Abstract

Given disjoint source and sink sets, \( S = \{s_1, \ldots, s_k\} \) and \( T = \{t_1, \ldots, t_k\} \), in a graph \( G \), an \emph{unpaired} \( k \)-disjoint path cover joining \( S \) and \( T \) is a set of pairwise vertex-disjoint paths \( \{P_1, \ldots, P_k\} \) that altogether cover every vertex of the graph, in which \( P_i \) is a path from source \( s_i \) to some sink \( t_j \). In terms of a generalized scattering number, named an \emph{\( r \)-scattering number}, we characterize interval graphs that have an unpaired 2-disjoint path cover joining \( S \) and \( T \) for any possible configurations of source and sink sets \( S \) and \( T \) of size 2 each. Also, it is shown that the \( r \)-scattering number of an interval graph can be computed in polynomial time.

Keywords: Disjoint path, path cover, path partition, scattering number, \( r \)-scattering number, interval graphs.

1. Introduction

Let \( G \) be a finite, simple undirected graph whose vertex and edge sets are denoted by \( V(G) \) and \( E(G) \), respectively. A \emph{path} from \( u \in V(G) \) to \( v \in V(G) \), referred to as a \emph{\( u \)-\( v \) path}, is a sequence \( \langle w_1, \ldots, w_l \rangle \) of distinct vertices of \( G \) such that \( w_1 = u \), \( w_l = v \), and \( (w_i, w_{i+1}) \in E(G) \) for all \( i \in \{1, \ldots, l - 1\} \). If \( l \geq 3 \) and \( (w_i, w_1) \in E(G) \), the sequence is called a \emph{cycle}. A \emph{path cover} of a graph \( G \) is a set of paths in \( G \) such that every vertex of \( G \) is contained in at least one path. A \emph{disjoint path cover} (DPC for short) of \( G \) is a set of vertex-disjoint paths that altogether cover every vertex of \( G \). This paper is concerned with a DPC in which each path runs from a prescribed source to a prescribed sink.

Given disjoint subsets \( S = \{s_1, \ldots, s_k\} \) and \( T = \{t_1, \ldots, t_k\} \) of \( V(G) \) for a positive integer \( k \), a \emph{many-to-many} \( k \)-disjoint path cover is a DPC composed of \( k \) paths, each of which joins a pair of source \( s_i \in S \) and sink \( t_j \in T \). If each...
source $s_i \in S$ must be joined to a specific sink $t_j \in T$, the DPC is called paired, and it is unpaired if no such constraint is imposed. The other DPC type is a one-to-many $k$-disjoint path cover for disjoint subsets $S = \{s\}$ and $T = \{t_1, \ldots, t_k\}$, in which each path runs from the common source $s$ to a sink $t_j, j \in \{1, \ldots, k\}$. When $S = \{s\}$ and $T = \{t\}$, a DPC composed of $k$ paths, each of which joins $s$ and $t$, is named one-to-one $k$-disjoint path cover.

**Definition 1** (Park et al. [20]). A graph $G$ of order $n \geq 2k$ is unpaired $k$-disjoint path coverable if $G$ has an unpaired (many-to-many) $k$-DPC joining $S$ and $T$ for any disjoint source and sink sets, $S$ and $T$, of size $k$ each.

Analogously, we can define paired $k$-disjoint path coverable graphs. Refer to Fig. 1 for examples. In addition, a graph $G$ of order $n \geq k + 1$ is one-to-many $k$-disjoint path coverable if for any disjoint subsets $S$ and $T$ of $V(G)$ with $|S| = 1$ and $|T| = k$, there exists a one-to-many $k$-DPC joining $S$ and $T$. A graph $G$ of order $n \geq k + 1$ is one-to-one $k$-disjoint path coverable if for any disjoint subsets $S$ and $T$ of $V(G)$ with $|S| = |T| = 1$, there exists a one-to-one $k$-DPC joining $S$ and $T$. The disjoint path coverability of a graph is closely related to the Hamiltonian properties (as well as the vertex connectivity) of the graph. A graph of order $n \geq 3$, for instance, is one-to-many 2-disjoint path coverable if and only if it is Hamiltonian-connected. Furthermore, a graph of order $n \geq 3$ is one-to-one 2-disjoint path coverable if and only if it is Hamiltonian.

An interval graph is the intersection graph of a family $I$ of intervals on the real line, where two vertices are connected with an edge if and only if their corresponding intervals intersect. The family $I$ is usually called an interval representation for the graph. A proper interval graph is an interval graph with an interval representation in which none of the intervals properly contains another. The interval graphs that are Hamiltonian/traceable were characterized by Deogun et al. [8]; also, those that are Hamiltonian-connected were characterized by Broersma et al. [5] (Theorem 3 below). The characterizations are all in terms of the scattering number. For a noncomplete graph $G$, the scattering number $sc(G)$ of $G$ is defined as

$$sc(G) = \max\{c(G - X) - |X| : X \subset V(G), c(G - X) \geq 2\},$$

where $c(G - X)$ denotes the number of connected components in $G - X$. A
vertex cut $X$ of $G$ that fulfills $c(G - X) - |X| = \text{sc}(G)$ is called a scattering set. For a complete graph $K_n$ of order $n$, we set $\text{sc}(K_n) = 3 - n$ in this paper, so that the scattering numbers of the $n$-vertex graphs form a consecutive set \{3 $- n, 4 - n, \ldots, n}\). \[21\]

**Theorem 1** (Broersma et al. [5]). An interval graph $G$ of order $n \geq 3$ is Hamiltonian-connected if and only if $\text{sc}(G) \leq -1$ or $G$ is isomorphic to a complete graph $K_3$.

Extending Theorem 1, the one-to-one $k$-disjoint path coverability and one-to-many $k$-disjoint path coverability of interval graphs were characterized by Li et al. [13] and by Park et al. [21] (Theorems 2 and 3 below). Both characterizations can be described in terms of the scattering number. In addition, a sufficient condition for an interval graph to be unpaired $k$-disjoint path coverable was established, as shown in Theorem 4.

**Theorem 2** (Li and Wu [13] and Park et al. [21]). For $k \geq 2$, an interval graph $G$ of order $n \geq k + 1$ is one-to-one $k$-disjoint path coverable if and only if $\text{sc}(G) \leq 2 - k$.

**Theorem 3** (Li and Wu [13] and Park et al. [21]). For $k \geq 2$, an interval graph $G$ of order $n \geq k + 1$ is one-to-many $k$-disjoint path coverable if and only if $\text{sc}(G) \leq 1 - k$ or $G$ is isomorphic to a complete graph $K_{k+1}$.

**Theorem 4** (Park [18]). Let $G$ be an interval graph of order $n \geq 2k$ for $k \geq 2$. If $\text{sc}(G) \leq -k$, then $G$ is unpaired $k$-disjoint path coverable.

The disjoint path cover problems are applicable in many areas such as software testing, database design, and code optimization [1, 16]. The problems have been studied for various classes of graphs, including recent studies on dense graphs [13], cube of connected graphs [19], balanced hypercubes [15], directed graphs [6], and torus networks [12, 17]. The one-to-one and one-to-many $k$-disjoint path coverability of interval graphs can be checked in linear time thanks to the linear-time algorithms for finding the scattering number of an interval graph devised by Broersma et al. [5] and by Li et al. [13]. For proper interval graphs, a characterization regarding the unpaired $k$-disjoint path coverability was derived by Lee et al. [11].

We begin our study on Unpaired $k$-Disjoint Path Coverability of interval graphs by reviewing the sufficient condition of Theorem 4 and the necessary condition of Lemma 1 below derived in terms of the connectivity.

**Lemma 1** (Park et al. [20]). If a graph $G$ of order $n \geq 2k$ for $k \geq 1$ is unpaired $k$-disjoint path coverable, then $G$ is $k$-connected.

If $G$ is a noncomplete interval graph, the conditions of Theorem 4 and of Lemma 1 respectively can be represented as

\[ \text{sc}(G) = \max_X (c(G - X) - |X|) \leq -k, \quad (1) \]
\[ -\kappa(G) = \max_X (0 - |X|) \leq -k, \quad (2) \]
where the maximums are taken over all vertex cuts $X$ of $G$. Comparing the above two in a hope of deriving a necessary and sufficient condition, it can be said that the term “$c(G - X)$” of Eq. 1 is too large whereas the term “$0$” of Eq. 2 is too small. For the graphs $G_{2,3}$ and $G_{3,4}$ of Fig. 1 and $k = 2$, for example, we have $sc(G_{2,3}) = "2" - 3 \nleq -2$ and $-\kappa(G_{3,4}) = "0" - 4 \leq -2$; also, both graphs have the same scattering number, but one is unpaired 2-disjoint path coverable and the other is not. Accordingly, we introduce a function $c_r(G, X)$, ranging over $\{0, 1, \ldots, c(G - X)\}$, which is defined to be the maximum number of connected components in $G - X$ each of which contains no marked vertices when marking exactly $r$ out of $n$ vertices in $G$, where $r$ is an integer between 0 and $n$. In other words, 

$$c_r(G, X) = \begin{cases} 
  c(G - X) & \text{if } |X| \geq r, \\
  c(G - X) - h_{r-|X|}(G - X) & \text{if } |X| < r,
\end{cases}$$

where $h_{r-|X|}(G - X)$ is the minimum number of connected components in $G - X$ whose total number of vertices is $r - |X|$ or more, i.e., $h_{r-|X|}(G - X)$ is equal to the integer $q$ such that $|X| + \sum_{i=1}^{q} |V(H_i)| \geq r > |X| + \sum_{i=1}^{q-1} |V(H_i)|$ for the connected components $H_1, \ldots, H_p$ of $G - X$ with $|V(H_1)| \geq \cdots \geq |V(H_p)|$, where $p = c(G - X)$.

Now, we introduce the notion of $r$-scattering number, a generalization of the scattering number in the sense that the 0-scattering number of a graph is equal to the scattering number of the graph.

**Definition 2** ($r$-Scattering number). For a noncomplete graph $G$ of order $n$ and a nonnegative integer $r$ with $r \leq n$, the $r$-scattering number, denoted $sc_r(G)$, of $G$ is the maximum of $c_r(G, X) - |X|$ over all vertex cuts $X$ of $G$.

A vertex cut $X$ of $G$ that fulfills $c_r(G, X) - |X| = sc_r(G)$ is called an $r$-scattering set. The $r$-scattering number of a complete graph $K_n$ is left undefined. See Fig. 2 for an example of calculating the $r$-scattering numbers. Also, the 4-scattering numbers of the graphs $G_{2,3}$ and $G_{3,4}$ of Fig. 1 are $sc_4(G_{2,3}) = "1" - 3 = -2$ (which is smaller than $sc_4(G_{2,3}) = "2" - 3$) and $sc_4(G_{3,4}) = "3" - 4 = -1$ (which is larger than $-\kappa(G_{3,4}) = "0" - 4$). In addition, the $r$-scattering numbers of $G_{3,5}$ are $sc_6(G_{3,5}) = -5$, $sc_7(G_{3,5}) = -4$, $sc_6(G_{3,5}) = -3$, and $sc_5(G_{3,5}) = sc_4(G_{3,5}) = sc_3(G_{3,5}) = sc_2(G_{3,5}) = sc_1(G_{3,5}) = sc_0(G_{3,5}) = -2 = sc(G_{3,5})$.

In this paper, we derive a necessary condition for a general graph to be unpaired $k$-disjoint path coverable, and then establish a characterization of interval graphs that are unpaired 2-disjoint path coverable. Specifically, we prove that a noncomplete graph $G$ of order $n \geq 2k$ for $k \geq 1$ is unpaired $k$-disjoint path coverable only if $sc_{2k}(G) \leq -k$, and that a noncomplete interval graph $G$ of order $n \geq 4$ is unpaired 2-disjoint path coverable if and only if $sc_4(G) \leq -2$. Moreover, we show that the $r$-scattering number of an interval graph can be computed in polynomial time, whereas deciding whether the $r$-scattering number of a general graph is greater than or equal to given $K$ is NP-complete for any fixed $r$. 


Fig. 2: An interval representation for an interval graph $G$, where $sc_4(G) = \max\{1 - |\{u_2, u_3, u_4\}|, 2 - |\{u_2, u_3, u_5, u_6\}|, \ldots, 3 - |\{u_2, u_3, u_4, u_5, u_6\}|\} = -2$ and $sc_6(G) = -2$. The graph $G$ is unpaired 2-disjoint path coverable but has no unpaired 3-DPC joining $S = \{u_2, u_3, u_4\}$ and $T = \{u_5, u_6, u_7\}$.

2. Preliminaries

A cycle that visits each vertex exactly once is a Hamiltonian cycle; a path that visits each vertex exactly once is a Hamiltonian path. A graph is Hamiltonian if a Hamiltonian cycle exists; a graph is traceable if a Hamiltonian path exists; a graph is Hamiltonian-connected if every two distinct vertices are joined by a Hamiltonian path. A connected component of $G$ is a maximal connected subgraph of $G$. A vertex cut of $G$ is a set $X \subseteq V(G)$ such that $G - X$ has two or more connected components, where $G - X$ is the subgraph obtained from $G$ by deleting all the vertices of $X$ (or equivalently, $G - X$ is the subgraph of $G$ induced by $V(G) \setminus X$). The connectivity of $G$, denoted $\kappa(G)$, is the minimum number of vertices whose removal results in a disconnected graph or a single vertex. So, $\kappa(G)$ is equal to the size of a minimum vertex cut if $G$ is a noncomplete graph; $\kappa(G) = n - 1$ if $G$ is a complete graph $K_n$. A graph $G$ is $k$-connected if $\kappa(G) \geq k$. In addition, $N_G(v)$, or $N(v)$ if the graph $G$ is clear in the context, represents the open neighborhood of a vertex $v \in V(G)$, i.e. $N_G(v) = \{u \in V(G) : (u, v) \in E(G)\}$, whereas $N_G[v]$, or $N[v]$, denotes the closed neighborhood of $v$, i.e. $N_G[v] = N_G(v) \cup \{v\}$. The graph-theoretic terms that are not defined here can be found in [9].

Interval graphs are a well-studied class of graphs. One of the early characterizations of interval graphs is the following:

**Theorem 5** (Gilmore and Hoffman [9]). A graph $G$ is an interval graph if and only if the maximal cliques of $G$ can be linearly ordered such that, for every vertex $v$ of $G$, the maximal cliques containing $v$ occur consecutively.

Let $C_1, \ldots, C_q$ be a linear ordering of the maximal cliques of an interval graph $G$ such that each vertex of $G$ appears in consecutive cliques only. Obviously, the graph $G$ is noncomplete if and only if $q \geq 2$. Since $C_1$ and $C_q$ are maximal cliques, $C_1$ and $C_q$ each contains at least one vertex that does not occur in any other maximal clique. Let $u_1$ be such a vertex in $C_1$ and let $u_n$ be such a vertex in $C_q$. The vertices $u_1$ and $u_n$, respectively, are referred to as a left tip and a right tip.

**Lemma 2** (Park [18]). Let $u_1$ be a left tip of an interval graph $G$ with $sc(G) \leq 0$. Then, a Hamiltonian $u_1$–$w$ path exists in $G$ for every vertex $w$ other than $u_1$. 


Finally, the following relationship between the scattering number and the connectivity of a noncomplete graph will also be used for our proof:

**Lemma 3** (Zhang and Wang [25]). Let $G$ be a noncomplete graph of order $n \geq 2$. Then, $\text{sc}(G) \geq 2 - \kappa(G)$.

### 3. Unpaired $k$-disjoint path coverability of general graphs

In this section, we establish a necessary condition for a noncomplete graph to be unpaired $k$-disjoint path coverable, and then show that the condition is also a sufficient one for two special cases. We begin with some basic properties of the $r$-scattering number. The $r$-scattering number of a graph $G$ can be seen as bridging the gap between the scattering number and the connectivity of $G$, in that the 0-scattering number becomes $\text{sc}(G)$ and the $n$-scattering number becomes $-\kappa(G)$.

**Lemma 4.** Let $G$ be a noncomplete graph of order $n \geq 2$.

(a) $\text{sc}_{r+1}(G) \leq \text{sc}_r(G) \leq \text{sc}_{r+1}(G) + 1$ for all $r \in \{0, \ldots, n-1\}$.

(b) $-\kappa(G) \leq \text{sc}_n(G) \leq \text{sc}_{n-1}(G) \leq \cdots \leq \text{sc}_1(G) \leq \text{sc}_0(G) = \text{sc}(G)$.

(c) If $G$ is $k$-connected, then $\text{sc}_r(G) = \text{sc}(G)$ for all $r \in \{0, \ldots, k\}$.

**Proof.** For the proof of (a), it suffices to show that $c_{r+1}(G,X) \leq c_r(G,X) \leq c_{r+1}(G,X) + 1$ for every vertex cut $X$ of $G$. If $|X| \geq r + 1$, then $c_r(G,X) = c_{r+1}(G,X) = c(G-X)$; secondly, if $|X| = r$, then $c_r(G,X) = c(G-X)$ and $c_{r+1}(G,X) = c(G-X)-1$; if $|X| < r$ finally, then $h_{r+1-|X|}(G-X) = h_r-|X|)(G-X)$ or $h_{r-|X|}(G-X) + 1$. This means that $h_{r+1-|X|}(G-X) \leq h_r-|X|)(G-X)$, leading to that $-h_{r+1-|X|}(G-X) \leq -h_r-|X|)(G-X) \leq -h_{r+1-|X|}(G-X) + 1$. Thus, (a) is proven. The inequalities of (b) is due to (a). In addition, we have $c_0(G,X) = 0$ and $c_0(G,X) = c(G-X)$, respectively leading to $\text{sc}_n(G) = -\kappa(G)$ and $\text{sc}_0(G) = \text{sc}(G)$, proving (b). Finally, if $G$ is $k$-connected and $r \leq k$, then every vertex cut $X$ of $G$ is of size $r$ or more, meaning $c_r(G,X) = c(G-X)$, proving (c).

**Lemma 5.** Let $G$ be a noncomplete graph of order $n \geq 2$. Then,

$$
\text{sc}_r(G) \geq \begin{cases} 
4 - n & \text{if } r \leq n - 2, \\
3 - n & \text{if } r = n - 1, \\
2 - n & \text{if } r = n.
\end{cases}
$$

**Proof.** For a pair of nonadjacent vertices $u$ and $v$ of $G$, let $X = V(G) \setminus \{u, v\}$. It follows that

$$
\text{sc}_r(G) \geq c_r(G,X) - |X| \geq \begin{cases} 
2 - (n - 2) = 4 - n & \text{if } r \leq n - 2, \\
1 - (n - 2) = 3 - n & \text{if } r = n - 1, \\
0 - (n - 2) = 2 - n & \text{if } r = n.
\end{cases}
$$

Thus, the lemma is proven.

\[\]
In terms of the $2k$-scattering number, we can establish a necessary condition for a general graph to be unpaired $k$-disjoint path coverable, as shown in Theorem 6 below. The new condition $\text{sc}_{2k}(G) \leq -k$ of Theorem 6 is stronger than the condition $\kappa(G) \geq k$ of Lemma 1 in the sense that every graph that satisfies $\text{sc}_{2k}(G) \leq -k$ also satisfies $\kappa(G) \geq k$ but the converse is not always true. Note that the hypothesis $\text{sc}_{2k}(G) \leq -k$ implies $\text{sc}_{c}(G) = -\kappa(G) \leq -k$ by Lemma 1(b), and that the interval graph of Fig. 2, which is not unpaired 3-disjoint path coverable, is 3-connected and $\text{sc}_{c}(G) = -2 \not\leq -3$.

**Theorem 6.** If a noncomplete graph $G$ of order $n \geq 2k$ for $k \geq 1$ is unpaired $k$-disjoint path coverable, then $\text{sc}_{2k}(G) \leq -k$.

**Proof.** Let $G$ be a noncomplete graph of order $n \geq 2k$ that is unpaired $k$-disjoint path coverable. Suppose $\text{sc}_{2k}(G) \geq 1 - k$ for a contradiction. Then, there exists a $2k$-scattering set $X$ such that $c_{2k}(G, X) - |X| = \text{sc}_{2k}(G) \geq 1 - k$; moreover, $|X| \geq k$ because $G$ is $k$-connected by Lemma 1. Let $H_1, \ldots, H_p$ be the connected components of $G - X$ such that $|V(H_i)| \geq \cdots \geq |V(H_p)|$, where $p = c(G - X) \geq 2$. There are two cases according to the size of $X$.

**Case 1:** $k \leq |X| < 2k$. Let $|X| = k + r$ for some $0 \leq r < k$; let $q$ be an integer (of Definition 2) such that $|X| + \sum_{i=1}^{q} |V(H_i)| \geq 2k > |X| + \sum_{i=1}^{k-1} |V(H_i)|$, so $c_{2k}(G, X) = p - q$. In particular, consider an unpaired $k$-DPC $\{P_1, \ldots, P_k\}$ that joins disjoint terminal sets $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ such that $S \subseteq X$, $X \subseteq S \cup T$, and $T \subseteq X \cup \bigcup_{i=1}^{q} V(H_i)$, as illustrated in Fig. 3(a). Note that $|S \cap X| = k$, $|T \cap X| = r$, and $V(H_i) \cap (S \cup T) = \emptyset$ for all $i \in \{q + 1, \ldots, p\}$. Then, there are $c_{2k}(G, X)$ connected components of $G - X$ each of whose intersection with $S \cup T$ is empty, where $c_{2k}(G, X) \geq |X| + 1 - k = r + 1$. A DPC path $P_i$ that runs to a sink in $X$ passes through at most one among the $c_{2k}(G, X)$ components $H_{q+1}, \ldots, H_p$, whereas a DPC path $P_j$ that runs to a sink in $\bigcup_{i=1}^{q} V(H_i)$ passes through no component among $H_{q+1}, \ldots, H_p$. It follows that the $k$ paths in the DPC altogether pass through at most $r$ components among the $c_{2k}(G, X)$ components $H_{q+1}, \ldots, H_p$. This contradicts the fact that $\{P_1, \ldots, P_k\}$ is an unpaired $k$-DPC joining $S$ and $T$, proving Case 1.

**Case 2:** $|X| \geq 2k$. In this case, we have $c_{2k}(G, X) = p$. Let $S$ and $T$ be disjoint terminal sets of size $k$ each such that $S \cup T \subseteq X$, as illustrated in
Fig. 3(b). Consider an unpaired $k$-DPC joining $S$ and $T$, in which each path $P_i$ passes through at most $|V(P_i) \cap X| - 1$ connected components. It follows that the DPC paths collectively pass through at most $|X| - k$ connected components in total, leading to $|X| - k \geq p$. This contradicts the hypothesis $p - |X| = c_{2k}(G, X) - |X| \geq 1 - k$, thereby completing the proof. ■

Theorem 6 allows us to restrict our attention to the class of graphs whose $2k$-scattering numbers are $-k$ or less. A graph in the class is $k$-connected; its scattering number is considered in the following lemma.

**Lemma 6.** Let $G$ be a noncomplete graph of order $n \geq 2k$ for $k \geq 1$. If $\text{sc}_{2k}(G) \leq -k$, then $\text{sc}(G) \leq 0$.

**Proof.** A graph $G$ with $\text{sc}_{2k}(G) \leq -k$ is $k$-connected by Lemma 4(b), leading to $\text{sc}(G) = \text{sc}_k(G)$ by Lemma 4(c). In addition, we have $\text{sc}_k(G) \leq \text{sc}_{k+1}(G) + 1 \leq \cdots \leq \text{sc}_{k+k}(G) + k \leq -k + k = 0$ by Lemma 4(a), completing the proof. ■

In the remaining part of this section, we show that the necessary condition $\text{sc}_{2k}(G) \leq -k$ of Theorem 6 becomes a sufficient one if $n = 2k$, or if $\text{sc}(G) = 0$. Firstly, let $G$ be a noncomplete graph of order $n = 2k$ for $k \geq 2$. An unpaired $k$-DPC of $G$ joining $S$ and $T$ in this case forms a perfect matching in which each edge joins a source and a sink. So, the graph $G$ is unpaired $k$-disjoint path coverable if and only if $G$ is strongly matchable, where a graph $G$ is said to be strongly matchable if $G$ has a perfect matching in which each edge joins two vertices, one in $S$ and the other in $T$, for every partition of $V(G)$ into $S$ and $T$ with $|S| = |T| = \frac{n}{2}$. In addition, the condition $\text{sc}_{2k}(G) \leq -k$ is equivalent to that $\kappa(G) \geq k$ because $\text{sc}_{2k}(G) = \text{sc}_n(G) = -\kappa(G)$ by Lemma 4(b).

**Theorem 7.** Let $G$ be a noncomplete graph of order $n = 2k$ for $k \geq 2$. Then, $G$ is unpaired $k$-disjoint path coverable if and only if $\kappa(G) \geq k$.

**Proof.** The necessity is due to Theorem 6. For the sufficiency proof, let $\kappa(G) \geq k$. Suppose for a contradiction that $G$ is not unpaired $k$-disjoint path coverable. Then, for some partition $S$ and $T$ of $V(G)$ with $|S| = |T| = k$, there is no perfect matching joining $S$ and $T$. By the well-known Hall’s marriage theorem, there exists a subset $W \subseteq S$ such that $|N(W) \cap T| < |W|$, where $N(W)$ denotes the open neighborhood of $W$ defined as $\bigcup_{w \in W} N(w) \setminus W$. It follows that $|W| \geq 1$ and $|N(W) \cap T| < k$, meaning $(S \setminus W) \cup (N(W) \cap T)$ is a vertex cut of $G$ (because $W \neq \emptyset$, $T \setminus (N(W) \cap T) \neq \emptyset$, and there is no edge between the two subsets). However, the size of the cut $|(S \setminus W) \cup (N(W) \cap T)| = |S| - |W| + |N(W) \cap T| < |S| = k$, which contradicts the fact $\kappa(G) \geq k$. Thus, the theorem is proven. ■

Now, we consider the other special case where $\text{sc}(G) = 0$.

**Theorem 8.** Let $G$ be a noncomplete graph of order $n \geq 2k$ for $k \geq 2$ with $\text{sc}(G) = 0$. Then, $G$ is unpaired $k$-disjoint path coverable if and only if $\text{sc}_{2k}(G) \leq -k$.

**Proof.** The proof is a direct consequence of Lemmas 7 and 8 below. ■
Lemma 7 (Park et al. [20]). A complete bipartite graph $K_{k,k}$, $k \geq 1$, is unpaired $k$-disjoint path coverable.

Lemma 8. Let $G$ be a noncomplete graph of order $n \geq 2k$ for $k \geq 2$. If $\text{sc}_{2k}(G) \leq -k$ and $\text{sc}(G) = 0$, then $G$ is isomorphic to a spanning supergraph of a complete bipartite graph $K_{k,k}$ and isomorphic to a spanning subgraph of a complete split graph $G_{k,k}$.

Proof. It holds that $k \leq \kappa(G) < 2k$ by Lemma 1(b) and (c). There is a scattering set $X$ of $G$ such that $c(G-X) = |X| = \text{sc}(G) = 0$. Then, $|X| \geq k$ and moreover $|X| < 2k$; supposing $|X| \geq 2k$ leads to that $\text{sc}_{2k}(G) \geq c_{2k}(G,X) - |X| = c(G-X) - |X| = 0$, which contradicts the hypothesis $\text{sc}_{2k}(G) \leq -k$. First we prove a claim: $|X| = k$ and every connected component of $G - X$ is a singleton. Suppose to the contrary that $|X| \geq k + 1$ or there is a connected component that contains two or more vertices. Then, $h_{2k-|X|}(G-X) \leq 2k - |X| \leq k - 1$ if $|X| \geq k + 1$; also, $h_{2k-|X|}(G-X) \leq (2k-|X|)-1 \leq k - 1$ if there is a connected component of order two or more. It follows that $c_{2k}(G,X) = c(G-X) - h_{2k-|X|}(G-X) \geq c(G-X) - (k-1)$, meaning $\text{sc}_{2k}(G) \geq c_{2k}(G,X) - |X| \geq c(G-X) - (k-1) - |X| = \text{sc}(G) - (k-1) = 1-k$, which is a contradiction. Thus, the claim is proved. In addition, we have $c(G-X) = |X| = k$. It remains to show $(x_i,y_j) \in E(G)$ for every pair $x_i \in X$ and $y_j \in V(G) \setminus X$. Suppose $(x_i,y_j) \notin E(G)$ for some $x_i$ and $y_j$, then $X \setminus \{x_i\}$ would be a vertex cut of $G$, which contradicts the fact $\kappa(G) \geq k$. Therefore, $G$ is isomorphic to a spanning supergraph of $K_{k,k}$ and to a spanning subgraph of $G_{k,k}$, completing the proof.

4. Unpaired 2-disjoint path coverability of interval graphs

In this section, we prove that the necessity $\text{sc}_{2k}(G) \leq -k$ of Theorem 6 is a sufficient one for an interval graph $G$ and $k = 2$; in other words, we will prove that a noncomplete interval graph $G$ of order $n \geq 4$ is unpaired 2-disjoint path coverable if and only if $\text{sc}_4(G) \leq -2$. The sufficiency proof proceeds by induction on $n$. In order to build an unpaired 2-DPC of $G$ joining prescribed source and sink sets in a recursive manner, we define several subgraphs of $G$ that admit an unpaired 2-DPC and/or a Hamiltonian path. The unpaired 2-DPCs and Hamiltonian paths of the subgraphs are then combined into a required 2-DPC. Thanks to Theorems 7 and 8, we assume $n \geq 5$ and $\text{sc}(G) \leq -1$. (Recall that $\text{sc}(G) \leq 0$ if $\text{sc}_4(G) \leq -2$ by Lemma 6.)

First of all, we reduce our problem on a graph $G$ with $\text{sc}_4(G) \leq -2$ and $\text{sc}(G) \leq -2$ to a problem on a spanning subgraph $G'$ of $G$ with $\text{sc}_4(G') \leq -2$ and $\text{sc}(G') = -1$ as follows: Consider a noncomplete interval graph $G$ with $\text{sc}_4(G) \leq -2$ and $\text{sc}(G) \leq -2$ (i.e., $\text{sc}_4(G) \leq \text{sc}(G) \leq -2$), for which its interval representation is denoted by $\mathcal{I}$. Also, we denote by $\text{lp}(I_v)$ and $\text{rp}(I_v)$, respectively, the left and right endpoints of an interval $I_v$ corresponding to a vertex $v$. Let $Z$ be the leftmost minimal cut of $G$, i.e., $Z = V(C_1) \cap V(C_2)$ for the linear ordering $C_1, \ldots, C_q$ of the maximal cliques of $G$ such that each vertex of $G$ appears in consecutive cliques only. Then, $|Z| \geq 4$ because $\kappa(G) \geq 2 - \text{sc}(G) \geq 4$ by Lemma 3.
Let \( Z = \{z_1, \ldots, z_r\} \) with \( \text{rp}(I_{z_1}) \geq \cdots \geq \text{rp}(I_{z_r}) \). We define \( G' \) to be the intersection graph of a family \( \mathcal{I}' := (\mathcal{I} \setminus \{I_{z_1}, \ldots, I_{z_{r-3}}\}) \cup \{I'_{z_1}, \ldots, I'_{z_{r-3}}\} \), where \( \text{lp}(I'_{z_i}) = \min_{v \in V(G) \setminus V(C_1)} \text{lp}(I_v) \) and \( \text{rp}(I'_{z_i}) = \text{rp}(I_{z_i}) \) for \( i \in \{1, \ldots, r-3\} \). Refer to Fig. 4 for an example. The graph \( G' \) is a spanning subgraph of \( G \) because \( I'_{z_i} \subseteq I_{z_i} \) for all \( i \in \{1, \ldots, r-3\} \), and moreover \( \kappa(G') = 3 \).

**Lemma 9.** Let \( G' \) be the spanning subgraph of a noncomplete interval graph \( G \) with \( \text{sc}_4(G) \leq \text{sc}(G) \leq -2 \) defined above. Then, \( \text{sc}_4(G') \leq -2 \) and \( \text{sc}(G') = -1 \).

**Proof.** For the proof of \( \text{sc}_4(G') \leq -2 \), we suppose to the contrary that there is a 4-scattering set \( Y \) of \( G' \) such that \( c_4(G', Y) - |Y| = \text{sc}_4(G') \geq -1 \). Let \( Z' = \{z_1, \ldots, z_{r-3}\} \) and \( Z'' = \{z_{r-2}, z_{r-1}, z_r\} \), so that \( Z' \cup Z'' = Z \) and \( \text{rp}(I_{z_i}) \geq \text{rp}(I_{z_j}) \) for all \( z_i \in Z' \) and \( z_j \in Z'' \).

**Claim 1.** (a) \( Z' \not\subseteq Y \). (b) \( Z'' \subseteq Y \). (c) \(|Y| \geq 4\).

**Proof of claim.** Suppose \( Z' \subseteq Y \), then \( Y \) would also be a vertex cut of \( G \) and \( G - Y \) would have the same connected components as \( G' - Y \). This means \( c_4(G,Y) = c_4(G',Y) \), leading to that \( \text{sc}_4(G) \geq c_4(G,Y) - |Y| = c_4(G',Y) - |Y| = \text{sc}_4(G') \geq -1 \), which is a contradiction, proving (a). Suppose \( Z'' \not\subseteq Y \), then every vertex \( v \in V(C_1) \setminus V(C_2) \) with \( v \not\in Y \) would be included in the connected component of \( G' - Y \) that includes a vertex \( z_j \in Z'' \setminus Y \), and thus the vertex sets of the connected components of \( G - Y \) would be the same as those of \( G' - Y \). This means \( c_4(G,Y) = c_4(G',Y) \) again, leading to that \( \text{sc}_4(G) \geq c_4(G,Y) - |Y| = c_4(G',Y) - |Y| = \text{sc}_4(G') \geq -1 \), which is a contradiction, proving (b). Suppose \(|Y| \leq 3\), then \(|Y| = 3\), \( Y = Z'' \), and \( G' - Y \) would have only 2 connected components (because \( \kappa(G) \geq 4 \) and \( G - Y \) is connected). This means that \( c_4(G',Y) - |Y| = 1 - 3 = -2 \), which contradicts the assumption \( c_4(G',Y) - |Y| = \text{sc}_4(G') \geq -1 \). Thus, the claim is proven.

Let \( H_1, \ldots, H_p \) be the connected components of \( G' - Y \), where \( H_i \) is assumed to be to the left of \( H_{i+1} \) for all \( i < p \). Then, \( p \geq 3 \) because \(|Y| \geq 4\) and \( c_4(G',Y) - |Y| = p - |Y| \geq -1 \). There exists a point \( \alpha \) on the real line (between \( H_{p-1} \) and \( H_p \)) such that \( \max_{v \in V(H_{p-1})} \text{rp}(I_v) < \alpha < \min_{v \in V(H_p)} \text{lp}(I_v) \). So, the vertex subset \( W \) defined as \( \{v \in V(G) : \alpha \in I_v\} \) forms a vertex cut of \( G' \) satisfying \( W \subseteq Y \). The existence of a vertex \( z_i \in Z'' \setminus Y \) by Claim 1(a) leads to that \( z_i \not\in W \), \( \text{rp}(z_i) < \alpha \), and \( Z'' \cap W = \emptyset \). Thus, the set \( Y' := Y \setminus Z'' \) is a
vertex cut of $G'$ and also of $G$. It follows that $|Y'| \geq 4$; moreover, $c(G' - Y') = c(G' - Y)$ if $V(C_1) \setminus V(C_2) \subseteq Y$; $c(G' - Y') = c(G' - Y) - 1$ otherwise. Therefore, $sc_4(G') \geq c_4(G', Y') - |Y'| + 2 = c(G' - Y') - |Y'| \geq (c(G' - Y) - 1) - (|Y| - 3) = c_4(G', Y) - |Y| + 2 = sc_4(G') + 2$, which is a contradiction, proving $sc_4(G') \leq -2$.

Now, we show $sc(G') = -1$. Let $Y$ be an arbitrary vertex cut of $G'$. Then, $Y = Z'' ((|Y| = 3)$ or $|Y| \geq 4$ by the construction of $G'$ and $\kappa(G) \geq 4$. If $Y = Z''$, then $c(G' - Y) - |Y| = 2 - 3 = -1$ because $G - Y$ is connected; if $|Y| \geq 4$, then $c(G' - Y) - |Y| = c_4(G', Y) - |Y| \leq sc_4(G') \leq -2$. It follows that $sc(G') = -1$. This completes the proof.

Owing to Lemma 9, we further assume $sc(G) = -1$. One natural way of choosing interval subgraphs of the graph $G$ on which our subproblems are defined would be to associate them with a vertex cut, especially a minimum cut or a 4-scattering set. We observe basics of a minimum vertex cut of $G$ in Lemma 10 below (where the converse of (d) of the lemma is not always true).

**Lemma 10.** Let $G$ be a noncomplete interval graph of order $n \geq 5$ with $sc_4(G) \leq -2$ and $sc(G) = -1$. Then, (a) $\kappa(G) = 3$; (b) $c(G - X) = 2$ for a vertex cut $X$ of $G$ with $|X| = 3$; (c) $X$ is a scattering set of $G$ if and only if $X$ is a vertex cut with $|X| = 3$; (d) $X$ is a 4-scattering set of $G$ if $X$ is a vertex cut with $|X| = 3$; (e) $sc_4(G) = -2$.

**Proof.** The graph $G$ is 3-connected by Lemma 3 because $sc(G) = -1$; also, supposing that $G$ is 4-connected leads to $sc_4(G) = sc(G)$, which contradicts the hypothesis of the lemma. Thus, $\kappa(G) = 3$, proving (a). Supposing $c(G - X) \geq 3$ leads to $sc(G) \geq c(G - X) - |X| \geq 0$, which is a contradiction, proving (b). To prove (c), it suffices to prove the necessity due to (b). Supposing $X$ is a scattering set with $|X| \geq 4$ leads to $sc_4(G) \geq c_4(G, X) - |X| = c(G - X) - |X| = sc(G) = -1$, which is a contradiction, proving (c). If $X$ is a vertex cut with $|X| = 3$, then $sc_4(G) \geq c_4(G, X) - |X| = c(G - X) - 1 - |X| = -2 \geq sc_4(G)$, proving (d) and (e).

Let $X = \{x_1, x_2, x_3\}$ denote a minimum vertex cut of $G$, and let $H_1, H_2$ be the two connected components of $G - X$. The vertex cut $X$ is also a 4-scattering set, as well as a scattering set, of $G$ by Lemma 10. An interval $I_c$ is said to be to the left of $I_w$ if $rp(I_c) < lp(I_w)$. We assume $H_1$ is to the left of $H_2$, i.e., an interval corresponding to a vertex of $H_1$ is to the left of an interval corresponding to a vertex of $H_2$. We further assume w.l.o.g. that $rp(I_{x_3}) \leq rp(I_{x_2}) \leq rp(I_{x_1})$, so that $N_G(x_3) \cap V(H_2) \subseteq N_G(x_2) \cap V(H_2) \subseteq N_G(x_1) \cap V(H_2)$.

**Lemma 11.** (a) $|N_G(x_3) \cap V(H_2)| \geq 1$. (b) For $a \in \{1, 2\}$, $|N_G(x_a) \cap V(H_2)| \geq 2$ if $|V(H_2)| \geq 2$.

**Proof.** Supposing $x_3$ has no neighbor in $H_2$ leads to that $\{x_1, x_2\}$ would be a vertex cut of $G$, which contradicts the fact $\kappa(G) = 3$, proving (a). So, $|N_G(x_a) \cap V(H_2)| \geq 1$ for $a \in \{1, 2\}$. It suffices to prove (b) for $a = 2$. Suppose $x_2$ has only one neighbor $v$ in $H_2$ with $|V(H_2)| \geq 2$, then $\{x_1, v\}$ would be a vertex cut of $G$, which also contradicts the fact $\kappa(G) = 3$, proving (b).
The first interval subgraph we consider is a connected component $H_i$ of $G - X$. The component $H_i$ is not always Hamiltonian-connected; for example, the cut $\{z_2, z_3, z_4\}$ of the graph in Fig. 4(b) produces a component isomorphic to $G_{2,3}$ (which is not Hamiltonian-connected). Nonetheless, $H_i$ has a scattering number no more than 0 if it is noncomplete, as shown in the following lemma. So, quite a few vertex pairs in $H_i$ are expected to be joined by a Hamilton path.

**Lemma 12.** Each connected component $H_i$ of $G - X$ is a complete graph, or $\text{sc}(H_i) \leq 0$.

**Proof.** Suppose $H_i$ is not a complete graph and $\text{sc}(H_i) \geq 1$. Then, there is a scattering set $Y$ of $H_i$ such that $c(H_i - Y) - |Y| = \text{sc}(H_i) \geq 1$. It follows that $X' := X \cup Y$ would be a vertex cut of $G$ with $|X'| \geq 4$. This leads to $\text{sc}_4(G) \geq c_4(G, X') - |X'| = c(G - X') - |X'| = (c(G - X) - 1 + c(H_i - Y)) - (|X| + |Y|) \geq c(G - X) - |X| = \text{sc}(G) = -1$, which contradicts the fact $\text{sc}_4(G) \leq -2$, completing the proof.

Hereafter in Lemmas 13 through 17 we discuss several subgraphs induced by a single connected component and a nonempty subset of $X$. The subproblems defined on the subgraphs are then used as building blocks to resolve our problem of constructing an unpaired 2-DPC in $G$. Let $H_i^*$ denote the subgraph of $G$ induced by $V(H_i) \cup X$. In addition, let $H_i^{\#}$ and $H_i^{a,b}$, where $a, b \in \{1, 2, 3\}$ and $a \neq b$, be the induced subgraphs of $G$ by $V(H_i) \cup \{x_a\}$ and by $V(H_i) \cup \{x_a, x_b\}$, respectively. Refer to Fig. 5 for examples.

**Lemma 13.** The subgraph $H_i^*$ is complete, or $\text{sc}_4(H_i^*) \leq -2$ and $\text{sc}(H_i^*) \leq -1$.

**Proof.** Let $H_j$ denote the connected component of $G - X$ other than $H_i$, so that $\{i, j\} = \{1, 2\}$. Assume $H_i^*$ is noncomplete. Then, $|V(H_i^*)| \geq 5$ by Lemma 11.
Suppose to the contrary that $c_4(H^*_i) \geq -1$, i.e., there is a 4-scattering set $Y$ of $H^*_i$ such that $c_4(H^*_i, Y) - |Y| = c_4(H^*_i) \geq -1$. The vertices belonging to $X \setminus Y$, if any, are adjacent to each other and so all belong to the same connected component. Thus, $H_i$ will form a connected component of $G - Y$ if $X \setminus Y = \emptyset$; otherwise, it is included in the connected component that contains $x \in X \setminus Y$. It follows that $Y$ is a vertex cut of $G$; moreover, $c_4(G) \geq c_4(G, Y) - |Y| \geq c_4(H^*_i, Y) - |Y| = c_4(H^*_i) \geq -1$, which contradicts the fact $c_4(G) \leq -2$, proving $c_4(H^*_i) \leq -2$. Now, suppose $c_4(H^*_i) \geq 0$ for a contradiction. Then, $c_4(H^*_i) = 0$ by Lemma 6, leading to that $H^*_i$ is isomorphic to $G_{2,2}$ by Lemma 8, which contradicts the fact $|V(H^*_i)| \geq 5$, completing the proof.

Lemma 13 allows us to enjoy that $H^*_i$ is unpaired 2-disjoint path coverable (by the induction hypothesis) and also is Hamiltonian-connected (by Theorem 1). The subgraph $H^*_{2,2}$ also has such good properties as $H^*_i$.

**Lemma 14.** The subgraph $H^*_{2,2}$ is complete, or $c_4(H^*_{2,2}) \leq -2$ and $c_4(H^*_{2,2}) \leq -1$.

**Proof.** Assume $H^*_{2,2}$ is noncomplete. Supposing $|V(H_2)| \leq 2$ leads to that $H^*_{2,2}$ is a complete graph by Lemma 14(b), so we have $|V(H_2)| \geq 3$ and $|V(H^*_{2,2})| \geq 5$. Suppose, for a contradiction, that there is a 4-scattering set $Y$ of $H^*_{2,2}$ such that $c_4(H^*_{2,2}, Y) - |Y| = c_4(H^*_{2,2}) \geq -1$. First we prove a claim: There exists $x_n, a \in \{1, 2\}$, such that $x_n \notin Y$. Suppose otherwise, i.e., $x_1, x_2 \in Y$. Then, $Y \setminus \{x_1, x_2\}$ would be a vertex cut of $H_2$, meaning $|Y| \geq 4$ by Lemma 12. In addition, $Y' := Y \cup \{x_3\}$ would be a vertex cut of $G$, leading to $c_4(G, Y') = c(G - Y') = c(H^*_{2,2} - Y) + 1 = c_4(H^*_{2,2}, Y) + 1$. It follows that $c_4(G) \geq c_4(G, Y') - |Y'| = (c_4(H^*_{2,2}, Y) + 1) - (|Y| + 1) = c_4(H^*_{2,2}) \geq -1$, which is a contradiction, thereby proving the claim. In addition, the set $Y$ would be a vertex cut of $H_2$, because the vertex $x_3$ of $H^*_2 - Y$ must be included in the connected component that contains $x_n$. (Recall that $N_G(x_3) \cap V(H_2) \subseteq N_G(x_n) \cap V(H_2)$.) It follows that $c_4(H^*_2, Y) \geq c_4(H^*_{2,2}, Y)$, implying $c_4(H^*_2) \geq c_4(H^*_2, Y) - |Y| \geq c_4(H^*_{2,2}, Y) - |Y| \geq -1$, which is a contradiction. Thus, $c_4(H^*_{2,2}) \leq -2$. Finally, supposing $c_4(H^*_{2,2}) \geq 0$ leads to that $c_4(H^*_{2,2}) = 0$ and $H^*_{2,2}$ is isomorphic to $G_{2,2}$ by Lemmas 3 and 8, contradicting the fact $|V(H^*_{2,2})| \geq 5$. Thus, $c_4(H^*_{2,2}) \leq -1$. This completes the proof.

Except for the subgraphs $H^*_2$ and $H^*_{1,2}$ (and symmetrical graphs with them), it is not easy to find other induced subgraphs with such good properties of the two. Instead, we study a Hamiltonian property of the subgraphs induced by $V(H_2)$ and a subset of $X$. Specifically, the existence of a Hamiltonian path running from a vertex of $X$ is dealt with in Lemmas 15 through 17 below.

**Lemma 15.** For $a \in \{1, 2\}$, the subgraph $H^*_2$ is complete or $c_4(H^*_2) \leq 0$. Moreover, if $|N_G(x_3) \cap V(H_2)| \geq 2$ or $|V(H_2)| = 1$, then the subgraph $H^*_2$ is complete or $c_4(H^*_2) \leq 0$.

**Proof.** It suffices to prove that for an arbitrary $b \in \{1, 2, 3\}$, the subgraph $H^*_b$ is complete or $c_4(H^*_b) \leq 0$ under the condition $|N_G(x_b) \cap V(H_2)| \geq 2$.
or $|V(H_2)| = 1$. This is because the condition holds true for $b \in \{1, 2\}$ by Lemma 11. Suppose to the contrary that $H^b_2$ is noncomplete and $sc(H^b_2) \geq 1$, i.e., there is a scattering set $Y$ of $H^b_2$ such that $c(H^b_2 - Y) - |Y| = sc(H^b_2) \geq 1$. Then, $H^b_2$ is a 2-connected graph, because $|N_G(x_b) \cap V(H_2)| \geq 2$ and $H_2$ is complete or $sc(H_2) \leq 0$ by Lemma 12 leading to $|Y| \geq 2$. Moreover, the connected components of $H^b_2 - Y$ coincide with those of $H_2 - Y'$ where $Y' = Y \cup (X \setminus \{x_b\})$. It follows that $c(H^b_2, Y) = c(H^b_2 - Y') = c(H^b_2 - Y) \geq 2$, leading to that $sc(H^b_2) \geq c_4(H^b_2, Y') - |Y'| = c(H^b_2 - Y) - (|Y| + 2) = sc(H^b_2) - 2 \geq -1$, which is a contradiction. Thus, $H^b_2$ is complete or $sc(H^b_2) \leq 0$.

**Lemma 16.** For $a \in \{1, 2\}$, the subgraph $H^a_2$ has a Hamiltonian $x_a - y$ path for every $y \in V(H_2)$. Moreover, if $|N_G(x_3) \cap V(H_2)| \geq 2$ or $|V(H_2)| = 1$, then $H^a_2$ has a Hamiltonian $x_3 - y$ path for every $y \in V(H_2)$.

**Proof.** The lemma holds true obviously if $|V(H_2)| = 1$ by Lemma 11 so, assume $|V(H_2)| \geq 2$. Since $|N_G(x_1) \cap V(H_2)|, |N_G(x_2) \cap V(H_2)| \geq 2$ again by Lemma 11 it suffices to prove that the subgraph $H^b_2$, $b \in \{1, 2, 3\}$, has a Hamiltonian $x_b - y$ path for every $y \in V(H_2)$ provided $|N_G(x_b) \cap V(H_2)| \geq 2$. If $H_2$ is a complete graph, such a Hamiltonian $x_b - y$ path exists obviously; so, assume $H_2$ is noncomplete. Also, $H^b_2$ is noncomplete, and $sc(H^b_2) \leq 0$ by Lemma 15. Let $w$ be a vertex of $H^b_2$ such that $rp(I_w) \leq rp(I_v)$ for all $v \in V(H^b_2)$, so that $w$ is a left tip of $H^b_2$. If $x_b = w$, then a required Hamiltonian $x_b - y$ path exists in $H^b_2$ by Lemma 17. Let $x_b \neq w$ now. Then, $x_b$ and $w$ are adjacent each other because $I_w \subseteq I_{x_b}$. Moreover, $w$ is a left tip of $H^b_2$, so there exists a Hamiltonian $w - y$ path in $H_2$ for every $y \in V(H_2)$ other than $w$. (Recall $sc(H_2) \leq 0$ by Lemma 12.) If $y \neq w$, a required Hamiltonian $x_b - y$ path is obtained by combining one-vertex path $I_{x_b}$ with a Hamiltonian $w - y$ path of $H_2$; if $y = w$, for a neighbor $z$ of $x_b$ other than $w$, combining $I_{x_b}$ and a Hamiltonian $z - w$ path of $H_2$ results in a required Hamiltonian $x_b - y$ path. The neighbor $z$ of $x_b$ exists by the hypothesis $|N_G(x_3) \cap V(H_2)| \geq 2$. Therefore, the lemma is proved.

**Lemma 17.** The subgraph $H^a_{2, b}$ has a Hamiltonian $x_a - x_b$ path for distinct $a, b \in \{1, 2, 3\}$.

**Proof.** It suffices to show that $H^2_{2, 3}$ has a Hamiltonian $x_2 - x_3$ path. For a neighbor $z \in V(H_2)$ of $x_3$, a Hamiltonian $x_2 - z$ path of $H^2_2$ exists by Lemma 10. Combining the Hamiltonian path with one-vertex path $I_{x_3}$ results in a required Hamiltonian path.

The final subgraph on which our subproblems are defined is a spanning subgraph of $H^*$, possibly not an induced subgraph but an interval subgraph of $G$. Let $z$ be a vertex in $N_{H^2_2}(x_3) \cap V(H_2)$ such that $lp(I_z) \leq lp(I_v)$ for all $v \in V(H_2) \setminus \{z\}$, because replacing $I_z$ with a new interval $I'_z$ such that $rp(I'_z) = rp(I_z)$ and $max_{v \in V(H_2)} rp(I_z) < lp(I'_z) < min_{v \in V(H_2)} lp(I_z)$ results in another interval representation for the graph $H^*_2$. (Supposing $I'_z \cap I_w \neq \emptyset$ and $I_z \cap I_w = \emptyset$ for some $w \in V(H^*_2)$) leads to $rp(I_w) < lp(I_z) \leq rp(I_{x_3})$, meaning
Claim such that 

Suppose to the contrary $sc(4)$. Suppose $H_2$ by replacing the interval $I_{x_3}$ with a new interval $I'_{x_3}$ having $lp(I_{x_3}) = lp(I_{x_3})$ and $rp(I'_{x_3}) = lp(I'_{x_3})$. Refer to Fig. 6 for an example. The graph $H_2^\dagger$ is obviously a spanning subgraph of $H_2^\dagger$.

**Lemma 18.** The subgraph $H_2^\dagger$ is complete, or $sc(H_2^\dagger) \leq -2$ and $sc(H_2^\dagger) \leq -1$.

**Proof.** Assume $H_2^\dagger$ is noncomplete. It follows that $|V(H_2)| \geq 2$ and $|V(H_2)| \geq 5$. Suppose to the contrary $sc_4(H_2) \geq -1$, i.e., there is a 4-scattering set $Y$ of $H_2^\dagger$ such that $c_4(H_2) \geq c_4(H_2) \geq -1$.

**Claim 2.** (a) $x_3 \notin Y$. (b) $N_{H_2}(x_3) = \{x_1, x_2, z\} \subseteq Y$. (c) $|Y| \geq 4$.

**Proof of claim.** Supposing $x_3 \in Y$ leads to that $Y$ would also be a vertex cut of $H_2^\dagger$. Moreover, $H_2^\dagger - Y$ and $H_2^\dagger - Y$ coincide, meaning $sc_4(H_2) \geq c_4(H_2) - |Y| = c_4(H_2, Y) - |Y| = sc_4(H_2) \geq -1$, which is a contradiction, proving (a). Suppose $N_{H_2}(x_3) \notin Y$ for a contradiction, i.e., the connected component, $R_1$, of $H_2^\dagger - Y$ that contains $x_3$, is of size two or more. Then, $R_1 - \{x_3\}$ is connected. (This is because $x_3 \in V(R_1)$ and $N_{R_1}(x_3) \subseteq N_{R_1}(x_3)$ if $x_3 \notin Y$ for some $a \in \{1, 2\}$; $|N_{R_1}(x_3)| = 1$ otherwise.) Moreover, $(H_2^\dagger - Y) - \{x_3\}$ and $H_2^\dagger - Y$ coincide. These lead to that $c(H_2^\dagger - Y) = c(H_2^\dagger - Y)$ and $Y$ is a vertex cut of $H_2^\dagger$, where $|Y| \geq 3$ by Lemmas 3 and 14. Thus, we have $sc_4(H_2^\dagger) \geq c_4(H_2^\dagger, Y) - |Y| = c_4(H_2, Y) - |Y| \geq -1$, which is a contradiction, completing the proof of (b). Finally, suppose $|Y| \leq 3$ for a contradiction. Then, $Y = \{x_1, x_2, z\}$ by Claim 2(b). So, $c(H_2^\dagger - Y) = c_4(H_2, Y) + 1 = |Y| + sc_4(H_2) + 1 \geq |Y| = 3$, implying that $Y' := Y \setminus \{x_1, x_2\} = \{z\}$ would be a vertex cut of $H_2^\dagger$ which contradicts the fact that $H_2^\dagger$ is complete or $sc(H_2^\dagger) \leq 0$. Therefore, the claim is proved.

Claim 2 leads to that $c(H_2^\dagger - Y) = c_4(H_2, Y) = |Y| + sc_4(H_2, Y) \geq 3$. Thus, $Z := Y \setminus \{x_1, x_2\}$ is a vertex cut of $H_2$, where $c(H_2 - Z) = c(H_2 - Y) - 1 \geq 2$.

**Claim 3.** (a) $c(H_2 - Z) - |Z| = sc(H_2) = 0$. (b) $N_{H_2}(x_3) \cap V(H_2) \subseteq Z$.

**Proof of claim.** The proof of (a) is a direct consequence of the following two inequalities: $sc(H_2) \leq 0$ and $sc(H_2) \geq c(H_2 - Z) - |Z| = (c(H_2 - Y) - 1) - (|Y| - 2)$.
2) = sc_4(H_2^1) + 1 ≥ 0. Now, suppose for a contradiction that there is a vertex y ∈ N_{H_2^1}(x_3) ∩ V(H_2) such that y ⋄ Z. Assume w.l.o.g. that lp(I_y) ≤ lp(I_v) for all v ∈ N_{H_2^1}(x_3) ∩ V(H_2) with v ⋄ Z. So, there is no interval I_u with u ∈ V(H_2) \ Z to the left of I_y. Moreover, lp(I_z) < lp(I_y) and rp(I_z) ≤ rp(I_y) by the choice of z. Then, Z′ := Z \ {z} would be a vertex cut of H_2, because z belongs to the connected component of H_2 − Z′ that includes y, or forms a one-vertex connected component, leading to c(H_2 − Z′) ≥ c(H_2 − Z). It follows that sc(H_2) ≥ c(H_2 − Z′) − |Z′| ≥ c(H_2 − Z) − (|Z| − 1) = 1, which is a contradiction, thereby proving (b).

Claims [2(b)] and [3(b)] lead to N_{H_2^1}(x_3) ⊆ Y. Thus, the connected components of H^*_2 − Y are the same as those of H^*_1 − Y, meaning Y is also a vertex cut of H^*_2. It follows that sc_4(H^*_2) ≥ c_4(H^*_2, Y) − |Y| = c_4(H^*_1, Y) − |Y| ≥ −1, which is a contradiction. Thus, sc_4(H_2^1) ≤ −2. Finally, supposing sc_4(H_2^1) ≥ 0 leads to that sc_4(H_2^1) = 0 and H_2^1 is isomorphic to G_2, by Lemmas [6] and [8], contradicting the fact |V(H_2^1)| ≥ 5. Thus, sc_4(H_2^1) ≤ −1. This completes the entire proof.

Now, we are ready to prove our main theorem.

**Theorem 9.** A noncomplete interval graph \( G \) of order \( n \geq 4 \) is unpaired 2-disjoint path coverable if and only if sc_4(G) ≤ −2.

**Proof.** The necessity part is due to Theorem [6]. The sufficiency proof proceeds by induction on \( n \). Assume sc_4(G) ≤ −2. Then, sc(G) ≤ 0 by Lemma [9]. The graph \( G \) is unpaired 2-disjoint path coverable if \( n = 4 \) by Theorem [7] the same follows if sc(G) = 0 by Theorem [8]. In addition, due to Lemma [9], there is a spanning subgraph \( G' \) of \( G \) with sc_4(G') ≤ −2 and sc(G') = −1. So, we further assume \( n \geq 5 \) and sc(G) = −1. As before, let \( X = \{x_1, x_2, x_3\} \) denote a minimum vertex cut of \( G \) with rp(I_{x_1}) ≤ rp(I_{x_2}) ≤ rp(I_{x_1}), and let \( H_1, H_2 \) be the connected components of \( G − X \), where \( H_1 \) is to the left of \( H_2 \). Given disjoint source and sink sets, \( S = \{s_1, s_2\} \) and \( T = \{t_1, t_2\} \) in \( G \), we will build an unpaired 2-DPC joining them. There are three cases depending on the distribution of the four terminals in \( S ∪ T \).

**Case 1:** There is a connected component, say \( H_1 \), that contains no terminal.

Firstly, suppose \( x_3 \) is a nonterminal. In the subgraph \( H_1^1 \) of \( G \), there exists an unpaired 2-DPC \( P \) joining \( S \) and \( T \) by the induction hypothesis because \( H_1^1 \) is complete or sc_4(H_1^1) ≤ −2 by Lemma [18]. A path in \( P \) that passes through \( x_3 \) as an intermediate vertex visits an edge \( (x_3, x_a) \) for some \( a \in \{1, 2\} \) because \( N_{H_1^1}(x_3) = \{x_1, x_2, x_3\} \) for some \( z \in V(H_2) \). Replacing the subpath \( (x_3, x_a) \) with a Hamiltonian \( x_3−x_a \) path of the subgraph induced by \( V(H_1) \cup \{x_3, x_a\} \), which exists by Lemma [17], results in an unpaired 2-DPC of \( G \) joining \( S \) and \( T \).

Now, suppose \( x_3 \) is a terminal, say a source. If \( x_a \) is a nonterminal for some \( a \in \{1, 2\} \), it suffices to combine a Hamiltonian \( x_3−x_a \) path of the subgraph induced by \( V(H_1) \cup \{x_3, x_a\} \) with an unpaired 2-DPC of \( H_2^1 \) joining \( S' := (S \setminus \{x_3\}) \cup \{x_a\} \) and \( T \), which exists by the induction hypothesis and Lemma [14], into a required 2-DPC of \( G \). Suppose otherwise, i.e., \( x_1, x_2 \in S \cup T \). For some
b ∈ {1, 2} with \( x_b \in T \), it suffices to build two Hamiltonian paths: a Hamiltonian \( x_3-x_b \) path in the subgraph induced by \( V(H_1) \cup \{x_3, x_b\} \) and a Hamiltonian \( x_a-y \) path in \( H_2^a \) for \( a \) with \( \{a, b\} = \{1, 2\} \) and a terminal \( y \) contained in \( H_2 \). The Hamiltonian \( x_a-y \) path exists by Lemma [16]

**Case 2:** There is a connected component, say \( H_2 \), that contains a single terminal, say a source \( s_1 \). The component \( H_1 \) contains one or more terminals in this case; so, \( X \) contains at most two terminals. Firstly, suppose there exists a nonterminal \( x_a \in X \) such that (i) \( a \in \{1, 2\} \) or (ii) \( a = 3 \) and either \( |N_G(x_3) \cap V(H_2)| \geq 2 \) or \( |V(H_2)| = 1 \). It suffices to combine a Hamiltonian \( s_1-x_a \) path of \( H_2^3 \) with an unpaired 2-DPC of \( H_1^s \) joining \( S' := (S \setminus \{s_1\}) \cup \{x_a\} \) and \( T \). The unpaired 2-DPC exists by the induction hypothesis and Lemma [13]

Now, there remains a case where \( x_1, x_2 \in S \cup T \), \( x_3 \notin S \cup T \), and moreover \( |N_G(x_3) \cap V(H_2)| = 1 \) and \( |V(H_2)| \geq 2 \). So, there is a single terminal, say \( y \), in \( H_1 \). We first prove a claim: \( |N_G(x_3) \cap V(H_1)| \geq 2 \) or \( |V(H_1)| = 1 \). Supposing \( |N_G(x_3) \cap V(H_1)| = 1 \) and \( |V(H_1)| \geq 2 \) for a contradiction, along with \( |N_G(x_3) \cap V(H_2)| = 1 \) and \( |V(H_2)| \geq 2 \), leads to that \( Z := N_G(x_3) \) would be a vertex cut of \( G \), leaving three connected components \( \{x_3\}, H_1-Z, H_2-Z \) in \( G-Z \). This implies that \( sc_4(G) \geq c_4(G, Z) - |Z| = c(G-Z) - |Z| = 3 - 4 = -1 \), which is a contradiction, thereby proving the claim. Thus, there exists a Hamiltonian \( x_3-y \) path in the subgraph induced by \( V(H_1) \cup \{x_3\} \) by Lemma [16] so, combining the Hamiltonian path with an unpaired 2-DPC joining \( S' \) and \( T' \) in \( H_2^y \) results in a required 2-DPC, where \( S' = (S \setminus \{y\}) \cup \{x_3\} \) and \( T' = T \) if \( y \in S' \); \( S' = S \) and \( T' = (T \setminus \{y\}) \cup \{x_3\} \) otherwise.

**Case 3:** Each connected component contains two terminals. Assume \( H_1 \) contains a source \( s_1 \) and a terminal \( y \) other than \( s_1 \), where \( y \) may be a source or a sink. There exists an unpaired 2-DPC joining \( \{s_1, y\} \) and \( \{x_1, x_2\} \) in \( H_1^s \) by the induction hypothesis and Lemma [13] Assume w.l.o.g. the DPC is composed of \( s_1-x_1 \) and \( y-x_2 \) paths. It suffices to combine the 2-DPC of \( H_1^s \) with an unpaired 2-DPC of \( H_2^{1,2} \) joining \( S' \) and \( T' \), where \( S' = \{x_1, x_2\} \) and \( T' = T \) if \( y \in S' \); \( S' = (S \setminus \{s_1\}) \cup \{x_1\} \) and \( T' = (T \setminus \{y\}) \cup \{x_2\} \) if \( y \in T \). This completes the entire proof. \( \blacksquare \)

5. Algorithm for computing the \( r \)-scattering number

The \( r \)-scattering number \( sc_r(G) \) of a noncomplete graph \( G \) of order \( n \) for a nonnegative integer \( r \) with \( r \leq n \) can be expressed as

\[
sc_r(G) = \max\{sc'_r(G), sc''_r(G)\},
\]

where \( sc'_r(G) \) is defined to be the maximum of \( c(G-X) - |X| \) over all vertex cuts \( X \) of \( G \) with \( |X| \geq r \) if such \( X \) exists; \( sc'_r(G) = -\infty \) otherwise. Also, \( sc''_r(G) \) is defined to be the maximum of \( c(G-X) - h_{r-|X|}(G-X) - |X| \) over all vertex cuts \( X \) of \( G \) with \( |X| \leq r \) if such \( X \) exists; \( sc''_r(G) = -\infty \) otherwise. Note that \( h_{r-|X|}(G-X) = 0 \) if \( |X| = r \), so including the vertex cuts of size \( r \) in defining \( sc''_r(G) \) does not cause a problem.
Fig. 7: An illustrative example of computing $sc''_r(G)$ for a noncomplete interval graph $G$ of order $n = 14$: (a) an interval representation for the graph $G$; (b) four connected components $H_1, \ldots, H_4$ are produced through the minimal cut $Z = \{u_{10}, u_{11}\}$.

For a graph $G$ of order $n$, possibly a complete graph, we define $c_i(G)$, $i \in \{0, \ldots, n - 1\}$, to be the largest number of connected components the graph $G$ can get after the removal of exactly $i$ vertices, i.e., $c_i(G) = \max\{c(G - X) : X \subseteq V(G), |X| = i\}$. Provided the numbers $c_1(G)$ are given, $sc'_r(G)$ can be calculated easily due to the following lemma. For an interval graph $G$, the numbers $c_i(G)$ can be computed in $O(n^3)$ time by Kratsch et al. [10]. So, we conclude that $sc'_r(G)$ of a noncomplete interval graph can be computed in $O(n^3)$ time.

**Lemma 19.** Let $G$ be a noncomplete graph of order $n$ and $r$ be a nonnegative integer with $r \leq n$. If $G$ has a vertex cut of size $r$ or more, then $sc'_r(G) = \max\{c_i(G) - i : \max\{r, \kappa(G)\} \leq i \leq n - 2\}$.

**Proof.** If $G$ has a vertex cut of size $r$ or more, then $sc'_r(G) = \max\{c_i(G) - i : i \geq r, c_i(G) \geq 2\}$ from the definition of $sc'_r(G)$. So, the lemma follows.

Now, consider how to compute $sc''_r(G)$ of a noncomplete interval graph. We first introduce a function $f_r(G)$ for a graph $G$ of order $n$ and a nonnegative integer $r$ with $r \leq n$, defined as the maximum of $c(G - X) - h_{r - |X|}(G - X) - |X|$ over all subsets $X \subseteq V(G)$ with $|X| \leq r$ (where $X$ is not necessarily a vertex cut of $G$). Let us take a look at the basic idea of an algorithm for computing $sc''_r(G)$ through an illustrative example shown in Fig. 7. Observe that for the vertex cut $X = \{u_1, u_{10}, u_{11}, u_{12}\}$ of the example graph $G$, we have $sc''_r(G) = c(G - X) - h_{r - |X|}(G - X) - |X| = (8 - 2) - 4 = 2$. Also, $sc''_r(G)$ can be obtained from $f_{r-|Z|}(G - Z)$ for some minimal vertex cut $Z \subseteq X$ of $G$. That is, if we pick up $Z = \{u_{10}, u_{11}\}$ and let $G_1 = H_1$ and $G_2 = H_2 \cup H_3 \cup H_4$, then $f_{r-|Z|}(G - Z) - |Z| = f_6(G_1 \cup G_2) - 2 = f_1(G_4) + f_5(G_2) - 2 = f_1(H_1) + (f_2(H_2) + f_3(H_3) + f_3(H_4)) - 2 = (3 - 1) + (0 + 1 + (2 - 1)) - 2 = 2 = sc''_r(G)$.

**Lemma 20.** Let $G$ be a noncomplete graph of order $n$ and $r$ be a nonnegative integer with $r \leq n$. If $G$ has a vertex cut of size $r$ or less, then

$$sc''_r(G) = \max_Z \left( f_{r - |Z|}(G - Z) - |Z| \right),$$

where the maximum is taken over all minimal vertex cuts $Z$ of $G$ with $|Z| \leq r$. 


Proof. Firstly, we show $sc''_r(G) \leq f_{r-\lvert Z \rvert}(G - Z) - \lvert Z \rvert$ for some minimal vertex cut $Z$ with $\lvert Z \rvert \leq r$. Let $X$ be a vertex cut of $G$ with $\lvert X \rvert \leq r$ such that $c(G - X) - h_{r-\lvert X \rvert}(G - X) - \lvert X \rvert = sc''_r(G)$. It is obvious that there is a subset $Y \subseteq V(G) \setminus X$ of size $r - \lvert X \rvert$ such that the number of connected components $H'_j$ of $G - X$ with $V(H'_j) \cap Y = \emptyset$ is equal to $c(G - X) - h_{r-\lvert X \rvert}(G - X)$. For a minimal vertex cut $Z$ of $G$ that is a subset of $X$, let $G' = G - Z$. Also, let $X' = X \setminus Z$ and $Y' = Y$, so $\lvert X' \rvert + \lvert Y' \rvert = r - \lvert Z \rvert$. Then,

$$sc''_r(G) = c(G - X) - h_{r-\lvert X \rvert}(G - X) - \lvert X \rvert$$

$$= \lvert \text{# of components } H'_j \text{ in } G - X \text{ with } V(H'_j) \cap Y = \emptyset \rvert - \lvert X \rvert$$

$$= \lvert \text{# of components } H'_j \text{ in } G' - X' \text{ with } V(H'_j) \cap Y' = \emptyset \rvert - (\lvert X' \rvert + \lvert Z \rvert)$$

$$\leq f_{r-\lvert Z \rvert}(G') - \lvert Z \rvert.$$

Now, we show $sc''_r(G) \geq f_{r-\lvert Z \rvert}(G - Z) - \lvert Z \rvert$ for every minimal vertex cut $Z$ with $\lvert Z \rvert \leq r$. Let $G' = G - Z$ and $r' = r - \lvert Z \rvert$. There exists a vertex subset $X'$ of $G'$ with $\lvert X' \rvert \leq r'$ such that $c(G' - X') - h_{r'-\lvert X' \rvert}(G' - X') - \lvert X' \rvert = f_{r'}(G')$. Moreover, there exists $Y' \subseteq V(G') \setminus X'$ with $\lvert Y' \rvert = r' - \lvert X' \rvert$ such that the number of connected components $H'_j$ of $G' - X'$ with $V(H'_j) \cap Y' = \emptyset$ is equal to $c(G' - X') - h_{r'-\lvert X' \rvert}(G' - X')$. Let $X' \cup Z$ and $Y' = Y'$, so $\lvert X' \rvert + \lvert Y' \rvert = r$. Then, $X$ is a vertex cut of $G$ with $\lvert X \rvert \leq r$, because $V(H_i) \not\subseteq X'$ for all connected components $H_i$ of $G'$. (Suppose for a contradiction $V(H_i) \subseteq X'$ for some $H_i$. For the sets $X'' = X' \setminus V(H_i)$ and $Y'' = Y' \cup V(H_i)$, the connected components $H'_j$ of $G' - X''$ with $V(H'_j) \cap Y'' = \emptyset$ coincide with the connected components $H'_j$ of $G' - X''$ with $V(H'_j) \cap Y'' = \emptyset$. It follows that $f_{r'}(G') \geq c(G' - X'') - h_{r'-\lvert X'' \rvert}(G' - X'') - \lvert X'' \rvert \geq c(G' - X') - h_{r'-\lvert X' \rvert}(G' - X') - \lvert X' \rvert > c(G' - X') - h_{r'-\lvert X' \rvert}(G' - X') - \lvert X' \rvert = f_{r'}(G')$, which is a contradiction.) So, we have

$$sc''_r(G) \geq c(G - X) - h_{r-\lvert X \rvert}(G - X) - \lvert X \rvert$$

$$\geq \lvert \text{# of components } H'_j \text{ in } G - X \text{ with } V(H'_j) \cap Y = \emptyset \rvert - \lvert X \rvert$$

$$= \lvert \text{# of components } H'_j \text{ in } G' - X' \text{ with } V(H'_j) \cap Y' = \emptyset \rvert - \lvert X \rvert$$

$$= c(G' - X') - h_{r'-\lvert X' \rvert}(G' - X') - (\lvert X' \rvert + \lvert Z \rvert)$$

$$\leq f_{r'}(G') - \lvert Z \rvert.$$

This completes the proof.

The graph $G - Z$ of Lemma 20 is disconnected, so it is the disjoint union $G_1 \cup G_2$ of two subgraphs $G_1$ and $G_2$ of $G$, i.e., $V(G) = V(G_1) \cup V(G_2)$, $E(G) = E(G_1) \cup E(G_2)$, and $V(G_1) \cap V(G_2) = \emptyset$. Lemma 21 below deals with how to compute the term $f_{r-|Z|}(G - Z)$ of Lemma 20.

Lemma 21. Let $G$ be a graph of order $n$ and $r$ be a nonnegative integer no more than $n$. (a) If $G$ is the disjoint union $G_1 \cup G_2$ of two subgraphs $G_1$ and $G_2$, then $f_r(G) = \max\{f_{r_1}(G_1) + f_{r_2}(G_2)\}$ over all pairs $(r_1, r_2)$ with $r_1 + r_2 = r$ and $0 \leq r_i \leq |V(G_i)|$ for $i \in \{1, 2\}$. (b) If $G$ is connected, then $f_0(G) = 1$ and $f_r(G) = \max\{sc''_r(G), 0\}$ for $r \geq 1$.}

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Proof. The proof of (a) is similar to that of Lemma 20. There are two disjoint sets \( X, Y \subseteq V(G) \) with \( |X| \leq r \) and \( |Y| = r - |X| \) such that \( f_r(G) + |X| = c(G - X) - h_{r-|X|}(G - X) - |X| = \# \text{ of components } H'_j \text{ in } G - X \) with \( V(H'_j) \cap Y = \emptyset \).

Let \( X_i = X \cap V(G_i) \) and \( Y_i = Y \cap V(G_i) \) for \( i \in \{1, 2\} \). Then,

\[
f_r(G) = \left[ \# \text{ of components } H'_j \text{ in } G - X \text{ with } V(H'_j) \cap Y = \emptyset \right] - |X|
\]

\[
= \sum_{i=1}^{2} \left( \left[ \# \text{ of components } H'_j \text{ in } G_i - X_i \text{ with } V(H'_j) \cap Y_i = \emptyset \right] - |X_i| \right)
\]

\[
\leq f_{r_1}(G_1) + f_{r_2}(G_2), \text{ where } r_1 = |X_1| + |Y_1| \text{ and } r_2 = |X_2| + |Y_2|.
\]

It remains to show \( f_r(G) \geq f_{r_1}(G_1) + f_{r_2}(G_2) \) for every pair \((r_1, r_2)\). For each \( G_i \), there are two disjoint sets \( X_i, Y_i \subseteq V(G_i) \) with \( |X_i| \leq r_i \) and \( |Y_i| = r_i - |X_i| \) such that \( f_{r_i}(G_i) = |X_i| = c(G_i - X_i) - h_{r_i-|X_i|}(G_i - X_i) = \# \text{ of components } H'_j \text{ in } G_i - X_i \text{ with } V(H'_j) \cap Y_i = \emptyset \). For the sets \( X = X_1 \cup X_2 \) and \( Y = Y_1 \cup Y_2 \), we have

\[
f_r(G) \geq \left[ \# \text{ of components } H'_j \text{ in } G - X \text{ with } V(H'_j) \cap Y = \emptyset \right] - |X|
\]

\[
= \sum_{i=1}^{2} \left( \left[ \# \text{ of components } H'_j \text{ in } G_i - X_i \text{ with } V(H'_j) \cap Y_i = \emptyset \right] - |X_i| \right)
\]

\[
= f_{r_1}(G_1) + f_{r_2}(G_2), \text{ proving (a)}.
\]

To prove (b), let \( G \) be a connected graph. It is obvious \( f_r(G) = 1 \) if \( r = 0 \); so, assume \( r \geq 1 \). There is a subset \( X \) of \( V(G) \) with \( |X| \leq r \) such that \( c(G - X) - h_{r-|X|}(G - X) - |X| = f_r(G) \). Moreover, the left term \( c(G - X) - h_{r-|X|}(G - X) - |X| \) is equal to \( sc''(G) \) if \( X \) is a vertex cut of \( G \); it is equal to 0 if \( X = \emptyset \); it is less than or equal to \( 1 - |X| \) if \( X \neq \emptyset \). So, the lemma follows.

Lemmas 20 and 21 suggest an algorithm for computing \( sc''(G) \) and \( f_r(G) \) of a general graph \( G \). The weakness of the algorithm is that all minimal vertex cuts must be taken into account. (A graph may have an exponential number of minimal vertex cuts; for example, think of a graph composed of \( \Theta(n) \) internally vertex-disjoint paths of length 3 between two vertices.) Fortunately, there are \( O(n) \) minimal vertex cuts in an interval graph. Moreover, the class of interval graphs is hereditary, i.e., every induced subgraph of a graph in the class is contained in the same class.

Restricting to interval graphs from now on, we discuss the details of the algorithm suggested by the two lemmas. Let \( G \) be an interval graph of order \( n \). By Theorem 5 there is a linear ordering \( C_1, \ldots, C_q \) of the maximal cliques of \( G \) such that each vertex of \( G \) appears in consecutive cliques only. Consequently, a vertex \( u_i \) of \( G \) can be represented by a closed interval \( I_{u_i} = [l_i, r_i] \), where \( l_i \) and \( r_i \) respectively are the minimum and maximum indices \( j \) such that \( u_i \in V(C_j) \).

So, \( I_{u_1} = [1, 1] \) and \( I_{u_n} = [q, q] \) for a left tip \( u_1 \) and a right tip \( u_n \) of \( G \). Such ‘compact’ interval representation can be built easily from an arbitrary interval representation by scanning the sorted list of \( 2n \) endpoints of \( n \) intervals.

Let \( G_{a,b}, a \leq b, \) be the subgraph of \( G \) induced by \( \bigcup_{i=a}^b V(C_i) - (V(C_{a-1}) \cup V(C_{b+1})) \), where \( V(C_0) \) and \( V(C_{a+1}) \) are defined to be empty sets. In other
words, \( G_{a,b} \) is the intersection graph of intervals whose left and right endpoints are between \( a \) and \( b \) inclusive. A minimal vertex cut of \( G \) is equal to \( Z_j \), defined as \( C_j \cap C_{j+1} \), for some \( 1 \leq j < q \), because each \( Z_j \) is a (not necessarily minimal) vertex cut and every vertex cut contains some \( Z_j \) as a subset. In addition, \( G = G_{1,q} \) and \( G - Z_j \) is the disjoint union of \( G_{1,j} \) and \( G_{j+1,q} \); also, a connected component of \( G - Z_j \) is represented as \( G_{a,b} \) for some \([a, b] \subseteq [1, q]\). These indicate an algorithm for computing \( f_r(G) \) (and eventually \( \text{sc}_{r'}(G) \)) using a dynamic programming approach.

Consider how to compute \( f_r(G_{a,b}) \) for \( a \leq b \) and \( 0 \leq r \leq n_{a,b} \), where \( n_{a,b} \) denotes the order of \( G_{a,b} \). The graph \( G_{a,b} \) is possibly a null graph of order 0; let \( f_0(G_{a,b}) = 0 \) if \( n_{a,b} = 0 \). For the base case \( a = b \), where \( G_{a,b} \) is a complete graph or a null graph, we have \( f_r(G_{a,b}) = 1 \) if \( n_{a,b} \geq 1 \) and \( r = 0 \); \( f_r(G_{a,b}) = 0 \) otherwise. Assume \( a < b \) hereafter, and let \( Z'_j = Z_j \cap V(G_{a,b}) \) for \( a \leq j < b \). We define \( f'_r(G_{a,b}) \) to be the maximum of \( c(G_{a,b} - X) - h_{r-1}(G_{a,b} - X) - |X| \) over all vertex subsets \( X \subseteq V(G_{a,b}) \) with \( |X| \leq r \) such that \( X \geq Z'_j \), if such \( X \) exists (i.e., \( r \geq |Z'_j| \)); \( f'_r(G_{a,b}) = -\infty \) otherwise. Also, let \( f''_r(G_{a,b}) \) be the maximum of \( c(G_{a,b} - X) - h_{r-1}(G_{a,b} - X) - |X| \) over all vertex subsets \( X \subseteq V(G_{a,b}) \) with \( |X| \leq r \) such that \( X \not\supseteq Z'_j \) for all \( j \). Then, by definition,

\[
f_r(G_{a,b}) = \begin{cases} 
\max_{a \leq j < b} f'_r(G_{a,b}) & \text{if } G_{a,b} \text{ is disconnected}, \\
\max\{\max_{a \leq j < b} f'_r(G_{a,b}), f''_r(G_{a,b})\} & \text{if } G_{a,b} \text{ is connected}.
\end{cases}
\]

Note that if \( G_{a,b} \) is disconnected, then \( Z'_j = \emptyset \) for some \( j \), hence \( X \geq Z'_j \) for every vertex subset \( X \); also, the graph \( G_{a,b} - X \), as well as \( G_{a,b} \), is connected if \( X \not\supseteq Z'_j \) for all \( j \). The graph \( G_{a,b} - Z'_j \) for \( a \leq j < b \) is the disjoint union of \( G_{a,j} \) and \( G_{j+1,b} \) (possibly, \( G_{a,j} \) and/or \( G_{j+1,b} \) are null graphs); so, if \( r \geq |Z'_j| \),

\[
f'_r(G_{a,b}) = f_r - |Z'_j| = \max_{(r_1, r_2)} (f_{r_1}(G_{a,j}) + f_{r_2}(G_{j+1,b})) - |Z'_j|, \text{ by Lemma } 21(a),
\]

where the maximum is taken over all pairs \((r_1, r_2)\) such that \( r_1 + r_2 = r - |Z'_j| \), \( 0 \leq r_1 \leq n_{a,j} \), and \( 0 \leq r_2 \leq n_{j+1,b} \). Finally, we have \( f''_r(G_{a,b}) = 1 \) if \( n_{a,b} \geq 1 \) and \( r = 0 \); \( f''_r(G_{a,b}) = 0 \) otherwise.

What we have discussed so far in this section is summarized in Algorithm 1 and Theorem 10. As a preprocessing stage, we can compute \( n_{a,b} \) and determine if \( G_{a,b} \) is connected for all \( a \leq b \), in \( O(n^2) \) time. As a result, \( |Z'_j| \) is computed in constant time because \( |Z'_j| = n_{a,b} - (n_{a,j} + n_{j+1,b}) \). The for loops of Steps 8 and 9 of Algorithm 1 iterate over all subgraphs \( G_{a,b} \) with \( a < b \). Given \( a \) and \( b \), all \( f_r(G_{a,b}) \) for \( 0 \leq r \leq n_{a,b} \) are computed in Steps 10 through 28 given \( a \), \( b \) and \( j \) with \( a \leq j < b \), all \( f'_r(G_{a,b}) \) for \( |Z'_j| \leq r \leq n_{a,b} \) are computed in Steps 13 through 22. Every for and foreach loop iterates \( O(n) \) times, so the algorithm runs in \( O(n^3) \) time.

**Theorem 10.** All \( r \)-scattering numbers \( \text{sc}_{r'}(G) \) of a noncomplete interval graph \( G \) of order \( n \) can be computed in \( O(n^3) \) time.
Algorithm 1: Computing \(sc''_r(G)\) of an interval graph \(G\)

**input** : An interval representation of an interval graph \(G\) of order \(n\);
**output**: \(sc''_r(G)\) for all \(r \in \{0, \ldots, n\}\);

/* preprocessing stage */

1. Build a 'compact' interval representation for \(G\) such that all endpoints of the intervals are integers between 1 and \(q\) inclusive.
2. Compute \(n_{a,b}\) and determine if \(G_{a,b}\) is connected for all \(a \leq b\).
3. \(\kappa(G) \leftarrow \min_{1 \leq j < q} (n - (n_{1,j} + n_{j+1,q}))\) if \(q \geq 2\); \(\kappa(G) \leftarrow n - 1\) otherwise.

/* compute all \(f_r(G_{a,b})\), where \(1 \leq a \leq b \leq q\) and \(0 \leq r \leq n_{a,b}\) */

4. for \(a \leftarrow 1\) to \(q\) do
5.   foreach \(r \in \{0, \ldots, n_{a,b}\}\) do \(f_r(G_{a,a}) \leftarrow 0\);
6.   if \(n_{a,a} \geq 1\) then \(f_{0}(G_{a,a}) \leftarrow 1\);
7. end

8. for \(d \leftarrow 1\) to \(q - 1\) do  // \(d = b - a\)
9.   for \(a \leftarrow 1\) to \(q - d\) do  // for each \(G_{a,b} = G_{a,a+d}\)
10.   \(b \leftarrow a + d\);
11.   foreach \(r \in \{0, \ldots, n_{a,b}\}\) do \(\text{maxSoFar}[r] \leftarrow -\infty\);
12.   for \(j \leftarrow a\) to \(b - 1\) do  // for each \(Z_{j}^\prime\) in \(G_{a,b}\)
13.     \(z_{j}^\prime \leftarrow n_{a,b} - (n_{a,j} + n_{j+1,b})\);  // \(z_{j}^\prime = |Z_{j}^\prime|\)
14.     foreach \(r \in \{z_{j}^\prime, \ldots, n_{a,b}\}\) do \(f_{j}^r(G_{a,b}) \leftarrow -\infty\);
15.     for \(r \leftarrow z_{j}^\prime\) to \(n_{a,b}\) do
16.       \(r' \leftarrow r - z_{j}^\prime\);
17.       for \(r_1 \leftarrow \max\{0, r' - n_{j+1,b}\}\) to \(\min\{n_{a,j}, r'\}\) do
18.       \(r_2 \leftarrow r' - r_1\);
19.       \(f_{j}^r(G_{a,b}) \leftarrow \max\{f_{j}^r(G_{a,b}), f_{r_1}(G_{a,j}), f_{r_2}(G_{j+1,b}) - z_{j}^\prime\};\)
20.     end
21.     \(\text{maxSoFar}[r] \leftarrow \max\{\text{maxSoFar}[r], f_{j}^r(G_{a,b})\}\);
22.   end
23. end
24. foreach \(r \in \{0, \ldots, n_{a,b}\}\) do \(f_r(G_{a,b}) \leftarrow \text{maxSoFar}[r]\);
25. if \(G_{a,b}\) is connected then
26.   foreach \(r \in \{0, \ldots, n_{a,b}\}\) do \(f_r(G_{a,b}) \leftarrow \max\{f_r(G_{a,b}), 0\}\);
27. if \(n_{a,b} \geq 1\) then \(f_{0}(G_{a,b}) \leftarrow \max\{f_{0}(G_{a,b}), 1\}\);
28. end
29. end
30. end
31. /* compute all \(sc''_r(G)\), where \(0 \leq r \leq n\) */
32. foreach \(r \in \{0, \ldots, n\}\) do \(sc''_r(G) \leftarrow -\infty\);
33. if \(q \geq 2\) then  // if \(G\) is noncomplete
34.   foreach \(r \in \{\kappa(G), \ldots, n\}\) do
35.     \(sc''_r(G) \leftarrow \text{maxSoFar}[r]\)  // the latest \(\text{maxSoFar}[r]\) for \(G_{1,q}\)
36. end
37. end
38. end
39. end
The $O(n^5)$-time algorithm for computing the $r$-scattering numbers $\text{sc}_r(G)$ of a noncomplete interval graph is implemented in C language. The source code and some running examples may be downloaded from http://tcs.catholic.ac.kr/~jhpark/papers/sc_r.zip.

It is worth mentioning that Steps 13–22 of Algorithm 1 actually compute a variation of convolution, named $(\max, +)$-convolution. To be specific, simply denoting $n_{a,j}$ by $n_1$, $n_{j+1,b}$ by $n_2$, $f_{r_1}(G_{a,j})$ by $a_{r_1}$, and $f_{r_2}(G_{j+1,b})$ by $b_{r_2}$, the steps calculate a vector $(c_0, \ldots, c_{n_1+n_2})$ from two vectors $(a_0, \ldots, a_{n_1})$ and $(b_0, \ldots, b_{n_2})$, where $c_{r'} = \max_{(r_1, r_2)}(a_{r_1} + b_{r_2})$ over all $(r_1, r_2)$ with $r_1 + r_2 = r'$, $0 \leq r_1 \leq n_1$, and $0 \leq r_2 \leq n_2$. Then, $f_r'(G_{a,b})$ will be $c_{r'-z'_j} - z'_j$ for $z'_j \leq r \leq n_{a,b}$.

In contrast to the standard convolution that admits an $O((n_1 + n_2) \log(n_1 + n_2))$-time algorithm, where $c_{r'}$ is defined to be $\sum_{(r_1, r_2)}(a_{r_1} \cdot b_{r_2})$, the $(\max, +)$-convolution is a hard problem for which there is no known truly subquadratic, $O((n_1 + n_2)^{2-\epsilon})$-time algorithm for $\epsilon > 0$ [22]. Bremner et al. [4] devised an $o(n^2)$-time algorithm for computing the $(\max, +)$-convolution of two vectors of size $n$. (Its running time is not $O(n^{2-\epsilon})$.) If the algorithm of Bremner et al. is employed in place of Steps 13–22 of Algorithm 1, the running time of our algorithm would be slightly improved to $o(n^2)$.

On the other hand, the problem of determining the $r$-scattering number of a general graph is NP-complete as shown below.

**Theorem 11.** Given a noncomplete graph $G$ and an integer $K$, the problem of deciding if $\text{sc}_r(G) \geq K$ is NP-complete for any fixed $r$.

**Proof.** It was proved by Zhang et al. [24] that the problem of deciding if $\text{sc}(G) \geq K$ for a graph $G$ and an integer $K$ is NP-complete. We show that the scattering number problem is polynomial-time reducible to the $r$-scattering number problem. Given a graph $G$ of order $n$, we define a graph $G'$ of order $n + r$ as follows:

$V(G') = V(G) \cup W$ for some set $W$ with $|W| = r$ such that $W \cap V(G) = \emptyset$;

$E(G') = E(G) \cup \{(u, v) : u \in V(G), v \in W\} \cup \{(v, v') : v, v' \in W, v \neq v'\}$. Then, every vertex cut of $G'$ contains $W$ as a subset; $X$ is a vertex cut of $G$ if and only if $X \cup W$ is a vertex cut of $G'$; and the connected components of $G - X$ are exactly the same as those of $G' - (X \cup W)$. It follows that

$$\text{sc}_r(G') = \max\{c(G' - X') - |X'| : X' \text{ is a vertex cut of } G'\}$$

(because $|X'| \geq r$)

$$= \max\{c(G' - (X \cup W)) - |X \cup W| : X \cup W \text{ is a vertex cut of } G'\}$$

$$= \max\{c(G - X) - |X| - |W| : X \text{ is a vertex cut of } G\}$$

$$= \text{sc}(G) - r,$$

i.e., $\text{sc}(G) \geq K$ if and only if $\text{sc}_r(G') \geq K - r$. It is clear that our reduction is polynomial in the input size, completing the proof.

6. Concluding remarks

In this paper, we proposed the notion of $r$-scattering number, a generalization of the scattering number, and investigated the unpaired $k$-disjoint path
coverability of an interval graph. It was proved that a noncomplete graph $G$ of order $n \geq 2k$ for $k \geq 1$ is unpaired $k$-disjoint path coverable only if $sc_{2k}(G) \leq -k$, and that a noncomplete interval graph $G$ of order $n \geq 4$ is unpaired 2-disjoint path coverable if and only if $sc_4(G) \leq -2$. According to the proofs given in this paper, we can design a polynomial-time algorithm for building an unpaired 2-DPC in an interval graph that is unpaired 2-disjoint path coverable. Furthermore, it was shown that the $r$-scattering number of an interval graph can be computed in polynomial time. It is open to characterize interval graphs that are unpaired $k$-disjoint path coverable for $k \geq 3$, or to settle down the following conjecture:

**Conjecture 1.** Let $G$ be a noncomplete interval graph of order $n \geq 2k$ for $k \geq 1$. Then, $G$ is unpaired $k$-disjoint path coverable if and only if $sc_{2k}(G) \leq -k$.

The conjecture turns out to be true if $k = 1$ or if $G$ is a proper interval graph, as shown in Appendix A.

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**Appendix A. Toward resolving Conjecture 1**

We show that Conjecture 1 holds true if $k = 1$ or if $G$ is a proper interval graph.

**Lemma 22.** For a noncomplete graph $G$, $sc_2(G) \leq -1$ if and only if $sc(G) \leq -1$.

**Proof.** Let $G$ be a noncomplete graph. Since $sc_2(G) \leq sc_0(G) = sc(G)$ by Lemma 4(b), it suffices to show the necessity part. Assume $sc_2(G) \leq -1$, meaning that $G$ is 1-connected by Lemma 4(b) again. Suppose $sc(G) \geq 0$ for a contradiction. Then, there is a scattering set $X$ such that $c(G - X) - |X| = sc(G) \geq 0$. If $|X| \geq 2$, then $sc_2(G) \geq c_2(G, X) - |X| = c(G - X) - |X| = sc(G) \geq 0$, contradicting the hypothesis $sc_2(G) \leq -1$. If $|X| = 1$, then $sc_2(G) \geq c_2(G, X) - |X| = (c(G - X) - 1) - |X| = c(G - X) - 2 \geq 0$, also contradicting the hypothesis. Note that $c(G - X) \geq 2$ because $X$ is a vertex cut of $G$. Thus, the lemma is proven.

**Theorem 12.** A noncomplete interval graph $G$ is unpaired 1-disjoint path coverable (or equivalently, Hamiltonian-connected) if and only if $sc_2(G) \leq -1$.

**Proof.** The theorem follows from Lemma 22 and Theorem 1. 

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Now, we turn to the second case where \( G \) is a proper interval graph. It has been known that the classes of proper interval graphs and unit interval graphs coincide \cite{23}, where a unit interval graph is an interval graph with an interval representation in which all the intervals have unit length. An ordering, \((v_1, \ldots, v_n)\), of the vertices of a graph \( G \) is a consecutive ordering if for each vertex \( v_i \), its closed neighbor \( N[v_i] \) is consecutive, i.e., \( N[v_i] = \{ v_j : l_i \leq j \leq r_i \} \) for some \( l_i \) and \( r_i \).

**Lemma 23** (Chen et al. \cite{7}). (a) A graph \( G \) is a proper interval graph if and only if \( G \) has a consecutive ordering. (b) Given a consecutive ordering \((v_1, \ldots, v_n)\) of a proper interval graph \( G \) of order \( n \geq k + 1 \) for a positive integer \( k \), \( G \) is \( k \)-connected if and only if \((v_i, v_j) \in E(G) \) whenever \( 1 \leq |i - j| \leq k \).

**Theorem 13** (Lee et al. \cite{11}). Let \( G \) be a proper interval graph of order \( n \geq 2k \) with a consecutive ordering \((v_1, \ldots, v_n)\) for \( k \geq 2 \). Then, \( G \) is unpaired \( k \)-disjoint path coverable if and only if \( G \) is \( k \)-connected and either \( n = 2k \) or \( n \geq 2k + 1 \) and \((v_i, v_{i+k+1}) \in E(G) \) for every \( i \), \( 1 \leq i \leq n - 2k \) or \( k \leq i \leq n - k - 1 \).

**Theorem 14.** A noncomplete proper interval graph \( G \) of order \( n \geq 2k \) for \( k \geq 2 \) is unpaired \( k \)-disjoint path coverable if and only if \( sc_{2k}(G) \leq -k \).

**Proof.** The necessity is due to Theorem \cite{6} For the sufficiency proof, assume \( sc_{2k}(G) \leq -k \). Suppose, for a contradiction, that \( G \) is not unpaired \( k \)-disjoint path coverable. Then, by Theorem \cite{13} (i) \( G \) is not \( k \)-connected, or (ii) \( n \geq 2k + 1 \) and \((v_i, v_{i+k+1}) \notin E(G) \) for some \( i \), where \( 1 \leq i \leq n - 2k \) or \( k \leq i \leq n - k - 1 \). If \( G \) is not \( k \)-connected, then \( sc_{2k}(G) \leq -k \) by Lemma \cite{1}(b), which contradicts the hypothesis. Now, suppose that \( G \) is \( k \)-connected and the condition (ii) is satisfied. Let \( X = \{ v_{i+1}, \ldots, v_{i+k} \} \). Then, \( X \) is a vertex cut of \( G \) and \( G - X \) has two connected components, which are subgraphs of \( G \) induced by \( \{ v_1, \ldots, v_i \} \) and \( \{ v_{i+k+1}, \ldots, v_n \} \), respectively. The larger connected component, say \( H_1 \), between the two has at least \( k \) vertices, because

\[
|V(H_1)| = \max\{i, n - i - k\} \geq \begin{cases} n - i - k & \text{if } 1 \leq i \leq n - 2k, \\ i & \text{if } k \leq i \leq n - k - 1. \end{cases}
\]

It follows that \( c_{2k}(G, X) = 2 - 1 = 1 \), leading to \( sc_{2k}(G) \geq 1 \), which also contradicts the hypothesis \( sc_{2k}(G) \leq -k \). Thus, the theorem is proven. \( \blacksquare \)

**References**


