Dihamiltonian Decomposition of Regular Graphs with Degree Three *

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Abstract. We consider the dihamiltonian decomposition problem for 3-regular graphs. A graph G is *dihamiltonian decomposable* if in the digraph obtained from G by replacing each edge of G as two directed edges, the set of edges are partitioned into 3 edge-disjoint directed hamiltonian cycles. We suggest some conditions for dihamiltonian decomposition of 3-regular graphs: for a 3-regular graph G, it is dihamiltonian decomposable only if it is bipartite, and it is not dihamiltonian decomposable if the number of vertices is a multiple of 4. Applying these conditions to interconnection network topologies, we investigate dihamiltonian decomposition of cube-connected cycles, chordal rings, etc.

1 Introduction

An r-regular graph G is *hamiltonian decomposable* if it is possible to partition the set of edges into k edge-disjoint hamiltonian cycles when r is $2k$ and it is possible to partition the set of edges into k edge-disjoint hamiltonian cycles and a 1-factor when r is $2k+1$. Here a 1-factor of a graph is a 1-regular spanning subgraph.

The symmetric digraph of an undirected graph G is defined as the digraph obtained from G by replacing each edge (u, v) of G as two directed edges, $\langle u, v\rangle$ and $\langle v, u\rangle$. An r-regular graph G is *dihamiltonian decomposable* if the set of edges in its symmetric digraph can be partitioned into r edge-disjoint directed hamiltonian cycles.

For a graph G to have either a hamiltonian decomposition or a dihamiltonian decomposition, it is necessary that G is loopless, connected, and regular. A hamiltonian decomposable r-regular graph with even r is also dihamiltonian decomposable, since each hamiltonian cycle in the decomposition can be regarded as two directed hamiltonian cycles of opposite direction. When r is odd, it is not always true.

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A survey on hamiltonian decomposition of graphs is provided in[3, 4]. Much of the focus of research has been directed towards proving that some special cases of Cayley graphs over an abelian group are hamiltonian decomposable, such as the product of any number of cycles, m-cubes, and recursive circulants[5, 9]. But still, the current status of the matter lies, for the most part, in the sphere of problems and conjectures. For the dihamiltonian decomposition of graphs, there is only a trivial result such that the hamiltonian decomposable graphs are dihamiltonian decomposable. It remains open whether m-cubes and recursive circulants with an odd degree are dihamiltonian decomposable.

The problem of dihamiltonian decomposition of graphs has an application in the design of reliable communication algorithms such as broadcasting and multicasting under the wormhole routing model. In the way of packing as many edge-disjoint directed hamiltonian cycles as possible, reliable communication algorithms are presented on m-cubes and tori[6], and m-dimensional meshes[7]. Note that the dihamiltonian decomposition of a regular graph results in the maximum number of edge-disjoint directed hamiltonian cycles.

In this paper, we consider the dihamiltonian decomposition of 3-regular graphs. We show that if a 3-regular graph is dihamiltonian decomposable, it is necessarily bipartite. We also show that every 3-regular graph with $n = 4k$ vertices is not dihamiltonian decomposable. Applying these results to interconnection network topologies with degree 3, we investigate dihamiltonian decomposition of cube-connected cycles[8], chordal rings[1], etc.

A 3-regular graph G with multiple edges is not dihamiltonian decomposable, since edge-connectivity of G is less than or equal to 2 and it is not possible for three edge-disjoint directed hamiltonian cycles to pass through the edge cut. Thus we assume a graph is simple. Graph theoretic terms not defined here can be found in[2].

This paper is organized as follows. We show that a dihamiltonian decomposable graph is bipartite in Section 2, and prove that a graph with $n = 4k$ vertices is not dihamiltonian decomposable in Section 3. In Section 4, we consider the dihamiltonian decomposition of the interconnection network topologies, and finally concluding remarks are given in Section 5.

2 Necessary Conditions

In this section we present some properties for dihamiltonian decomposition of 3-regular graphs. Let G be a 3-regular graph with n vertices which is dihamiltonian decomposable. Note that the number of vertices of 3-regular graphs is even. Let G_S denote the symmetric digraph of G. Then the set of edges of G_S is partitioned into 3 directed hamiltonian cycles. Let C, which is one of such cycles, be (v_{n-1}, v_{n-2}, $\cdots$, v_1, v_0, v_{n-1}). Let C_0 and C_1 be the other two directed hamiltonian cycles. For the symmetric digraph G_S of G, $G_S - C$ is defined as the digraph which is obtained by deleting all the edges in C from G_S. From now on, it is assumed that all arithmetic operations on the indices of vertices is performed with mod n.

We call $\langle v_i, v_{i+1}\rangle$, $0 \le i \le n-1$, to be a boundary edge w.r.t. C, and $\langle v_i, v_j\rangle$ with $j \ne i+1$ to be a chord edge w.r.t. C(see Figure 1). An edge (v_i, v_j) in G is called a boundary (resp. chord) edge w.r.t. C if $\langle v_i, v_j\rangle$ or $\langle v_j, v_i\rangle$ in G_S is a boundary (resp. chord) edge w.r.t. C. The size of chord edge $\langle v_i, v_j\rangle$ is defined as $\min\{i-j, j-i\}$. We will now investigate the properties of two edge-disjoint hamiltonian cycles C_0 and C_1. In the following, C_0 denotes the hamiltonian cycle which contains $\langle v_0, v_1\rangle$ unless it is otherwise specified.

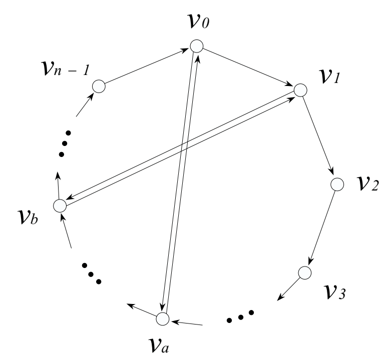

Fig. 1. $G_S - C$

Lemma 1 *(a) In each of C_0 and C_1, the boundary edge and the chord edge w.r.t. C appear alternately. (b) For each boundary edge $\langle v_i, v_{i+1}\rangle$ w.r.t. C in C_0 (resp. C_1), i is even (resp. odd).*

Proof We first show that the lemma is true for C_0. Let $\langle v_i, v_{i+1}\rangle$ be an arbitrary boundary edge w.r.t. C in C_0. Since both $\langle v_{i+1}, v_i\rangle$ and $\langle v_{i+2}, v_{i+1}\rangle$ are in C, the edge which immediately follows the edge $\langle v_i, v_{i+1}\rangle$ on C_0 should be the chord edge for C_1 to be a hamiltonian cycle. For every chord edge $\langle v_i, v_j\rangle$ in C_0, the edge which immediately follows it on C_0 is the boundary edge $\langle v_j, v_{j+1}\rangle$ since $\langle v_j, v_{j-1}\rangle$ is in C. Thus the boundary edge and the chord edge appear alternately in C_0. The proof of part (b) is as follows. Let $\langle v_i, v_{i+1}\rangle$ be an arbitrary boundary edge w.r.t. C in C_0. For each j with $0 \le j < i$, one of $\langle v_{j-1}, v_j\rangle$ and $\langle v_j, v_{j+1}\rangle$ is in C_0 and the other is not in C_0. Therefore, i should be even since $\langle v_0, v_1\rangle$ is in C_0. It can be similarly shown that the lemma holds for C_1. □

Using Lemma 1, we present a necessary condition for dihamiltonian decomposition of 3-regular graphs.

Theorem 1 *If a 3-regular graph G is dihamiltonian decomposable, G is a bipartite graph.*

Proof Let C, C_0, and C_1 be 3 edge-disjoint hamiltonian cycles in G_S. Let $C = (v_{n-1}, v_{n-2}, \cdots, v_1, v_0, v_{n-1})$, and let C_0 be the hamiltonian cycle which contains $\langle v_0, v_1 \rangle$. A vertex v_a is called even vertex if a is even; otherwise it is called an odd vertex. Every boundary edge joins even vertex and odd vertex since n is even. Therefore we have only to show that every chord edge w.r.t. C join an even vertex and an odd vertex. Consider an arbitrary chord edge $\langle v_i, v_j \rangle$ w.r.t. C in C_0. The edge which immediately precedes $\langle v_i, v_j \rangle$ on C_0, is a boundary edge $\langle v_{i-1}, v_i \rangle$ by Lemma 1. Thus $i-1$ is even, that is, i is odd, by Lemma 1. Since the edge which immediately follows $\langle v_i, v_j \rangle$ on C_0, is a boundary edge $\langle v_j, v_{j+1} \rangle$, j is even. It can be similarly shown that every chord edge w.r.t. C in C_1 joins an even vertex and an odd vertex. Therefore G is a bipartite graph. □

We denote by G_0^C the graph obtained by deleting all the boundary edges w.r.t. C and contracting two vertices $\{v_i, v_{i+1}\}$ in the digraph $G_S - C$ for every even i, $0 \leq i \leq n-1$. We denote by G_1^C the graph obtained by deleting all the boundary edges w.r.t. C and contracting two vertices $\{v_i, v_{i+1}\}$ in $G_S - C$ for every odd i, $0 \leq i \leq n-1$. Then we have the following properties. The proof is straightforward, and omitted here.

Property 1 *Each of G_0^C and G_1^C is a cycle with length $n/2$.*

Property 2 *A 3-regular graph G is dihamiltonian decomposable if and only if for some hamiltonian cycle C in the symmetric digraph G_S of G, each of G_0^C and G_1^C is a cycle with length $n/2$.*

3 Case of $n = 4k$

Some 3-regular graphs are not dihamiltonian decomposable. In this section, we will show that every 3-regular graphs with $n = 4k$ vertices can not be dihamiltonian decomposable by utilizing the following lemma.

Lemma 2 *If there exists a 3-regular graph with $n \geq 8$ vertices which is dihamiltonian decomposable, then there exists a 3-regular graph with $n-4$ vertices which is dihamiltonian decomposable.*

Proof Let G be a 3-regular graph with n vertices. Let C be a hamiltonian cycle $(v_{n-1}, v_{n-2}, \cdots, v_1, v_0, v_{n-1})$ in dihamiltonian decomposition of G. Let C_0 and C_1 be two edge-disjoint hamiltonian cycles in $G_S - C$.

Case 1 There is a chord edge of size 3 w.r.t. C in G.

Let (v_{i+1}, v_{i+4}) be a chord edge of size 3 w.r.t. C in G, and let v'_{i+2} and v'_{i+3} be the vertices connected with v_{i+2} and v_{i+3} by a chord edge, respectively(see Figure 2). Let C_0 and C_1 be the hamiltonian cycles which contain the edge $\langle v_i, v_{i+1} \rangle$ and $\langle v_{i+1}, v_{i+2} \rangle$, respectively. We have $C_0 = (v_i, v_{i+1}, v_{i+4}, v_{i+5}, P_{1,1}, v'_{i+2}, v_{i+2}, v_{i+3}, v'_{i+3}, P_{1,2}, v_i)$ where $P_{1,1}$ is the path from v_{i+5} to v'_{i+2} and $P_{1,2}$ is the path from v'_{i+3} to v_i. We have $C_1 = (v'_{i+3}, v_{i+3}, v_{i+4}, v_{i+1}, v_{i+2}, v'_{i+2},$

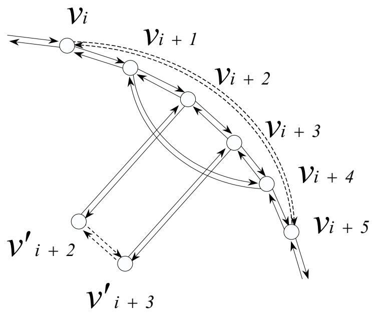

Fig. 2. Illustration of the proof of Lemma 2 (Case 1)

$P_{1,3}$, v'_{i+3}) where $P_{1,3}$ is the path from v'_{i+2} to v'_{i+3} in C_1. We first consider the case that all vertices in $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}, v'_{i+2}, v'_{i+3}\}$ are distinct.

We define a graph G' with $n-4$ vertices as $G' = (G - \{v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}\}) \cup \{(v_i, v_{i+5}), (v'_{i+2}, v'_{i+3})\}$(see Figure 2). We will show that G' is a simple 3-regular graph by proving that (i) (v_i, v_{i+5}) is not a chord edge w.r.t. C in G, and that (ii) (v'_{i+2}, v'_{i+3}) is not a boundary edge w.r.t. C in G.

If (v_i, v_{i+5}) is a chord edge, C_0 becomes a cycle $(v_i, v_{i+1}, v_{i+4}, v_{i+5}, v_i)$ with length 4 by Lemma 1, which is a contradiction to C_0 being a hamiltonian cycle. If (v'_{i+2}, v'_{i+3}) is a boundary edge, then for some k, either $v'_{i+2} = v_k$ and $v'_{i+3} = v_{k+1}$ or $v'_{i+2} = v_{k+1}$ and $v'_{i+3} = v_k$. In the first case, C_1 becomes a cycle $(v_{i+1}, v_{i+2}, v'_{i+2} = v_k, v'_{i+3} = v_{k+1}, v_{i+3}, v_{i+4}, v_{i+1})$ with length 6 by Lemma 1, which is a contradiction. In the second case, C_0 becomes a cycle $(v_{i+2}, v_{i+3}, v'_{i+3} = v_k, v'_{i+2} = v_{k+1}, v_{i+2})$ with length 4, which is also a contradiction.

Using C, C_0, and C_1, we construct 3 edge-disjoint hamiltonian cycles C', C'_0, and C'_1 in the symmetric digraph of G'. Let $C' = (v_{n-1}, v_{n-2}, \cdots, v_{i+5}, v_i, v_{i-1}, \cdots, v_0, v_{n-1})$, and $C'_0 = (v_i, v_{i+5}, P_{1,1}, v'_{i+2}, v'_{i+3}, P_{1,2}, v_i)$ and $C'_1 = (v'_{i+3}, v'_{i+2}, P_{1,3}, v'_{i+3})$. Then C', C'_0 and C'_1 are edge-disjoint hamiltonian cycles in the symmetric digraph of G'.

We consider the case that some vertices in $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}, v'_{i+2}, v'_{i+3}\}$ are not distinct. Since G is bipartite, either $v'_{i+2} = v_{i+5}$ or $v'_{i+3} = v_i$. If $v'_{i+2} = v_{i+5}$, $P_{1,1}$ is a path with length 0. If $v'_{i+3} = v_i$, $P_{1,2}$ is a path with length 0. It does not hold that both $v'_{i+2} = v_{i+5}$ and $v'_{i+3} = v_i$; otherwise C_0 becomes a cycle with length 6. In either case, C', C'_0 and C'_1 are edge-disjoint hamiltonian cycles in G'.

Case 2 There are no chord edges of size 3 w.r.t. C in G and either C_0 or C_1 has a boundary edge w.r.t. C, $\langle v_i, v_{i+1}\rangle$, $0 \le i \le n-1$, such that when we traverse that hamiltonian cycle starting at $\langle v_i, v_{i+1}\rangle$, $\langle v_{i+4}, v_{i+5}\rangle$ precedes $\langle v_{i+2}, v_{i+3}\rangle$.

Let C_0 and C_1 be the hamiltonian cycles which contain the edge $\langle v_i, v_{i+1}\rangle$ and $\langle v_{i+1}, v_{i+2}\rangle$, respectively(see Figure 3). We have $C_0 = (v_i, v_{i+1}, v'_{i+1}, P_{2,1}, v'_{i+4}, v_{i+4}, v_{i+5}, P_{2,2}, v'_{i+2}, v_{i+2}, v_{i+3}, v'_{i+3}, P_{2,3}, v_i)$ where $P_{2,1}$ is the path from

v'_{i+1} to v'_{i+4} and $P_{2,2}$ is the path from v_{i+5} to v'_{i+2} and $P_{2,3}$ is the path from v'_{i+3} to v_i. We have $C_1 = (v'_{i+1}, v_{i+1}, v_{i+2}, v'_{i+2}, P_{2,4}, v'_{i+3}, v_{i+3}, v_{i+4}, v'_{i+4}, P_{2,5}, v'_{i+1})$ where $P_{2,4}$ is the path from v'_{i+2} to v'_{i+3} and $P_{2,5}$ is the path from v'_{i+4} to v'_{i+1}. Since there are no chord edges of size 3 w.r.t. C, all vertices in $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}, v'_{i+1}, v'_{i+2}, v'_{i+3}, v'_{i+4}\}$ are distinct.

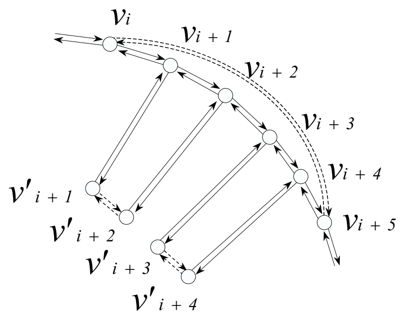

Fig. 3. Illustration of the proof of Lemma 2 (Case 2)

Let G' be the graph such that $G' = (G - \{v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}\}) \cup \{(v_i, v_{i+5}), (v'_{i+1}, v'_{i+2}), (v'_{i+3}, v'_{i+4})\}$(see Figure 3). Then we can see that G' is a simple 3-regular graph by proving that (i) (v_i, v_{i+5}) is not a chord edge w.r.t. C, that (ii) (v'_{i+1}, v'_{i+2}) is not a boundary edge w.r.t. C, and that (iii) (v'_{i+3}, v'_{i+4}) is not a boundary edge w.r.t. C.

If (v_i, v_{i+5}) is a chord edge, then C_0 is $(v'_{i+4}, v_{i+4}, v_{i+5}, v_i, v_{i+1}, v'_{i+1}, \cdots, v'_{i+4})$, that is, C_0 is $(v_i, v_{i+1}, \cdots, v_{i+2}, v_{i+3}, \cdots, v_{i+4}, v_{i+5}, v_i)$. Since $\langle v_{i+4}, v_{i+5} \rangle$ appears after $\langle v_{i+2}, v_{i+3} \rangle$ in C_0, it contradicts the assumption of Case 2. If (v'_{i+1}, v'_{i+2}) is a boundary edge, then for some k, either $v'_{i+1} = v_k$ and $v'_{i+2} = v_{k+1}$ or $v'_{i+1} = v_{k+1}$ and $v'_{i+2} = v_k$. In the first case, C_0 is $(v_i, v_{i+1}, v'_{i+1} = v_k, v'_{i+2} = v_{k+1}, v_{i+2}, v_{i+3}, v'_{i+3}, \cdots, v_i)$, which contradicts the assumption of Case 2. In the second case, C_1 becomes a cycle $(v_{i+1}, v_{i+2}, v'_{i+2} = v_k, v'_{i+1} = v_{k+1}, v_{i+1})$ with length 4, which is a contradiction.

If (v'_{i+3}, v'_{i+4}) is a boundary edge, then for some k, either $v'_{i+3} = v_k$ and $v'_{i+4} = v_{k+1}$ or $v'_{i+3} = v_{k+1}$ and $v'_{i+4} = v_k$. In the first case, C_0 is $(v_{i+2}, v_{i+3}, v'_{i+3} = v_k, v'_{i+4} = v_{k+1}, v_{i+4}, v_{i+5}, \cdots, v_i, v_{i+1}, \cdots, v_{i+2})$, that is, C_0 is $(v_i, v_{i+1}, \cdots, v_{i+2}, v_{i+3}, v'_{i+3}, v'_{i+4}, v_{i+4}, v_{i+5}, \cdots, v_i)$, which contradicts the assumption of Case 2. In the second case, C_1 becomes a cycle $(v_{i+3}, v_{i+4}, v'_{i+4} = v_k, v'_{i+3} = v_{k+1}, v_{i+3})$ with length 4, which is a contradiction.

The dihamiltonian decomposition of G' is as follows. Let C' be $(v_{n-1}, v_{n-2}, \cdots, v_{i+5}, v_i, v_{i-1}, \cdots, v_0, v_{n-1})$, and C'_0 be $(v_i, v_{i+5}, P_{2,2}, v'_{i+2}, v'_{i+1}, P_{2,1}, v'_{i+4}, v'_{i+3}, P_{2,3}, v_i)$ and C'_1 be $(v'_{i+1}, v'_{i+2}, P_{2,4}, v'_{i+3}, v'_{i+4}, P_{2,5}, v'_{i+1})$. Then C', C'_0 and C'_1 are edge-disjoint hamiltonian cycles in G'.

Case 3 Both Case 1 and Case 2 do not hold.

Let v_i be an arbitrary vertex and let C_0 and C_1 be the disjoint hamiltonian cycles in $G_S - C$ which contain the edge $\langle v_i, v_{i+1}\rangle$ and $\langle v_{i+1}, v_{i+2}\rangle$, respectively(see Figure 4). We first consider the case that all vertices in $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}, v_{i+6}, v'_{i+1}, v'_{i+2}, v'_{i+3}, v'_{i+4}, v'_{i+5}\}$ are distinct. When we traverse C_0 starting at $\langle v_i, v_{i+1}\rangle$, $\langle v_{i+2}, v_{i+3}\rangle$ precedes $\langle v_{i+4}, v_{i+5}\rangle$ in C_0. Also, when we traverse C_1 starting at $\langle v_{i+1}, v_{i+2}\rangle$, $\langle v_{i+3}, v_{i+4}\rangle$ precedes $\langle v_{i+5}, v_{i+6}\rangle$ in C_1; otherwise it is Case 2. Therefore C_0 and C_1 can be represented as follows: $C_0 = (v_i, v_{i+1}, v'_{i+1}, P_{3,1}, v'_{i+2}, v_{i+2}, v_{i+3}, v'_{i+3}, P_{3,2}, v'_{i+4}, v_{i+4}, v_{i+5}, v'_{i+5}, P_{3,3}, v_i)$ where $P_{3,1}$ is the path from v'_{i+1} to v'_{i+2} and $P_{3,2}$ is the path from v'_{i+3} to v'_{i+4} and $P_{3,3}$ is the path from v'_{i+5} to v_i; $C_1 = (v'_{i+1}, v_{i+1}, v_{i+2}, v'_{i+2}, P_{3,4}, v'_{i+3}, v_{i+3}, v_{i+4}, v'_{i+4}, P_{3,5}, v'_{i+5}, v_{i+5}, v_{i+6}, P_{3,6}, v'_{i+1})$ where $P_{3,4}$ is the path from v'_{i+2} to v'_{i+3} and $P_{3,5}$ is the path from v'_{i+4} to v'_{i+5} and $P_{3,6}$ is the path from v_{i+6} to v'_{i+1}.

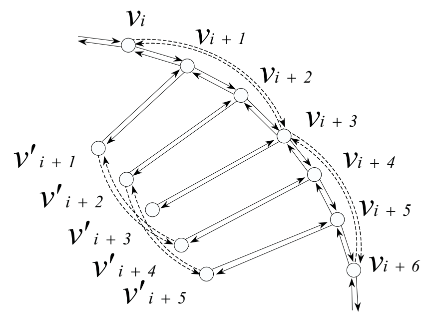

Fig. 4. Illustration of the proof of Lemma 2 (Case 3)

Let G' be the graph such that $G' = (G - \{v_{i+1}, v_{i+2}, v_{i+4}, v_{i+5}\}) \cup \{(v_i, v_{i+3}), (v_{i+3}, v_{i+6}), (v'_{i+1}, v'_{i+4}), (v'_{i+2}, v'_{i+5})\}$(see Figure 4). Then we can see that G' is a simple 3-regular graph by proving that (i) both (v_i, v_{i+3}) and (v_{i+3}, v_{i+6}) are not chord edges, that (ii) (v'_{i+1}, v'_{i+4}) is not a boundary edge, and that (iii) (v'_{i+2}, v'_{i+5}) is not a boundary edge.

Since there are no chord edges of size 3 in Case 3, (i) holds. If (v'_{i+1}, v'_{i+4}) is a boundary edge, then for some k, either $v'_{i+1} = v_k$ and $v'_{i+4} = v_{k+1}$ or $v'_{i+1} = v_{k+1}$ and $v'_{i+4} = v_k$. In the first case, C_0 is $(v_i, v_{i+1}, v'_{i+1} = v_k, v'_{i+4} = v_{k+1}, v_{i+4}, v_{i+5}, \cdots, v_{i+2}, v_{i+3}, \cdots, v_i)$, which is Case 2. In the second case, C_1 is $(v_{i+3}, v_{i+4}, v'_{i+4} = v_k, v'_{i+1} = v_{k+1}, v_{i+1}, v_{i+2}, \cdots, v_{i+5}, v_{i+6}, \cdots, v_{i+3}) = (v_{i+1}, v_{i+2}, \cdots, v_{i+5}, v_{i+6}, \cdots, v_{i+3}, v_{i+4}, v'_{i+4}, v'_{i+1}, v_{i+1})$, which is also Case 2.

If (v'_{i+2}, v'_{i+5}) is a boundary edge, then for some k, either $v'_{i+2} = v_k$ and $v'_{i+5} = v_{k+1}$ or $v'_{i+2} = v_{k+1}$ and $v'_{i+5} = v_k$. In the first case, C_1 is $(v_{i+1}, v_{i+2}, v'_{i+2} = v_k, v'_{i+5} = v_{k+1}, v_{i+5}, v_{i+6}, \cdots, v_{i+3}, v_{i+4}, \cdots, v_{i+1})$, which is Case 2. In the second case, C_0 is $(v_{i+4}, v_{i+5}, v'_{i+5} = v_k, v'_{i+2} = v_{k+1}, v_{i+2}, v_{i+3}, \cdots, v_i, v_{i+1}, \cdots, v_{i+4}) = (v_i, v_{i+1}, \cdots, v_{i+4}, v_{i+5}, v'_{i+5}, v'_{i+2}, v_{i+2}, v_{i+3}, \cdots, v_i)$, which is also Case 2.

The dihamiltonian decomposition C', C'_0 and C'_1 of G' is similar to the previous cases. Let C' be $(v_{n-1}, v_{n-2}, \cdots, v_{i+6}, v_{i+3}, v_i, v_{i-1}, \cdots, v_0, v_{n-1})$, and C'_0 be $(v_i, v_{i+3}, v'_{i+3}, P_{3,2}, v'_{i+4}, v'_{i+1}, P_{3,1}, v'_{i+2}, v'_{i+5}, P_{3,3}, v_i)$, and C'_1 be $(v'_{i+1}, v'_{i+4}, P_{3,5}, v'_{i+5}, v'_{i+2}, P_{3,4}, v'_{i+3}, v_{i+3}, v_{i+6}, P_{3,6}, v'_{i+1})$. Then C', C'_0 and C'_1 are edge-disjoint hamiltonian cycles in the symmetric digraph of G'.

It remains to consider the case that some vertices in $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}, v_{i+6}, v'_{i+1}, v'_{i+2}, v'_{i+3}, v'_{i+4}, v'_{i+5}\}$ are not distinct. In case of $v'_{i+1} = v_{i+6}$, $P_{3,6}$ is a path with length 0. In case of $v'_{i+5} = v_i$, $P_{3,3}$ is a path with length 0. In either case, C', C'_0 and C'_1 are edge-disjoint hamiltonian cycles in G'. □

Now we are ready to present our main theorem.

Theorem 2 *Every* 3*-regular graph* G *with* $n = 4k$ *vertices is not dihamiltonian decomposable.*

Proof We use induction on the number of vertices, n. When $n = 4$, G is a complete graph. Since G is not a bipartite graph, it is not dihamiltonian decomposable by Theorem 1. Assume that the theorem is true for $n = 4k \geq 4$. We will show that the theorem is true for $n = 4(k+1)$. Suppose for a contradiction that there exists a 3-regular graph with $n = 4(k+1)$ vertices which is dihamiltonian decomposable. Then from Lemma 2, there is a 3-regular digraph with $n = 4k$ vertices which is dihamiltonian decomposable. This leads to a contradiction. □

4 Applications to Interconnection Network Topologies

In this section, we consider the dihamiltonian decomposition problem of interconnection network topologies whose vertex degree is 3 such as CCC(cube-connected cycles)[8], chordal rings[1] and $C_m \oplus C_m$.

4.1 CCC

The m-dimensional cube-connected cycles (CCC) is constructed from the m-dimensional hypercube by replacing each node of the hypercube with a cycle of m nodes in the CCC. The ith dimension edge incident to a node of the hypercube is then connected to the ith node of the corresponding cycle of the CCC. The m-dimensional CCC is a 3-regular graph.

Theorem 3 *Every* m*-dimensional CCC is not dihamiltonian decomposable,* $m \geq 3$.

Proof The number of vertices in m-dimensional CCC is $m2^m$, which is a multiple of 4. Therefore it is not dihamiltonian decomposable by Theorem 2. □

4.2 Chordal ring

For an even n and odd r, chordal ring $CR(n,r)$ is a graph G such that its vertex set is $\{v_0, v_1, \cdots, v_{n-2}, v_{n-1}\}$ and the edge set is $\{(v_i, v_j) \mid j =_{\text{mod } n} (i+1), 0 \leq i \leq n-1\} \cup \{(v_i, v_j) \mid j =_{\text{mod } n} (i+r)$, for even i, $0 \leq i \leq n-1\}$. The chordal ring is a 3-regular graph.

Theorem 4 *(a) Every chordal ring $CR(n,r)$ with $n = 4k$ is not dihamiltonian decomposable. (b) Chordal ring $CR(n,r)$ with $n = 4k+2$ is dihamiltonian decomposable if $n/2$ is relatively prime to both $\lfloor r/2 \rfloor$ and $\lceil r/2 \rceil$.*

Proof The proof of (a) follows from Theorem 2. Let G be a chordal ring $CR(n,r)$ with $n = 4k+2$. Let C be a hamiltonian cycle $(v_{n-1}, v_{n-2}, \cdots, v_1, v_0, v_{n-1})$ in the symmetric digraph of G. Since G_0^C and G_1^C are cycles with length $n/2$ if and only if $n/2$ is relatively prime to $\lfloor r/2 \rfloor$ and $\lceil r/2 \rceil$ respectively, G is dihamiltonian decomposable by Property 2. □

$CR(4k+2,3)$, $CR(4k+2,2k+1)$, and $CR(4k+2,4k-1)$ are dihamiltonian decomposable by Theorem 4. The condition of Theorem 4 (b) is not a necessary condition. There are some dihamiltonian decomposable chordal rings $CR(4k+2,r)$ without a dihamiltonian decomposition such that one of three hamiltonian cycles in the decomposition is $(v_{n-1}, v_{n-2}, \cdots, v_1, v_0, v_{n-1})$. $CR(30,5)$ is such a chordal ring, which has the minimum number of vertices.

4.3 $C_m \oplus C_m$

$C_m \oplus C_m$ is a class of 3-regular graphs such that some edges between the vertices of two cycles with length m are added so that the degree of every vertex is 3. Petersen graph belongs to $C_m \oplus C_m$ where m is 5. The product of C_m and complete graph K_2, $C_m \times K_2$, also belongs to $C_m \oplus C_m$.

Theorem 5 *Every graph in $C_m \oplus C_m$ is not dihamiltonian decomposable, $m \geq 3$.*

Proof Let G be a graph in $C_m \oplus C_m$. G has $2m$ vertices. If m is odd, G is not dihamiltonian decomposable since it is not a bipartite graph. If m is even, G is not dihamiltonian decomposable since the number of vertices is a multiple of 4. □

5 Concluding Remarks

The dihamiltonian decomposition of regular graphs can be used to design reliable communication algorithms for broadcasting and multicasting under the wormhole routing model. We suggested some conditions for 3-regular graphs to be dihamiltonian decomposable. For a 3-regular graph G, it is bipartite if it is dihamiltonian decomposable. If the number of vertices is $4k$, then G is not dihamiltonian decomposable. Using these results, we showed that chordal rings with $4k$ vertices, m-dimensional CCCs, and $C_m \oplus C_m$ are not dihamiltonian decomposable. We also proposed a class of chordal rings with $4k+2$ vertices which are dihamiltonian decomposable. The characterization for 3-regular graphs to have a dihamiltonian decomposition should be further studied.

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