subgraph $H_2^\dagger$ of $G$, there exists an unpaired 2-DPC $\mathcal{P}$ joining $S$ and $T$ by the induction hypothesis because $H_2^\dagger$ is complete or $\mathrm{sc}_4(H_2^\dagger) \leq -2$ by Lemma 18. A path in $\mathcal{P}$ that passes through $x_3$ as an intermediate vertex visits an edge $(x_3, x_a)$ for some $a \in \{1, 2\}$ because $N_{H_2^\dagger}(x_3) = \{x_1, x_2, z\}$ for some $z \in V(H_2)$. Replacing the subpath $\langle x_3, x_a \rangle$ with a Hamiltonian $x_3$–$x_a$ path of the subgraph induced by $V(H_1) \cup \{x_3, x_a\}$, which exists by Lemma 17, results in an unpaired 2-DPC of $G$ joining $S$ and $T$.

Now, suppose $x_3$ is a terminal, say a source. If $x_a$ is a nonterminal for some $a \in \{1, 2\}$, it suffices to combine a Hamiltonian $x_3$–$x_a$ path of the subgraph induced by $V(H_1) \cup \{x_3, x_a\}$ with an unpaired 2-DPC of $H_2^{1,2}$ joining $S' := (S \setminus \{x_3\}) \cup \{x_a\}$ and $T$, which exists by the induction hypothesis and Lemma 14, into a required 2-DPC of $G$. Suppose otherwise, i.e., $x_1, x_2 \in S \cup T$. For some

$b \in \{1,2\}$ with $x_b \in T$, it suffices to build two Hamiltonian paths: a Hamiltonian $x_3$–$x_b$ path in the subgraph induced by $V(H_1) \cup \{x_3, x_b\}$ and a Hamiltonian $x_a$–$y$ path in $H_2^a$ for $a$ with $\{a,b\} = \{1,2\}$ and a terminal $y$ contained in $H_2$. The Hamiltonian $x_a$–$y$ path exists by Lemma 16.

***Case 2:*** *There is a connected component, say $H_2$, that contains a single terminal, say a source $s_1$.* The component $H_1$ contains one or more terminals in this case; so, $X$ contains at most two terminals. Firstly, suppose there exists a nonterminal $x_a \in X$ such that (i) $a \in \{1,2\}$ or (ii) $a = 3$ and either $|N_G(x_3) \cap V(H_2)| \geq 2$ or $|V(H_2)| = 1$. It suffices to combine a Hamiltonian $s_1$–$x_a$ path of $H_2^a$ with an unpaired 2-DPC of $H_1^*$ joining $S' := (S \setminus \{s_1\}) \cup \{x_a\}$ and $T$. The unpaired 2-DPC exists by the induction hypothesis and Lemma 13.

Now, there remains a case where $x_1, x_2 \in S \cup T$, $x_3 \notin S \cup T$, and moreover $|N_G(x_3) \cap V(H_2)| = 1$ and $|V(H_2)| \geq 2$. So, there is a single terminal, say $y$, in $H_1$. We first prove a claim: $|N_G(x_3) \cap V(H_1)| \geq 2$ or $|V(H_1)| = 1$. Supposing $|N_G(x_3) \cap V(H_1)| = 1$ and $|V(H_1)| \geq 2$ for a contradiction, along with $|N_G(x_3) \cap V(H_2)| = 1$ and $|V(H_2)| \geq 2$, leads to that $Z := N_G(x_3)$ would be a vertex cut of $G$, leaving three connected components $\{x_3\}$, $H_1 - Z$, $H_2 - Z$ in $G - Z$. This implies that $\mathrm{sc}_4(G) \geq c_4(G,Z) - |Z| = c(G - Z) - |Z| = 3 - 4 = -1$, which is a contradiction, thereby proving the claim. Thus, there exists a Hamiltonian $x_3$–$y$ path in the subgraph induced by $V(H_1) \cup \{x_3\}$ by Lemma 16; so, combining the Hamiltonian path with an unpaired 2-DPC joining $S'$ and $T'$ in $H_2^*$ results in a required 2-DPC, where $S' = (S \setminus \{y\}) \cup \{x_3\}$ and $T' = T$ if $y \in S$; $S' = S$ and $T' = (T \setminus \{y\}) \cup \{x_3\}$ otherwise.

***Case 3:*** *Each connected component contains two terminals.* Assume $H_1$ contains a source $s_1$ and a terminal $y$ other than $s_1$, where $y$ may be a source or a sink. There exists an unpaired 2-DPC joining $\{s_1, y\}$ and $\{x_1, x_2\}$ in $H_1^*$ by the induction hypothesis and Lemma 13. Assume w.l.o.g. the DPC is composed of $s_1$–$x_1$ and $y$–$x_2$ paths. It suffices to combine the 2-DPC of $H_1^*$ with an unpaired 2-DPC of $H_2^{1,2}$ joining $S'$ and $T'$, where $S' = \{x_1, x_2\}$ and $T' = T$ if $y \in S$; $S' = (S \setminus \{s_1\}) \cup \{x_1\}$ and $T' = (T \setminus \{y\}) \cup \{x_2\}$ if $y \in T$. This completes the entire proof. ■

## 5. Algorithm for computing the $r$-scattering number

The $r$-scattering number $\mathrm{sc}_r(G)$ of a noncomplete graph $G$ of order $n$ for a nonnegative integer $r$ with $r \leq n$ can be expressed as

$$\mathrm{sc}_r(G) = \max\{\mathrm{sc}'_r(G), \mathrm{sc}''_r(G)\},$$

where $\mathrm{sc}'_r(G)$ is defined to be the maximum of $c(G - X) - |X|$ over all vertex cuts $X$ of $G$ with $|X| \geq r$ if such $X$ exists; $\mathrm{sc}'_r(G) = -\infty$ otherwise. Also, $\mathrm{sc}''_r(G)$ is defined to be the maximum of $c(G - X) - h_{r-|X|}(G - X) - |X|$ over all vertex cuts $X$ of $G$ with $|X| \leq r$ if such $X$ exists; $\mathrm{sc}''_r(G) = -\infty$ otherwise. Note that $h_{r-|X|}(G - X) = 0$ if $|X| = r$, so including the vertex cuts of size $r$ in defining $\mathrm{sc}''_r(G)$ does not cause a problem.

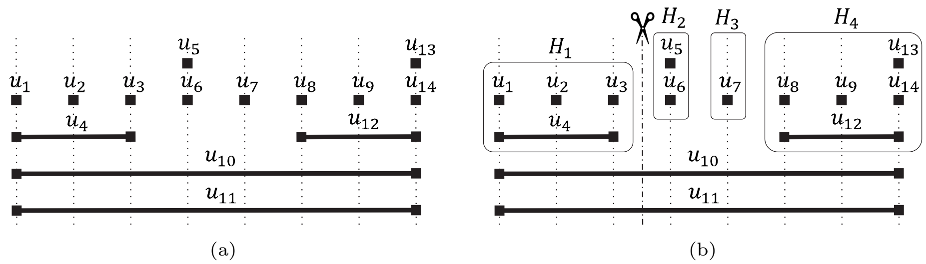



Fig. 7: An illustrative example of computing $sc''_r(G)$ for a noncomplete interval graph $G$ of order $n = 14$: (a) an interval representation for the graph $G$; (b) four connected components $H_1, \ldots, H_4$ are produced through the minimal cut $Z = \{u_{10}, u_{11}\}$.

For a graph $G$ of order $n$, possibly a complete graph, we define $c_i(G)$, $i \in \{0, \ldots, n-1\}$, to be the largest number of connected components the graph $G$ can get after the removal of exactly $i$ vertices, i.e., $c_i(G) = \max\{c(G-X) : X \subseteq V(G), |X| = i\}$. Provided the numbers $c_i(G)$ are given, $\mathrm{sc}'_r(G)$ can be calculated easily due to the following lemma. For an interval graph $G$, the numbers $c_i(G)$ can be computed in $O(n^3)$ time by Kratsch et al. [10]. So, we conclude that $\mathrm{sc}'_r(G)$ of a noncomplete interval graph can be computed in $O(n^3)$ time.

**Lemma 19.** *Let $G$ be a noncomplete graph of order $n$ and $r$ be a nonnegative integer with $r \leq n$. If $G$ has a vertex cut of size $r$ or more, then* $\mathrm{sc}'_r(G) = \max\{c_i(G) - i : \max\{r, \kappa(G)\} \leq i \leq n-2\}$.

*Proof.* If $G$ has a vertex cut of size $r$ or more, then $\mathrm{sc}'_r(G) = \max\{c_i(G) - i : i \geq r, c_i(G) \geq 2\}$ from the definition of $\mathrm{sc}'_r(G)$. So, the lemma follows. ■

Now, consider how to compute $\mathrm{sc}''_r(G)$ of a noncomplete interval graph. We first introduce a function $f_r(G)$ for a graph $G$ of order $n$ and a nonnegative integer $r$ with $r \leq n$, defined as the maximum of $c(G-X) - h_{r-|X|}(G-X) - |X|$ over all subsets $X \subseteq V(G)$ with $|X| \leq r$ (where $X$ is not necessarily a vertex cut of $G$). Let us take a look at the basic idea of an algorithm for computing $\mathrm{sc}''_r(G)$ through an illustrative example shown in Fig. 7. Observe that for the vertex cut $X = \{u_4, u_{10}, u_{11}, u_{12}\}$ of the example graph $G$, we have $\mathrm{sc}''_8(G) = c(G-X) - h_{r-|X|}(G-X) - |X| = (8-2) - 4 = 2$. Also, $\mathrm{sc}''_8(G)$ can be obtained from $f_{r-|Z|}(G-Z)$ for some minimal vertex cut $Z \subseteq X$ of $G$. That is, if we pick up $Z = \{u_{10}, u_{11}\}$ and let $G_1 = H_1$ and $G_2 = H_2 \cup H_3 \cup H_4$, then $f_{r-|Z|}(G-Z) - |Z| = f_6(G_1 \cup G_2) - 2 = f_1(G_1) + f_5(G_2) - 2 = f_1(H_1) + (f_2(H_2) + f_0(H_3) + f_3(H_4)) - 2 = (3-1) + (0+1+(2-1)) - 2 = 2 = \mathrm{sc}''_8(G)$.

**Lemma 20.** *Let $G$ be a noncomplete graph of order $n$ and $r$ be a nonnegative integer with $r \leq n$. If $G$ has a vertex cut of size $r$ or less, then*

$$\mathrm{sc}''_r(G) = \max_Z \left(f_{r-|Z|}(G-Z) - |Z|\right),$$

*where the maximum is taken over all minimal vertex cuts $Z$ of $G$ with $|Z| \leq r$.*

*Proof.* Firstly, we show $\mathrm{sc}''_r(G) \le f_{r-|Z|}(G-Z) - |Z|$ for some minimal vertex cut $Z$ with $|Z| \le r$. Let $X$ be a vertex cut of $G$ with $|X| \le r$ such that $c(G-X) - h_{r-|X|}(G-X) - |X| = \mathrm{sc}''_r(G)$. It is obvious that there is a subset $Y \subseteq V(G) \setminus X$ of size $r - |X|$ such that the number of connected components $H'_j$ of $G - X$ with $V(H'_j) \cap Y = \emptyset$ is equal to $c(G-X) - h_{r-|X|}(G-X)$. For a minimal vertex cut $Z$ of $G$ that is a subset of $X$, let $G' = G - Z$. Also, let $X' = X \setminus Z$ and $Y' = Y$, so $|X'| + |Y'| = r - |Z|$. Then,

$$\begin{aligned}
\mathrm{sc}''_r(G) &= c(G-X) - h_{r-|X|}(G-X) - |X| \\
&= \big[\#\text{ of components } H'_j \text{ in } G-X \text{ with } V(H'_j)\cap Y = \emptyset\big] - |X| \\
&= \big[\#\text{ of components } H'_j \text{ in } G'-X' \text{ with } V(H'_j)\cap Y' = \emptyset\big] - (|X'| + |Z|) \\
&\le f_{r-|Z|}(G') - |Z|.
\end{aligned}$$

Now, we show $\mathrm{sc}''_r(G) \ge f_{r-|Z|}(G-Z) - |Z|$ for every minimal vertex cut $Z$ with $|Z| \le r$. Let $G' = G - Z$ and $r' = r - |Z|$. There exists a vertex subset $X'$ of $G'$ with $|X'| \le r'$ such that $c(G'-X') - h_{r'-|X'|}(G'-X') - |X'| = f_{r'}(G')$. Moreover, there exists $Y' \subseteq V(G') \setminus X'$ with $|Y'| = r' - |X'|$ such that the number of connected components $H'_j$ of $G' - X'$ with $V(H'_j) \cap Y' = \emptyset$ is equal to $c(G'-X') - h_{r'-|X'|}(G'-X')$. Let $X = X' \cup Z$ and $Y = Y'$, so $|X| + |Y| = r$. Then, $X$ is a vertex cut of $G$ with $|X| \le r$, because $V(H_i) \not\subseteq X'$ for all connected components $H_i$ of $G'$. (Suppose for a contradiction $V(H_i) \subseteq X'$ for some $H_i$. For the sets $X'' = X' \setminus V(H_i)$ and $Y'' = Y' \cup V(H_i)$, the connected components $H'_j$ of $G' - X'$ with $V(H'_j) \cap Y' = \emptyset$ coincide with the connected components $H'_j$ of $G' - X''$ with $V(H'_j) \cap Y'' = \emptyset$. It follows that $f_{r'}(G') \ge c(G'-X'') - h_{r'-|X''|}(G'-X'') - |X''| \ge c(G'-X') - h_{r'-|X'|}(G'-X') - |X''| > c(G'-X') - h_{r'-|X'|}(G'-X') - |X'| = f_{r'}(G')$, which is a contradiction.) So, we have

$$\begin{aligned}
\mathrm{sc}''_r(G) &\ge c(G-X) - h_{r-|X|}(G-X) - |X| \\
&\ge \big[\#\text{ of components } H'_j \text{ in } G-X \text{ with } V(H'_j)\cap Y = \emptyset\big] - |X| \\
&= \big[\#\text{ of components } H'_j \text{ in } G'-X' \text{ with } V(H'_j)\cap Y' = \emptyset\big] - |X| \\
&= c(G'-X') - h_{r'-|X'|}(G'-X') - (|X'| + |Z|) \\
&= f_{r'}(G') - |Z|.
\end{aligned}$$

This completes the proof. ■

The graph $G - Z$ of Lemma 20 is disconnected, so it is the disjoint union $G_1 \cup G_2$ of two subgraphs $G_1$ and $G_2$ of $G$, i.e., $V(G) = V(G_1) \cup V(G_2)$, $E(G) = E(G_1) \cup E(G_2)$, and $V(G_1) \cap V(G_2) = \emptyset$. Lemma 21 below deals with how to compute the term $f_{r-|Z|}(G-Z)$ of Lemma 20.

**Lemma 21.** *Let $G$ be a graph of order $n$ and $r$ be a nonnegative integer no more than $n$. (a) If $G$ is the disjoint union $G_1 \cup G_2$ of two subgraphs $G_1$ and $G_2$, then $f_r(G) = \max_{(r_1,r_2)} (f_{r_1}(G_1) + f_{r_2}(G_2))$ over all pairs $(r_1, r_2)$ with $r_1 + r_2 = r$ and $0 \le r_i \le |V(G_i)|$ for $i \in \{1,2\}$. (b) If $G$ is connected, then $f_0(G) = 1$ and $f_r(G) = \max\{\mathrm{sc}''_r(G), 0\}$ for $r \ge 1$.*

*Proof.* The proof of (a) is similar to that of Lemma 20. There are two disjoint sets $X, Y \subseteq V(G)$ with $|X| \le r$ and $|Y| = r - |X|$ such that $f_r(G) + |X| = c(G - X) - h_{r-|X|}(G - X) = \big[\#$ of components $H'_j$ in $G - X$ with $V(H'_j) \cap Y = \emptyset\big]$. Let $X_i = X \cap V(G_i)$ and $Y_i = Y \cap V(G_i)$ for $i \in \{1, 2\}$. Then,

$$\begin{aligned} f_r(G) &= \big[\#\text{ of components } H'_j \text{ in } G - X \text{ with } V(H'_j) \cap Y = \emptyset\big] - |X| \\ &= \textstyle\sum_{i=1}^{2} \big(\big[\#\text{ of components } H'_j \text{ in } G_i - X_i \text{ with } V(H'_j) \cap Y_i = \emptyset\big] - |X_i|\big) \\ &\le f_{r_1}(G_1) + f_{r_2}(G_2), \text{ where } r_1 = |X_1| + |Y_1| \text{ and } r_2 = |X_2| + |Y_2|. \end{aligned}$$

It remains to show $f_r(G) \ge f_{r_1}(G_1) + f_{r_2}(G_2)$ for every pair $(r_1, r_2)$. For each $G_i$, there are two disjoint sets $X_i, Y_i \subseteq V(G_i)$ with $|X_i| \le r_i$ and $|Y_i| = r_i - |X_i|$ such that $f_{r_i}(G_i) + |X_i| = c(G_i - X_i) - h_{r_i - |X_i|}(G_i - X_i) = \big[\#$ of components $H'_j$ in $G_i - X_i$ with $V(H'_j) \cap Y_i = \emptyset\big]$. For the sets $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$, we have

$$\begin{aligned} f_r(G) &\ge \big[\#\text{ of components } H'_j \text{ in } G - X \text{ with } V(H'_j) \cap Y = \emptyset\big] - |X| \\ &= \textstyle\sum_{i=1}^{2} \big(\big[\#\text{ of components } H'_j \text{ in } G_i - X_i \text{ with } V(H'_j) \cap Y_i = \emptyset\big] - |X_i|\big) \\ &= f_{r_1}(G_1) + f_{r_2}(G_2), \text{ proving (a).} \end{aligned}$$

To prove (b), let $G$ be a connected graph. It is obvious $f_r(G) = 1$ if $r = 0$; so, assume $r \ge 1$. There is a subset $X$ of $V(G)$ with $|X| \le r$ such that $c(G - X) - h_{r-|X|}(G - X) - |X| = f_r(G)$. Moreover, the left term $c(G - X) - h_{r-|X|}(G - X) - |X|$ is equal to $\mathrm{sc}''_r(G)$ if $X$ is a vertex cut of $G$; it is equal to 0 if $X = \emptyset$; it is less than or equal to $1 - 0 - |X| \le 0$ if $X \ne \emptyset$ and $X$ is not a vertex cut of $G$. So, the lemma follows. ■

Lemmas 20 and 21 suggest an algorithm for computing $\mathrm{sc}''_r(G)$ and $f_r(G)$ of a general graph $G$. The weakness of the algorithm is that all minimal vertex cuts must be taken into account. (A graph may have an exponential number of minimal vertex cuts; for example, think of a graph composed of $\Theta(n)$ internally vertex-disjoint paths of length 3 between two vertices.) Fortunately, there are $O(n)$ minimal vertex cuts in an interval graph. Moreover, the class of interval graphs is hereditary, i.e., every induced subgraph of a graph in the class is contained in the same class.

Restricting to interval graphs from now on, we discuss the details of the algorithm suggested by the two lemmas. Let $G$ be an interval graph of order $n$. By Theorem 5, there is a linear ordering $C_1, \ldots, C_q$ of the maximal cliques of $G$ such that each vertex of $G$ appears in consecutive cliques only. Consequently, a vertex $u_i$ of $G$ can be represented by a closed interval $I_{u_i} = [l_i, r_i]$, where $l_i$ and $r_i$ respectively are the minimum and maximum indices $j$ such that $u_i \in V(C_j)$. So, $I_{u_1} = [1, 1]$ and $I_{u_n} = [q, q]$ for a left tip $u_1$ and a right tip $u_n$ of $G$. Such 'compact' interval representation can be built easily from an arbitrary interval representation by scanning the sorted list of $2n$ endpoints of $n$ intervals.

Let $G_{a,b}$, $a \le b$, be the subgraph of $G$ induced by $\bigcup_{i=a}^{b} V(C_i) - (V(C_{a-1}) \cup V(C_{b+1}))$, where $V(C_0)$ and $V(C_{q+1})$ are defined to be empty sets. In other words, $G_{a,b}$ is the intersection graph of intervals whose left and right endpoints are between $a$ and $b$ inclusive. A minimal vertex cut of $G$ is equal to $Z_j$, defined as $C_j \cap C_{j+1}$, for some $1 \le j < q$, because each $Z_j$ is a (not necessarily minimal) vertex cut and every vertex cut contains some $Z_j$ as a subset. In addition, $G = G_{1,q}$ and $G - Z_j$ is the disjoint union of $G_{1,j}$ and $G_{j+1,q}$; also, a connected component of $G - Z_j$ is represented as $G_{a,b}$ for some $[a, b] \subsetneq [1, q]$. These indicate an algorithm for computing $f_r(G)$ (and eventually $\mathrm{sc}''_r(G)$) using a dynamic programming approach.

Consider how to compute $f_r(G_{a,b})$ for $a \le b$ and $0 \le r \le n_{a,b}$, where $n_{a,b}$ denotes the order of $G_{a,b}$. The graph $G_{a,b}$ is possibly a null graph of order 0; let $f_0(G_{a,b}) = 0$ if $n_{a,b} = 0$. For the base case $a = b$, where $G_{a,b}$ is a complete graph or a null graph, we have $f_r(G_{a,b}) = 1$ if $n_{a,b} \ge 1$ and $r = 0$; $f_r(G_{a,b}) = 0$ otherwise. Assume $a < b$ hereafter, and let $Z'_j = Z_j \cap V(G_{a,b})$ for $a \le j < b$. We define $f_r^j(G_{a,b})$ to be the maximum of $c(G_{a,b} - X) - h_{r-|X|}(G_{a,b} - X) - |X|$ over all vertex subsets $X \subseteq V(G_{a,b})$ with $|X| \le r$ such that $X \supseteq Z'_j$, if such $X$ exists (i.e., $r \ge |Z'_j|$); $f_r^j(G_{a,b}) = -\infty$ otherwise. Also, let $f_r^0(G_{a,b})$ be the maximum of $c(G_{a,b} - X) - h_{r-|X|}(G_{a,b} - X) - |X|$ over all vertex subsets $X \subseteq V(G_{a,b})$ with $|X| \le r$ such that $X \not\supseteq Z'_j$ for all $j$. Then, by definition,

$$f_r(G_{a,b}) = \begin{cases} \max_{a \le j < b} f_r^j(G_{a,b}) & \text{if } G_{a,b} \text{ is disconnected,} \\ \max\{\max_{a \le j < b} f_r^j(G_{a,b}), f_r^0(G_{a,b})\} & \text{if } G_{a,b} \text{ is connected.} \end{cases}$$

Note that if $G_{a,b}$ is disconnected, then $Z'_j = \emptyset$ for some $j$, hence $X \supseteq Z'_j$ for every vertex subset $X$; also, the graph $G_{a,b} - X$, as well as $G_{a,b}$, is connected if $X \not\supseteq Z'_j$ for all $j$. The graph $G_{a,b} - Z'_j$ for $a \le j < b$ is the disjoint union of $G_{a,j}$ and $G_{j+1,b}$ (possibly, $G_{a,j}$ and/or $G_{j+1,b}$ are null graphs); so, if $r \ge |Z'_j|$,

$$\begin{aligned} f_r^j(G_{a,b}) &= f_{r-|Z'_j|}(G_{a,b} - Z'_j) - |Z'_j| \\ &= \max_{(r_1,r_2)} \left(f_{r_1}(G_{a,j}) + f_{r_2}(G_{j+1,b})\right) - |Z'_j|, \text{by Lemma 21(a),} \end{aligned}$$

where the maximum is taken over all pairs $(r_1, r_2)$ such that $r_1 + r_2 = r - |Z'_j|$, $0 \le r_1 \le n_{a,j}$, and $0 \le r_2 \le n_{j+1,b}$. Finally, we have $f_r^0(G_{a,b}) = 1$ if $n_{a,b} \ge 1$ and $r = 0$; $f_r^0(G_{a,b}) = 0$ otherwise.

What we have discussed so far in this section is summarized in Algorithm 1 and Theorem 10. As a preprocessing stage, we can compute $n_{a,b}$ and determine if $G_{a,b}$ is connected for all $a \le b$, in $O(n^2)$ time. As a result, $|Z'_j|$ is computed in constant time because $|Z'_j| = n_{a,b} - (n_{a,j} + n_{j+1,b})$. The **for** loops of Steps 8 and 9 of Algorithm 1 iterate over all subgraphs $G_{a,b}$ with $a < b$. Given $a$ and $b$, all $f_r(G_{a,b})$ for $0 \le r \le n_{a,b}$ are computed in Steps 10 through 28; given $a$, $b$ and $j$ with $a \le j < b$, all $f_r^j(G_{a,b})$ for $|Z'_j| \le r \le n_{a,b}$ are computed in Steps 13 through 22. Every **for** and **foreach** loop iterates $O(n)$ times, so the algorithm runs in $O(n^5)$ time.

**Theorem 10.** *All $r$-scattering numbers $\mathrm{sc}_r(G)$ of a noncomplete interval graph $G$ of order $n$ can be computed in $O(n^5)$ time.*

**Algorithm 1:** Computing $\mathrm{sc}''_r(G)$ of an interval graph $G$

**input** : An interval representation of an interval graph $G$ of order $n$;
**output:** $\mathrm{sc}''_r(G)$ for all $r \in \{0, \dots, n\}$;

```
/* preprocessing stage                                                        */
Build a 'compact' interval representation for G such that all endpoints
  of the intervals are integers between 1 and q inclusive.
Compute n_{a,b} and determine if G_{a,b} is connected for all a ≤ b.
κ(G) ← min_{1≤j<q}(n − (n_{1,j} + n_{j+1,q})) if q ≥ 2; κ(G) ← n − 1 otherwise.
/* compute all f_r(G_{a,b}), where 1 ≤ a ≤ b ≤ q and 0 ≤ r ≤ n_{a,b}          */
for a ← 1 to q do
    foreach r ∈ {0, ..., n_{a,a}} do f_r(G_{a,a}) ← 0;
    if n_{a,a} ≥ 1 then f_0(G_{a,a}) ← 1;
end
for d ← 1 to q − 1 do                                              // d = b − a
    for a ← 1 to q − d do                            // for each G_{a,b} = G_{a,a+d}
        b ← a + d;
        foreach r ∈ {0, ..., n_{a,b}} do maxSoFar[r] ← −∞;
        for j ← a to b − 1 do                              // for each Z'_j in G_{a,b}
            z'_j ← n_{a,b} − (n_{a,j} + n_{j+1,b}) ;                 // z'_j = |Z'_j|
            foreach r ∈ {z'_j, ..., n_{a,b}} do f_r^j(G_{a,b}) ← −∞;
            for r ← z'_j to n_{a,b} do
                r' ← r − z'_j;
                for r_1 ← max{0, r' − n_{j+1,b}} to min{n_{a,j}, r'} do
                    r_2 ← r' − r_1;
                    f_r^j(G_{a,b}) ← max{f_r^j(G_{a,b}), f_{r_1}(G_{a,j})+f_{r_2}(G_{j+1,b})−z'_j};
                end
                maxSoFar[r] ← max{maxSoFar[r], f_r^j(G_{a,b})};
            end
        end
        foreach r ∈ {0, ..., n_{a,b}} do f_r(G_{a,b}) ← maxSoFar[r];
        if G_{a,b} is connected then
            foreach r ∈ {0, ..., n_{a,b}} do f_r(G_{a,b}) ← max{f_r(G_{a,b}), 0};
            if n_{a,b} ≥ 1 then f_0(G_{a,b}) ← max{f_0(G_{a,b}), 1};
        end
    end
end
/* compute all sc''_r(G), where 0 ≤ r ≤ n                                     */
foreach r ∈ {0, ..., n} do sc''_r(G) ← −∞;
if q ≥ 2 then                                                // if G is noncomplete
    foreach r ∈ {κ(G), ..., n} do
        sc''_r(G) ← maxSoFar[r]     // the latest maxSoFar[r] for G_{1,q}
    end
end
```

The $O(n^5)$-time algorithm for computing the $r$-scattering numbers $\mathrm{sc}_r(G)$ of a noncomplete interval graph is implemented in C language. The source code and some running examples may be downloaded from `http://tcs.catholic.ac.kr/~jhpark/papers/sc_r.zip`.

It is worth mentioning that Steps 13–22 of Algorithm 1 actually compute a variation of convolution, named $(\max, +)$-convolution. To be specific, simply denoting $n_{a,j}$ by $n_1$, $n_{j+1,b}$ by $n_2$, $f_{r_1}(G_{a,j})$ by $a_{r_1}$, and $f_{r_2}(G_{j+1,b})$ by $b_{r_2}$, the steps calculate a vector $(c_0, \ldots, c_{n_1+n_2})$ from two vectors $(a_0, \ldots, a_{n_1})$ and $(b_0, \ldots, b_{n_2})$, where $c_{r'} = \max_{(r_1,r_2)}(a_{r_1} + b_{r_2})$ over all $(r_1, r_2)$ with $r_1 + r_2 = r'$, $0 \leq r_1 \leq n_1$, and $0 \leq r_2 \leq n_2$. Then, $f_r^j(G_{a,b})$ will be $c_{r-z'_j} - z'_j$ for $z'_j \leq r \leq n_{a,b}$. In contrast to the standard convolution that admits an $O((n_1+n_2)\log(n_1+n_2))$-time algorithm, where $c_{r'}$ is defined to be $\sum_{(r_1,r_2)}(a_{r_1} \cdot b_{r_2})$, the $(\max, +)$-convolution is a hard problem for which there is no known truly subquadratic, $O((n_1 + n_2)^{2-\epsilon})$-time algorithm for $\epsilon > 0$ [2, 22]. Bremner et al. [4] devised an $o(n^2)$-time algorithm for computing the $(\max, +)$-convolution of two vectors of size $n$. (Its running time is not $O(n^{2-\epsilon})$.) If the algorithm of Bremner et al. is employed in place of Steps 13–22 of Algorithm 1, the running time of our algorithm would be slightly improved to $o(n^5)$.

On the other hand, the problem of determining the $r$-scattering number of a general graph is NP-complete as shown below.

**Theorem 11.** *Given a noncomplete graph $G$ and an integer $K$, the problem of deciding if $\mathrm{sc}_r(G) \geq K$ is NP-complete for any fixed $r$.*

*Proof.* It was proved by Zhang et al. [24] that the problem of deciding if $\mathrm{sc}(G) \geq K$ for a graph $G$ and an integer $K$ is NP-complete. We show that the scattering number problem is polynomial-time reducible to the $r$-scattering number problem. Given a graph $G$ of order $n$, we define a graph $G'$ of order $n + r$ as follows: $V(G') = V(G) \cup W$ for some set $W$ with $|W| = r$ such that $W \cap V(G) = \emptyset$; $E(G') = E(G) \cup \{(u, v) : u \in V(G), v \in W\} \cup \{(v, v') : v, v' \in W, v \neq v'\}$. Then, every vertex cut of $G'$ contains $W$ as a subset; $X$ is a vertex cut of $G$ if and only if $X \cup W$ is a vertex cut of $G'$; and the connected components of $G - X$ are exactly the same as those of $G' - (X \cup W)$. It follows that

$$\begin{aligned}
\mathrm{sc}_r(G') &= \max\{c(G' - X') - |X'| : X' \text{ is a vertex cut of } G'\} \text{ (because } |X'| \geq r) \\
&= \max\{c(G' - (X \cup W)) - |X \cup W| : X \cup W \text{ is a vertex cut of } G'\} \\
&= \max\{c(G - X) - |X| - |W| : X \text{ is a vertex cut of } G\} \\
&= \mathrm{sc}(G) - r,
\end{aligned}$$

i.e., $\mathrm{sc}(G) \geq K$ if and only if $\mathrm{sc}_r(G') \geq K - r$. It is clear that our reduction is polynomial in the input size, completing the proof. ■

## 6. Concluding remarks

In this paper, we proposed the notion of $r$-scattering number, a generalization of the scattering number, and investigated the unpaired $k$-disjoint path coverability of an interval graph. It was proved that a noncomplete graph $G$ of order $n \geq 2k$ for $k \geq 1$ is unpaired $k$-disjoint path coverable only if $\mathrm{sc}_{2k}(G) \leq -k$, and that a noncomplete interval graph $G$ of order $n \geq 4$ is unpaired 2-disjoint path coverable if and only if $\mathrm{sc}_4(G) \leq -2$. According to the proofs given in this paper, we can design a polynomial-time algorithm for building an unpaired 2-DPC in an interval graph that is unpaired 2-disjoint path coverable. Furthermore, it was shown that the $r$-scattering number of an interval graph can be computed in polynomial time. It is open to characterize interval graphs that are unpaired $k$-disjoint path coverable for $k \geq 3$, or to settle down the following conjecture:

**Conjecture 1.** Let $G$ be a noncomplete interval graph of order $n \geq 2k$ for $k \geq 1$. Then, $G$ is unpaired $k$-disjoint path coverable if and only if $\mathrm{sc}_{2k}(G) \leq -k$.

The conjecture turns out to be true if $k = 1$ or if $G$ is a proper interval graph, as shown in Appendix A.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## Appendix A. Toward resolving Conjecture 1

We show that Conjecture 1 holds true if $k = 1$, or if $G$ is a proper interval graph.

**Lemma 22.** *For a noncomplete graph $G$, $\mathrm{sc}_2(G) \leq -1$ if and only if $\mathrm{sc}(G) \leq -1$.*

*Proof.* Let $G$ be a noncomplete graph. Since $\mathrm{sc}_2(G) \leq \mathrm{sc}_0(G) = \mathrm{sc}(G)$ by Lemma 4(b), it suffices to show the necessity part. Assume $\mathrm{sc}_2(G) \leq -1$, meaning that $G$ is 1-connected by Lemma 4(b) again. Suppose $\mathrm{sc}(G) \geq 0$ for a contradiction. Then, there is a scattering set $X$ such that $c(G-X)-|X| = \mathrm{sc}(G) \geq 0$. If $|X| \geq 2$, then $\mathrm{sc}_2(G) \geq c_2(G,X)-|X| = c(G-X)-|X| = \mathrm{sc}(G) \geq 0$, contradicting the hypothesis $\mathrm{sc}_2(G) \leq -1$. If $|X| = 1$, then $\mathrm{sc}_2(G) \geq c_2(G,X)-|X| = (c(G-X)-1)-|X| = c(G-X)-2 \geq 0$, also contradicting the hypothesis. Note that $c(G-X) \geq 2$ because $X$ is a vertex cut of $G$. Thus, the lemma is proven. ■

**Theorem 12.** *A noncomplete interval graph $G$ is unpaired 1-disjoint path coverable (or equivalently, Hamiltonian-connected) if and only if $\mathrm{sc}_2(G) \leq -1$.*

*Proof.* The theorem follows from Lemma 22 and Theorem 1. ■

Now, we turn to the second case where $G$ is a proper interval graph. It has been known that the classes of proper interval graphs and unit interval graphs coincide [23], where a *unit interval graph* is an interval graph with an interval representation in which all the intervals have unit length. An ordering, $(v_1, \ldots, v_n)$, of the vertices of a graph $G$ is a *consecutive ordering* if for each vertex $v_i$, its closed neighbor $N[v_i]$ is consecutive, i.e., $N[v_i] = \{v_j : l_i \leq j \leq r_i\}$ for some $l_i$ and $r_i$.

**Lemma 23** (Chen et al. [7]). *(a) A graph $G$ is a proper interval graph if and only if $G$ has a consecutive ordering. (b) Given a consecutive ordering $(v_1, \ldots, v_n)$ of a proper interval graph $G$ of order $n \geq k+1$ for a positive integer $k$, $G$ is $k$-connected if and only if $(v_i, v_j) \in E(G)$ whenever $1 \leq |i-j| \leq k$.*

**Theorem 13** (Lee et al. [11]). *Let $G$ be a proper interval graph of order $n \geq 2k$ with a consecutive ordering $(v_1, \ldots, v_n)$ for $k \geq 2$. Then, $G$ is unpaired $k$-disjoint path coverable if and only if $G$ is $k$-connected and either $n = 2k$ or $n \geq 2k+1$ and $(v_i, v_{i+k+1}) \in E(G)$ for every $i$, $1 \leq i \leq n-2k$ or $k \leq i \leq n-k-1$.*

**Theorem 14.** *A noncomplete proper interval graph $G$ of order $n \geq 2k$ for $k \geq 2$ is unpaired $k$-disjoint path coverable if and only if $\mathrm{sc}_{2k}(G) \leq -k$.*

*Proof.* The necessity is due to Theorem 6. For the sufficiency proof, assume $\mathrm{sc}_{2k}(G) \leq -k$. Suppose, for a contradiction, that $G$ is not unpaired $k$-disjoint path coverable. Then, by Theorem 13, (i) $G$ is not $k$-connected, or (ii) $n \geq 2k+1$ and $(v_i, v_{i+k+1}) \notin E(G)$ for some $i$, where $1 \leq i \leq n-2k$ or $k \leq i \leq n-k-1$. If $G$ is not $k$-connected, then $\mathrm{sc}_{2k}(G) \not\leq -k$ by Lemma 4(b), which contradicts the hypothesis. Now, suppose that $G$ is $k$-connected and the condition (ii) is satisfied. Let $X = \{v_{i+1}, \ldots, v_{i+k}\}$. Then, $X$ is a vertex cut of $G$ and $G - X$ has two connected components, which are subgraphs of $G$ induced by $\{v_1, \ldots, v_i\}$ and $\{v_{i+k+1}, \ldots, v_n\}$, respectively. The larger connected component, say $H_1$, between the two has at least $k$ vertices, because

$$|V(H_1)| = \max\{i, n-i-k\} \geq \begin{cases} n-i-k & \geq k \quad \text{if } 1 \leq i \leq n-2k, \\ i & \geq k \quad \text{if } k \leq i \leq n-k-1. \end{cases}$$

It follows that $c_{2k}(G, X) = 2 - 1 = 1$, leading to $\mathrm{sc}_{2k}(G) \geq c_{2k}(G, X) - |X| = 1 - k$, which also contradicts the hypothesis $\mathrm{sc}_{2k}(G) \leq -k$. Thus, the theorem is proven. ■

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