Paths and Cycles in d-Dimensional Tori with Faults *

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Abstract. This paper is concerned with the paths and the cycles in d-dimensional tori with faulty vertices and/or edge. Let f_v be the number of faulty vertices and f_e be the number of faulty edges. It is shown that in any non-bipartite d-dimensional $k_1 \times k_2 \times \cdots k_d$ torus with $k_i \geq 3$ for each $1 \leq i \leq d$, (a) if $f_v + f_e \leq 2d-3$, there is a fault-free spanning path between any pair of non-faulty vertices, and (b) if $f_v + f_e \leq 2d-2$, there is a fault-free spanning cycle. It is also shown that in any bipartite d-dimensional $k_1 \times k_2 \times \cdots k_d$ torus with $k_i \geq 4$ for each $1 \leq i \leq d$, (a) if $f_v + f_e \leq 2d-2$, there is a fault-free path of length at least $N - 2f_v - 1$ between any pair of non-faulty vertices which belong to the different partite sets, and there is a fault-free path of length at least $N - 2f_v - 2$ between any pair of non-faulty vertices which belong to the same partite set, and (b) if $f_v + f_e \leq 2d-1$ and $f_e \leq 2d-2$, there is a fault-free cycle of length at least $N - 2f_v$, and if $f_v = 0$ and $f_e = 2d-1$ and all the fault edges are not incident to a common vertex, there is a fault-free spanning cycle, and if $f_v = 0$ and $f_e = 2d-1$ and all the fault edges are incident to a common vertex, there is a fault-free cycle of length $N-2$ where N is the number of vertices.

1 Introduction

As the interconnection networks for massively parallel computing become more complex, the possibility of occurrence of faulty elements increases. In this paper, we address some issues related to ring or linear array embedding in d-dimensional torus networks with faulty vertices(nodes) and/or faulty edges(links). Ring or path embedding in the interconnection networks is closely related to a hamiltonian problem which is one of the well known problems in the graph theory. If an interconnection network has a hamiltonian cycle or a hamiltonian path, ring or linear array can be implemented in this network. Ring embedding in the interconnection networks with faulty vertices and faulty edges has been widely studied in [2, 3, 5, 7, 8].

A graph G is hamiltonian-connected if there is a spanning path between any pair of vertices. A graph G is called *k-fault hamiltonian-connected* if for each pair of vertices, u and v, there is a path between u and v which contains only all non-faulty vertices and contains only non-faulty edges when there are k or less faulty elements (vertices and edges). This path is called to be a *fault-free spanning path* or a *fault-free hamiltonian path* between u and v. A graph G is called *k-fault hamiltonian* if there is a cycle which contains only all non-faulty vertices and contains

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only non-faulty edges when there are k or less faulty elements (vertices and edges). This cycle is called to be a *fault-free spanning cycle* or a *fault-free hamiltonian cycle* in G. Note that if a graph G is k-fault hamiltonian, then $k \leq \delta(G) - 2$ where $\delta(G)$ is the minimum degree of vertices of G. For a graph G which is not complete, if G is k-fault hamiltonian-connected, then $k \leq \delta(G) - 3$.

The bipartite graph cannot be hamiltonian-connected because for bipartite sets X and Y, there is no hamiltonian path between u and v for $u, v \in Y$ such that $|X| \geq |Y|$. It is also neither k-fault hamiltonian for any $k \geq 1$, nor k-fault hamiltonian-connected for any $k \geq 0$. Therefore, in the bipartite graph, it is required to find the longer fault-free path between any pair of vertices and the longer fault-free cycle.

Let f_v denote the number of faulty vertices and f_e be the number of vertices. In an n-dimensional hypercube Q_n with $f_v + f_e \leq n - 1$, there is a fault-free cycle of length at least $2^n - 2f_v$ if $f_e < n - 1$, and there is a fault-free cycle of length at least $2^n - 2$ if $f_v = 0$ and $f_e = n - 1$ [6]. In an n-dimensional star graph, there is a fault-free cycle of length at least $n! - 2f_v$ if $f_v + f_e \leq n - 3$ [3]. Note that both the hypercubes and the star graphs are bipartite.

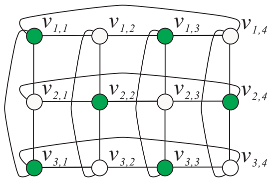
Kim *et al.* [5] considered the problem of finding a fault-free cycle in a 2-dimensional bipartite torus with faulty vertices. They suggested an algorithm of finding the longest fault-free cycle in a 2-dimensional torus with $f_v \leq 4$ such that both the number of rows and the number of columns are a multiple of 4. A two-dimensional torus with faulty vertices and/or edges, was shown to be 1-fault hamiltonian-connected and 2-fault hamiltonian if it is not bipartite [4].

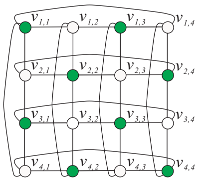
In this paper, we consider $d(\geq 2)$-dimensional tori with faulty vertices and/or edges. A d-dimensional $k_1 \times k_2 \times \cdots \times k_d$ torus with $k_i \geq 3$ for all $1 \leq i \leq d$, is $2d$-regular graph. If k_i for every i $(1 \leq i \leq d)$ is even, then the torus is bipartite; otherwise it is non-bipartite. It will be shown that any non-bipartite d-dimensional torus is $2d-3$-fault hamiltonian-connected and $2d - 2$-fault hamiltonian. In a bipartite d-dimensional torus, it will be shown that if $f_v + f_e \leq 2d - 2$, there is a fault-free path of length at least $N - 2f_v - 1$ between any pair of non-faulty vertices which belong to different partite sets, and there is a fault-free path of length at least $N - 2f_v - 2$ between any pair of vertices which belong to the same partite set where N is the number of vertices. We will also show that if $f_v + f_e \leq 2d - 1$ and $f_e \leq 2d - 2$, there is a fault-free cycle of length at least $N - 2f_v$, and if $f_v = 0$ and $f_e = 2d - 1$ and all the fault edges are not incident to a common vertex, there is a fault-free spanning cycle, and if $f_v = 0$ and $f_e = 2d - 1$ and all the fault edges are incident to a common vertex, there is a fault-free cycle of length $N - 2$.

The rest of this paper is organized as follows. In section 2, notation and terminologies are introduced. In section 3, we will show that every non-bipartite $d \geq 2$-dimensional torus is $2d-3$-fault hamiltonian-connected and $2d-2$-fault hamiltonian. In section 4, the lower bound for the length of fault-free path between any pair of non-faulty vertices, and the lower bound for the length of fault-free cycle are given in bipartite d-dimensional torus with faults. Finally, we conclude the paper in Section 5.

2 Notation and Terminologies

For a positive integer n, let $+_n$ denote cyclic addition. That is, for i, j with $1 \leq i, j \leq n$, $i +_n j = k$ where $k = i + j$ if $i + j \leq n$; otherwise, $k = (i + j) - n$. A d-dimensional $k_1 \times k_2 \times \cdots \times k_d$ torus with $k_i \geq 3$ for each i $(1 \leq i \leq d)$ denoted by $T(k_1, k_2, \cdots, k_d)$ is a graph consisting of $k_1 k_2 \cdots k_d$ vertices, each vertex is identified by $v_{i_1, i_2, \cdots, i_d}$ where $1 \leq i_j \leq k_j$ for every $1 \leq j \leq d$. Two vertices $v_{i_1, i_2, \cdots, i_d}$ and $v_{i'_1, i'_2, \cdots, i'_d}$, are adjacent if for some k with $1 \leq k \leq d$, $i'_k = i_k +_n 1$, and $i'_j = i_j$ for each j with $1 \leq j (\neq k) \leq d$. The edge which connects two vertices whose labels differ in the jth coordinate is called j-dimensional edge. Figure 1 shows 2-dimensional tori, $T(3, 4)$ which is non-bipartite, and $T(4, 4)$ which is bipartite.





(a) (b)

Figure 1: $T(3,4)$ and $T(4,4)$

Note that d-dimensional torus $T(k_1, k_2, \cdots, k_d)$ is vertex-symmetric but not edge-symmetric. In Figure 1 (a), there is a cycle of length 3 which contains the edge $(v_{1,2}, v_{2,2})$, but there is no cycle of length 3 which contains the edge $(v_{1,2}, v_{1,3})$. However, in d-dimensional torus, any two i-dimensional edges with $1 \leq i \leq d$, are similar, i.e. for any two j-dimensional edges, (u, v) and (u', v'), there is an automorphism g such that $g(u) = u'$ and $g(v) = v'$.

For a d-dimensional torus $T(k_1, k_2, \cdots, k_d)$, T^p_q denotes the subgraph induced by $\{v_{i_1,i_2,\cdots,i_{p-1},i_p,\cdots,i_d} | i_p = q, 1 \leq i_j \leq k_d \text{ for } 1 \leq j \neq p \leq d\}$. $T^p_{q:r}$ denotes the subgraph induced by $\{v_{i_1,i_2,\cdots,i_{p-1},i_p,\cdots,i_d} | p \leq i_p \leq r, 1 \leq i_j \leq k_d \text{ for } 1 \leq j \neq p \leq d\}$.

For a graph G, $V(G)$ denotes the set of vertices of G, and $E(G)$ denotes the set of edges of G. Graph theoretic terms not defined here can be found in [1].

3 Fault-Hamiltonicity of Non-Bipartite d-Dimensional Torus

For non-bipartite d-dimensional torus $T(k_1, k_2, \cdots, k_d)$ with $k_i \geq 3$ for each i $(1 \leq i \leq d)$, it will be shown that there is a fault-free spanning path joining each pair of non-faulty vertices when there are at most $2d-3$ faulty vertices and/or edges, and there is a fault-free hamiltonian cycle when there are at most $2d-2$ faulty vertices and/or edges. It is based on the results for 2-dimensional tori.

Lemma 1 *[4] 2-dimensional torus $T(m, n)$ with $m \geq 3$ and odd $n \geq 3$ is 1-fault hamiltonian-connected, and is 2-fault hamiltonian.*

The following lemma is useful for showing the fault-hamiltonicity of non-bipartite d-dimensional tori with faults.

Lemma 2 *Let m be an integer greater than or equal to 2. Let $G_1, G_2, \cdots, G_m$ be m copies of non-bipartite $d(\geq 2)$-dimensional torus $T(k_1, k_2, \cdots, k_d)$ with $k_i \geq 3$ for each i $(1 \leq i \leq d)$ where the vertices are labelled so that $V(G_i) \cap V(G_j) = \emptyset$ for $1 \leq i \neq j \leq m$. Let G be a graph such that $V(G) = \bigcup_{i=1}^{m} V(G_i)$ and $E(G) = (\bigcup_{i=1}^{m} E(G_i)) \cup (\bigcup_{i=1}^{m-1} E_{i,i+1})$ where $E_{i,i+1}$ is an isomorphism from G_i to G_{i+1}. If for each i, $(1 \leq i \leq m)$, G_i is $2d-3$-fault hamiltonian connected, there is a fault-free hamiltonian path between any two non-faulty vertices in $V(G)$ when there are no more than $2d-3$ faulty elements (vertices and edges) in G_i for each i $(1 \leq i \leq m)$, and there are no more than $2d-3$ faulty edges between $V(G_i)$ and $V(G_{i+1})$ for each i $(1 \leq i \leq m-1)$.*

Proof We will prove by induction on m. Let F^v be the set of faulty vertices and F^e be the set of faulty edges in G. For $1 \leq i \leq m$, let $F^v_i = F^v \cap V(G_i)$ and $F^e_i = F^e \cap E(G_i)$, and for $1 \leq i < m$, let $F_{i,i+1} = E_{i,i+1} \cap F^e$. Let u and v be arbitrary two non-faulty vertices of $V(G)$.

The proof for the base case $m = 2$ is similar to the proof for the inductive step, which is omitted. For the inductive step, assume $m > 2$. If both u and v are in $V(G_i)$ for some i $(1 \leq i \leq m)$, it may be assumed to be $i < m$ since the case of $u, v \in V(G_m)$ is the same situation as the case of $u, v \in V(G_1)$. By the induction hypothesis, there is a fault-free spanning path P between u and v in the induced subgraph by $\cup_{j=1}^{i} V(G_j)$. Let $P = (u_1 = u, u_2, \cdots, u_k = v)$, and let $S = \{j | u_j, u_{j+1} \in V(G_i), 1 \leq j \leq k-1\}$. Then there is l $(1 \leq l \leq k-1)$ such that u_l and u_{l+1} in S are connected to non-faulty vertices u' and v' in $V(G_{i+1})$ by non-faulty edges, respectively since we have $\lfloor \frac{|S|}{2} \rfloor \geq \lfloor \frac{3^d - |F_i^v| - |\{u,v\}|}{2} \rfloor \geq \lfloor \frac{3^d - 2d + 1}{2} \rfloor > (4d - 6) \geq |F_{i+1}^v| + |F_{i,i+1}|$. Since there is also a fault-free spanning path between u' and v' in the induced subgraph by $\bigcup_{j=i+1}^{m} G(V_j)$, we can obtain a fault-free hamiltonian path between u and v. Next we consider $u \in V(G_i)$ and $v \in V(G_j)$, $1 \leq i \neq j \leq m$. Let $i < j$. There is a non-faulty vertex $u' \in V(G_i) - \{u\}$ such that u' is connected to a non-faulty vertex $v' (\neq v)$ in $V(G_{i+1})$ by a non-faulty edge. By the induction hypothesis, there are fault-free spanning path P_1 between u and u' in the induced subgraph by $\bigcup_{k=1}^{i} V(G_k)$ and fault-free spanning path P_2 between v' and v in the induced subgraph by $\bigcup_{k=i+1}^{m} V(G_k)$. Thus we can obtain a fault-free hamiltonian path (u, P_1, u', v', P_2, v) in G. $\square$

Theorem 1 *Any non-bipartite d-dimensional torus $T(k_1, k_2, \cdots, k_d)$ with $k_i \geq 3$ for each i $(1 \leq i \leq d)$, is $2d-2$-fault hamiltonian and is $2d-3$-fault hamiltonian-connected.*

Proof Without loss of generality, let k_1 be odd. We will prove by induction on d. For the base step $d = 2$, it follows from Lemma 1 that the theorem holds true. For the inductive step, assume $d > 2$. $T(k_1, k_2, \cdots, k_d)$ is partitioned into $d-1$-dimensional tori $T_1^d, T_2^d, \cdots, T_{k_d}^d$. Let F^v be the set of faulty vertices and F^e be the set of faulty edges in $T(k_1, k_2, \cdots, k_d)$. For $1 \leq i \leq k_d$, let $F_i^v = F^v \cap V(T_i^d)$, and let $F_i^e = F^e \cap E(T_i^d)$. For $1 \leq i \leq k_d$, let $F_{i,i+_{k_d}1}$ be the set of faulty edges between $V(T_i^d)$ and $V(T_{i+_{k_d}1}^d)$. We can assume that T_1^d is the subtorus which has the most faulty elements among $T_1^d, T_2^d, \cdots, T_{k_d}^d$.

First We will show that $T(k_1, k_2, \cdots, k_d)$ is $2d-2$-fault hamiltonian.

Case 1 $|F_1^v| + |F_1^e| = 2d-2$, i.e., all faulty vertices and all faulty edges are in T_1^d.
Since T_1^d is $2d-4$-fault hamiltonian by the induction hypothesis, there are two fault-free vertex-disjoint paths P_1 and P_2 in T_1^d such that $V(P_1) \cup V(P_2) = V(T_1^d) - F_1^v$. Let u and v be two endvertices of P_1, and let w and x be two ednvertices of P_2. Let v' and w' be the vertices in $V(T_2^d)$ adjacent to v and w, respectively. Let u' and x' be the vertices in $V(T_{k_d}^d)$ adjacent to u and x, respectively. Since there are a spanning path P_3 between v' and w' in T_2^d and a spanning path P_4 between x' and u' in $T_{3:k_d}^d$ by Lemma 2, we can obtain a fault-free hamiltonian cycle by traversing $P_1, (v, v'), P_3, (w', w), P_2, (x, x'), P_4$, and (u', u).

Case 2 $|F_1^v| + |F_1^e| = 2d-3$.
Outside T_1^d, there is at most one faulty vertex or faulty edge. Since T_1^d is $2d-4$-fault hamiltonian, there is a fault-free spanning path P_1 in T_1^d. Let u and v be two endvertices of P_1. There are two non-faulty vertices $u', v' \in V(T_{2:k_d}^d)$ such that u and v are connected to u' and v' by non-faulty edges, respectively. Since there is a fault-free spanning path P_2 between v' and u' in the $T_{2:k_d}^d$ by Lemma 2, we can obtain a fault-free hamiltonian cycle by traversing $P_1, (v, v'), P_2$, and (u', u).

Case 3 For each $1 \leq i \leq k_d$, $|F_i^v| + |F_i^e| \leq 2d-4$.
If $|F_1^v| + |F_1^e| = 2d-4$, there is a fault-free spanning cycle C in the T_1^d by the induction hypothesis. Let $C = (u_1, u_2, \cdots, u_k, u_1)$. Then there is some l $(1 \leq l \leq k)$ such that u_l and u_{l+_k1} are connected to non-faulty vertices u' and v' in $V(T_2^d)$ by non-faulty edges, respectively. By Lemma 2, there is a fault-free spanning path P between u' and v' in the $T_{2:k_d}^d$. Thus we can obtain a fault-free hamiltonian cycle by replacing an edge (u_l, u_{l+_k1}) in C as the path $(u_l, u', P, v', u_{l+_k1})$. Consider the case that $|F_1^v| + |F_1^e| < 2d-4$. Let $m(1 \leq m \leq k_d)$ be an

integer such that $|F_{m,m+_{k_d}1}| = \max_{1\le i\le k_d}\{|F_{i,i+_{k_d}1}|\}$. When all edges connecting $V(T^d_m)$ and $V(T^d_{m+_{k_d}1})$ are deleted from $T(k_1,k_2,\cdots,k_d)$, the resulting graph is $(2d-5)$-fault hamiltonian-connected by Lemma 2, which means that there is a fault-free hamiltonian cycle .

Next we will show that $T(k_1,k_2,\cdots,k_d)$ is $2d-3$-fault hamiltonian-connected. Let u and v be arbitrary two non-faulty vertices.

Case 1 $|F^v_1| + |F^e_1| = 2d-3$.
We first consider $u, v \in V(T^d_1)$. Since T^d_1 is $2d-4$-fault hamiltonian by the induction hypothesis, there are three fault-free vertex-disjoint paths P_1, P_2, and P_3 such that u is the starting vertex of P_1, and v is the ending vertex of P_2, and $V(P_1)\cup V(P_2)\cup V(P_3) = (V(T^d_1) - F^v_1)$. Let u' and v' be the ending vertex of P_1 and the starting vertex of P_2, respectively. Let w' and w be two endvertices of P_3. Let u'' and w'' be the vertices in $V(T^d_2)$ which are adjacent to u' and w', respectively. Let v'' and w^* be the vertices in $V(T^d_{k_d})$ which are adjacent to v' and w, respectively. By the induction hypothesis and Lemma 2, there are a spanning path P_4 between u'' and w'' in T^d_2 and a spanning path P_5 between w^* and v'' in $T^d_{3:k_d}$. Thus we can obtain a fault-free hamiltonian path P between u and v in $T(k_1,k_2,\cdots,k_d)$ where $P = (u, P_1, u', u'', P_4, w'', w', P_3, w, w^*, P_5, v'', v', P_2, v)$.

Secondly, we consider $u \in V(T^d_1)$, $v \in V(T^d_i)$ with $2 \le i \le k_d$. It can be assumed that $i \ne k_d$. There are two fault-free vertex-disjoint paths P_1 starting at u, and P_2 such that $V(P_1)\cup V(P_2) = V(T^d_1) - F^v_1$. Let u' be the ending vertex of P_1, and let u'' be a vertex in $V(T^d_{k_d})$ adjacent to u'. Let w be one of two endvertices of P_2 which is not adjacent to v, and let w' be the other endvertex of P_2. Let w'' be a vertex in $V(T^d_{k_d})$ adjacent to w', and let v' be a vertex in $V(T^d_2)$ adjacent to w. By Lemma 2, we have a spanning path P_3 between v' and v in $T^d_{2:k_d-1}$. Since $T^d_{k_d}$ is $2d-5$-fault hamiltonian-connected by the induction hypothesis, we have a spanning path P_4 between u'' and w'' in $T^d_{k_d}$. Thus we can obtain a fault-free hamiltonian path P between u and v where $P = (u, P_1, u', u'', P_4, w'', w', P_2, w, v', P_3, v)$.

It remains to consider $u \in V(T^d_i)$ and $v \in V(T^d_j)$ with $i \ne 1, j \ne 1$. Since T^d_1 is $2d-4$-fault hamiltonian by the induction hypothesis, there is a fault-free spanning path P_1 in T^d_1. Let w and w' be two endvertices of P_1. It will be shown by dividing into two cases: $i = j$ and $i \ne j$. If $i = j$, we can assume $i < k_d$. If u or v is adjacent to one of w and w', we assume that u is adjacent to w. Let v' be a vertex in $V(T^d_{k_d})$ adjacent to w'. By Lemma 2, there is a spanning path P_2 between v' and v in $T^d_{2:k_d} - \{u\}$. Thus we can obtain a fault-free hamiltonian path $(u, w, P_1, w', v', P_2, v)$ between u and v. If neither u nor v is adjacent to w and w', let u' be a vertex $V(T^d_{i+1})$ adjacent to u. Let u'' be a vertex ($\ne u'$) in $V(T^d_{k_d})$ which is adjacent to w or w'. u'' is assumed to be adjacent to w. Let v' be a vertex in $V(T^d_2)$ which is adjacent to w'. By the induction hypothesis and Lemma 2, we have a spanning path P_2 between u' and u'' in $T^d_{i+1:k_d}$ and a spanning path P_3 from v' to v in $T^d_{2:i} - \{u\}$. Therefore we can obtain a fault-free hamiltonian path P between u and v such that $P = (u, u', P_2, u'', w, P_1, w', v', P_3, v)$. Finally, we consider $i \ne j$. We let $i < j$. Let w and w' be the endvertices of a fault-free spanning path P_1 in T^d_1. If between w and w' there is a vertex (say w) which is adjacent to both u and v, we have that $i = 2$ and $j = k_d$. Then we can obtain a fault-free hamiltonian path P between u and v such that $P = (u, w, P_1, w', w'', P_2, v$ where P_2 is a fault-free hamiltonian path between w'' and v in $T^d_{2:k_d} - \{u\}$. If neither w nor w' is adjacent to both u and v, we assume that w is not adjacent to u and w' is not adjacent to v. We can obtain a fault-free hamiltonian path P between u and v such that $P = u, P_1, u', w, P, w', v', P_2, v$ where $u' \in V(T^d_2)$ and $v' \in V(T^d_{k_d}$ are vertices adjacent to w and w', respectively, and P_1 is a spanning path between u and u' in $T^d_{2:i}$ and P_2 is a spanning path between v' and v in $T^d_{i+1:k_d}$.

Case 2 $|F^v_1| + |F^e_1| = 2d-4$.
Outside T^d_1, there is at most one faulty vertex or faulty edge. By the induction hypothesis,

there is a fault-free spanning cycle C in T_1^d. First consider $u, v \in V(T_1^d)$. From C, we can obtain two fault-free vertex-disjoint paths P_1, P_2 such that u is the starting vertex of P_1, and v is the ending vertex of P_2, and $V(P_1) \cup V(P_2) = (V(T_1^d) - F_1^v)$. Let u' and v' be the ending vertex of P_1 and the starting vertex of P_2, respectively. There are two non-faulty vertices u'' and v'' in $V(T_2^d) \cup V(T_{k_d}^d)$ which are connected to u' and v' by non-faulty edges, respectively. Since there is a fault-free spanning path between u'' and v'' in $T_{2:k_d}^d$ by the induction hypothesis and Lemma 2, we can obtain a fault-free hamiltonian path between u and v.

Secondly, we consider $u \in V(T_1^d)$ and $v \in V(T_i^d)$ with $2 \leq i \leq k_d$. We can assume $i \neq k_d$. From the fault-free spanning cycle C in T_1^d, we can obtain a fault-free spanning path P_1 starting at u in T_1^d such that the ending vertex u' of P_1 is connected to non-faulty vertex v' in $V(T_{k_d}^d)$ by non-faulty edge. Since there is either one faulty vertex or one faulty edge outside T_1^d, such path P_1 exists. We can obtain a fault-free hamiltonian path between u and v since there is a fault-free spanning path between v' and v in $T_{2:k_d}^d$ by Lemma 2.

Next we consider $u \in V(T_i^d)$ and $v \in V(T_j^d)$ with $i, j \neq 1$. Let C be a fault-free spanning cycle $(u_1, u_2, \cdots, u_k, u_1)$ in T_1^d. We consider by dividing into two cases:$i = j$ and $i \neq j$. If $i = j$, we can assume $i < k_d$. Between u and v, there is a vertex (say u) which is connected to a non-faulty vertex $u' \in V(T_{i+1}^d)$ by a non-faulty edge. Then there is some l $(1 \leq l \leq k)$ such that neither u_l nor $u_{l+_k 1}$ is adjacent to u, v, and u' since we have $k \geq 3^{d-1} - (2d-4) \geq 7$. Let P_1 be a fault-free spanning path between u_l and $u_{l+_k 1}$ in T_1^d. Since there is either one faulty vertex or one faulty edge outside T_1^d, there are two non-faulty vertices $u'' \in V(T_{k_d}^d)$ and $v' \in V(T_2^d)$ such that u'' and v' are connected to u_l and $u_{l+_k 1}$ by non-faulty edges, respectively. By the induction hypothesis and Lemma 2, we have a fault-free spanning path P_2 between u' and u'' in $T_{i+1:k_d}^d$, and a fault-free spanning path P_3 between v' and v in $T_{2:i}^d - \{u\}$. The path $P = (u, u', P_2, u'', u_l, P_1, u_{l+_k 1}, v', P_3, v)$ becomes a fault-free hamiltonian path between u and v. If $i \neq j$, let $i < j$. There is $l(1 \leq l \leq k)$ such that u_l and $u_{l+_k 1}$ are connected to a non-faulty vertex u' in $V(T_2^d) - \{u\}$ and a non-faulty vertex v' in $V(T_{k_d}^d) - \{v\}$ by non-faulty edges, respectively. By the induction hypothesis and Lemma 2, we have a fault-free spanning path P_2 between u and u' in $T_{2:i}^d$, and a fault-free spanning P_3 between v' and v in $T_{i+1:k_d}^d$. Therefore we can obtain a fault-free hamiltonian path between u and v.

Case 3 For each $1 \leq i \leq k_d$, $|F_i^v| + |F_i^e| \leq 2d - 5$.
Let $m (1 \leq m \leq k_d)$ be an integer such that $|F_{m,m+_{k_d}1}| = \max_{1 \leq i \leq k_d}\{|F_{i,i+_{k_d}1}|\}$. When all edges connecting $V(T_m^d)$ and $V(T_{m+1}^d)$ are deleted in $T(k_1, k_2, \cdots, k_d)$, the resulting graph is $(2d-5)$-fault hamiltonian-connected by Lemma 2. Therefore there is a fault-free hamiltonian path between u and v. $\square$

4 Paths and Cycles in Bipartite d-Dimensional Torus with Faults

In a bipartite d-dimensional torus with faults, there does not necessarily exist a fault-free path between any pair of vertices which includes all non-faulty vertices, and also there does not necessarily exist a fault-free cycle which includes all non-faulty vertices. Therefore it is required to find the longer fault-free path or the longer fault-free cycle. The vertices of the bipartite d-dimensional torus are colored as black or white in such a way that two adjacent vertices have different colors. Let N be the number of vertices. A fault-free path P between a pair of non-faulty vertices, u and v, is called to be $L-path$ if the length of P is greater than or equal to $N - 2f_v - 1$ when the colors of u and v are different, and the length of P is greater than or equal to $N - 2f_v - 2$ when the colors of u and v are the same. A fault-free cycle C is called to be $L-cycle$ if the length of C is greater than or equal to $N - 2f_v$. L-path and L-cycle have the meaning that when the colors of all the fault vertices are same, there is no fault-free path (fault-free cycle) of length greater than the length of L-path (L-cycle), respectively. Thus the length of L-path (L-cycle) is the upper bound for the length of fault-free path (fault-free cycle), respectively, in the worst case.

To show that there is an L-path between any pair of non-faulty vertices and there is an L-cycle in a d-dimensional torus with faults, we begin with the following lemma for a 2-dimensional bipartite torus, which plays a role as the base case.

Lemma 3 *In a 2-dimensional torus $T(m,n)$ with even $m,n \geq 4$, (a) there is an L-path between any pair of non-faulty vertices if $f_v + f_e \leq 2$, and (b) there is an L-cycle if $f_v + f_e \leq 3$ and $f_e \leq 2$, and there is a fault-free spanning cycle if $f_v = 0$ and $f_e = 3$ and all the faulty edges are not incident to a common vertex, and there is a fault-free cycle of length $mn - 2$ if $f_v = 0$ and $f_e = 3$ and all the faulty edges are incident to a common vertex.*

Theorem 2 *In any bipartite d-dimensional torus $T(k_1, k_2, \cdots, k_d)$ with $k_i \geq 4$ for each i $(1 \leq i \leq d)$, (a) there is an L-path between any pair of non-faulty vertices if $f_v + f_e \leq 2d - 2$, and (b) there is an L-cycle if $f_v + f_e \leq 2d - 1$ and $f_e \leq 2d - 2$, and there is a fault-free spanning cycle if $f_v = 0$ and $f_e = 2d - 1$ and all the faulty edges are not incident to a common vertex, and there is a fault-free cycle of length $N - 2$ if $f_v = 0$ and $f_e = 2d - 1$ and all the faulty edges are incident to a common vertex.*

Proof we can prove the theorem by the similar procedure to the proof of Theorem 1, which is omitted here.

An n-dimensional hypercube Q_n is $\frac{n}{2}$-dimensional torus $T(4, 4, \cdots, 4)$ if n is even and is greater than or equal to 4. Therefore, we have the following result for a faulty n-dimensional hypercube with even $n \geq 4$.

Corollary 1 *In an n-dimensional hypercube with even $n \geq 4$, there is an L-path between any pair of vertices if $f_v + f_e \leq n - 2$. If $f_v + f_e \leq n - 1$ and $f_e \leq n - 2$, there is an L-cycle. If $f_v = 0$ and $f_e = n - 1$ and all the faulty edges are not incident to a common vertex, there is a fault-free spanning cycle. If $f_v = 0$ and $f_e = n - 1$ and all the faulty edges are incident to a common vertex, there is a fault-free cycle of length $2^n - 2$.*

Corollary 1 gives not only the existence of a fault-free cycle with the same length as [6] but also the upper bound for the length of fault-free paths between any pair of non-faulty vertices in a faulty n-dimensional hypercube Q_n with even $n \geq 4$.

5 Concluding Remarks

In this paper, we investigated the fault-free path between any pair of non-faulty vertices and the fault-free cycle in a d-dimensional torus with faulty vertices and/or edges. We showed that any non-bipartite d-dimensional $k_1 \times k_2 \times \cdots \times k_d$ torus with $k_i \geq 3$ for all $1 \leq i \leq d$, is $2d-3$-fault hamiltonian-connected, and it is $2d-2$-fault hamiltonian. In a $d(\geq 4)$-dimensional bipartite torus with $f_v + f_e \leq 2d - 2$, it was shown that for any pair of vertices u, v, there is a fault-free path of the length at least $N - 2f_v - 1$ between u and v if u and v have different colors, and there is a fault-free path of the length at least $N - 2f_v - 2$ between u and v if u and v have the same color where N is the number of vertices, f_v is the number of faulty vertices, and f^e is the number of faulty edges. We also showed that in a d-dimensional bipartite torus, there is a fault-free cycle of length at least $N - 2f_v$ if $f_v + f_e \leq 2d-1$ and $f_e \leq 2d-2$, and there is a fault-free spanning cycle if $f_v = 0$ and $f_e = 2d-1$ and all the faulty edges are not incident to a common vertex, and there is a fault-free path of length $N - 2$ if $f_v = 0$ and $f_e = 2d - 1$ and all the faulty edges are incident to a common vertex.

Let f_v^b (f_v^w) be the number of faulty vertices colored as black (white) in a d-dimensional bipartite torus, respectively. The upper bound for the length of fault-free cycle is $N - 2\max\{f_v^b, f_v^w\}$. When $f_v^b = f_v^w$, the upper bound for the length of fault-free hamiltonian path between a pair of vertices, u and v, is $N - 2\max\{f_v^b, f_v^w\} - 1$ if the colors of u and v are different, and is $N - 2\max\{f_v^b, f_v^w\} - 2$ if u and v have the same color. When $f_v^b \neq f_v^w$, let $f_v^b < f_v^w$. Then the upper bound for the length of fault-free hamiltonian path between a pair of vertices, u and v, is $N - 2\max\{f_v^b, f_v^w\}$ if u and v are colored as black, and is $N - 2\max\{f_v^b, f_v^w\} - 1$ if the colors of u and v are different, and is $N - 2\max\{f_v^b, f_v^w\} - 2$ if u and v are colored as white. It remains to be open whether there exist a fault-free path of the length with the upper bound and a fault-free cycle of the length with the upper bound in a d-dimensional bipartite torus with faults.

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