Fault-Hamiltonicity of Hypercube-Like Interconnection Networks*

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Abstract

*We call a graph G to be f-*fault hamiltonian *(resp. f-*fault hamiltonian-connected*) if there exists a hamiltonian cycle (resp. if each pair of vertices are joined by a hamiltonian path) in $G \backslash F$ for any set F of faulty elements with $|F| \leq f$. In this paper, we deal with the graph $G_0 \oplus G_1$ obtained from connecting two graphs G_0 and G_1 with n vertices each by n pairwise nonadjacent edges joining vertices in G_0 and vertices in G_1. Provided each G_i is f-fault hamiltonian-connected and $f+1$-fault hamiltonian, $0 \leq i \leq 3$, we show that $G_0 \oplus G_1$ is $f+1$-fault hamiltonian-connected for any $f \geq 2$ and $f+2$-fault hamiltonian for any $f \geq 1$, and that for any $f \geq 0$, $H_0 \oplus H_1$ is $f+2$-fault hamiltonian-connected and $f+3$-fault hamiltonian, where $H_0 = G_0 \oplus G_1$ and $H_1 = G_2 \oplus G_3$. Many interconnection networks such as hypercube-like interconnection networks can be represented in the form of $G_0 \oplus G_1$ connecting two lower dimensional networks G_0 and G_1. Applying our main results to a subclass of hypercube-like interconnection networks, called* restricted HL-graphs*, which include twisted cubes, crossed cubes, multiply twisted cubes, Möbius cubes, Mcubes, and generalized twisted cubes, we show that every restricted HL-graph of degree $m(\geq 3)$ is $m-3$-fault hamiltonian-connected and $m-2$-fault hamiltonian.*

1. Introduction

The embedding of linear arrays and rings into a faulty interconnection network is one of the important problems in parallel processing. An interconnection network is often modelled as a graph, in which vertices and edges correspond to nodes and communication links, respectively. Thus, the embedding problem can be modelled as finding as long fault-free paths and cycles as possible in the graph with some faulty elements (vertices and/or edges). The problem is closely related to well-known hamiltonian problems in graph theory. In the rest of this paper, we will use standard terminology in graphs (see ref. [1]).

Definition 1 *A graph G is called f-*fault hamiltonian *(resp. f-*fault hamiltonian-connected*) if there exists a hamiltonian cycle (resp. if each pair of vertices are joined by a hamiltonian path) in $G \backslash F$ for any set F of faulty elements with $|F| \leq f$.*

For a graph G to be f-fault hamiltonian (resp. f-fault hamiltonian-connected), it is necessary that $f \leq \delta(G) - 2$ (resp. $f \leq \delta(G) - 3$), where $\delta(G)$ is the minimum degree of G.

Fault-hamiltonicity of various interconnection networks was investigated in the literature. Concerned with nonbipartite interconnection networks, most of the works were about proving that each network G is $\delta(G) - 3$-fault hamiltonian-connected and $\delta(G) - 2$-fault hamiltonian whenever $\delta(G) \geq 3$; for example, arrangement graphs[12, 13], recursive circulants[27], crossed cubes[14], twisted cubes[15], Möbius cubes[4], and meshes with wraparound edges[23]. Concerned with bipartite interconnection networks, most of the works were about constructing the longest path/cycle (precisely speaking, L^{opt}-path and L^{opt}-cycle defined in [20, 21]) or the longest path/cycle in a sense of worst case (L-path and L-cycle[20, 21]); for example, star graphs[2, 11, 16, 21, 28], hypercubes[9, 24, 26], product of path and even cycle[20].

We are given two graphs G_0 and G_1 with n vertices each. We denote by V_i and E_i the vertex set and edge set

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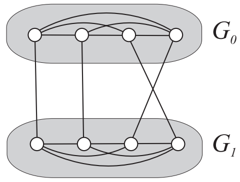
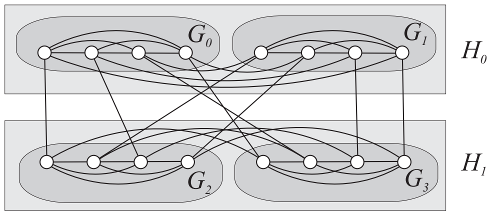


(a) $G_0 \oplus G_1$



(b) $H_0 \oplus H_1$

Figure 1. Examples of $G_0 \oplus G_1$ and $H_0 \oplus H_1$, where each G_i is isomorphic to K_4, the complete graph with four vertices.

of G_i, $i = 0, 1$, respectively. We let $V_0 = \{v_1, v_2, \ldots, v_n\}$ and $V_1 = \{w_1, w_2, \ldots, w_n\}$. With respect to a permutation $M = (i_1, i_2, \ldots, i_n)$ of $\{1, 2, \ldots, n\}$, we can "merge" the two graphs into a graph $G_0 \oplus_M G_1$ with $2n$ vertices in such a way that the vertex set $V = V_0 \cup V_1$ and the edge set $E = E_0 \cup E_1 \cup E_2$, where $E_2 = \{(v_j, w_{i_j}) | 1 \leq j \leq n\}$. We denote by $G_0 \oplus G_1$ a graph obtained by merging G_0 and G_1 w.r.t. an arbitrary permutation M. Here, G_0 and G_1 are called *components* of $G_0 \oplus G_1$. Figure 1 shows examples of $G_0 \oplus G_1$ and $H_0 \oplus H_1$, where $H_0 = G_0 \oplus G_1$ and $H_1 = G_2 \oplus G_3$.

We investigate fault-hamiltonicity of $G_0 \oplus G_1$ and $H_0 \oplus H_1$, where $H_0 = G_0 \oplus G_1$ and $H_1 = G_2 \oplus G_3$, provided that each G_i, $0 \leq i \leq 3$, is f-fault hamiltonian-connected and $f + 1$-fault hamiltonian. As a result, under the condition, we will show that for any $f \geq 2$, $G_0 \oplus G_1$ is $f + 1$-fault hamiltonian-connected and for any $f \geq 1$, it is $f + 2$-fault hamiltonian. Recently, it was shown in [4] that if G_i is δ-regular, $\delta - 3$-fault hamiltonian-connected, and $\delta - 2$-fault hamiltonian, $i = 0, 1$, then for any $\delta \geq 5$, $G_0 \oplus G_1$ is $\delta - 2$-fault hamiltonian-connected and for any $\delta \geq 4$, it is $\delta - 1$-fault hamiltonian. In this paper, we do not have the requirements of G_i's being regular and $f = \delta - 3$. Thus, our result is an extension of the work in [4]. Furthermore, we will show under the condition that for any $f \geq 0$, $H_0 \oplus H_1$ is $f + 2$-fault hamiltonian-connected and $f + 3$-fault hamiltonian.

We apply our main results to recursive circulant $G(2^m, 4)$ and a subclass of hypercube-like interconnection networks, called *restricted HL-graphs*. The subclass includes twisted cubes[10], crossed cubes[7], multiply twisted cubes[6], Mӧbius cubes[5], Mcubes[25], and generalized twisted cubes[3]. We will show that all these networks of degree $m(\geq 3)$ are $m - 3$-fault hamiltonian-connected and $m - 2$-fault hamiltonian. The results obtained in this paper are extensions of some works on fault-hamiltonicity of hypercube-like interconnection networks such as recursive circulants[27], crossed cubes[14], twisted cubes[15], and Mӧbius cubes[4]. Also, we will discuss that "near" bipartite graphs which belong to hypercube-like interconnection networks can have only limited fault-hamiltonicity by the illustration of twisted m-cubes.

2. Fault-Hamiltonicity of $G_0 \oplus G_1$ and $H_0 \oplus H_1$

In this section, we are to discuss about fault-hamiltonicity of $G_0 \oplus G_1$ and $H_0 \oplus H_1$, where $H_0 = G_0 \oplus G_1$ and $H_1 = G_2 \oplus G_3$. The degree of $G_0 \oplus G_1$ is increased by one from the minimum degree over all G_i's, and the degree of $H_0 \oplus H_1$ is increased by two. Provided each G_i is f-fault hamiltonian-connected and $f + 1$-fault hamiltonian, $i = 0, 1$, we have interests in whether $G_0 \oplus G_1$ is $f + 1$-fault hamiltonian-connected and $f + 2$-fault hamiltonian and whether $H_0 \oplus H_1$ is $f + 2$-fault hamiltonian-connected and $f + 3$-fault hamiltonian. That is, the problems we are concerned with are whether or not the bound on the number of faulty elements in $G_0 \oplus G_1$ (resp. in $H_0 \oplus H_1$) is increased by one (resp. by two) to preserve its fault-hamiltonicity.

For a vertex v in $G_0 \oplus G_1$, we denote by $\bar{v}$ the vertex adjacent to v which is in a component different from the component in which v is contained. We denote by F the set of faulty elements. When we are to construct a hamiltonian path from s to t, s and t are called a *source* and a *sink*, respectively, and both of them are called *terminals*. Throughout this paper, a path in a graph is represented as a sequence of vertices.

Definition 2 *A vertex v in $G_0 \oplus G_1$ is called* free *if v is fault-free and not a terminal, that is, $v \notin F$ and v is neither a source nor a sink. An edge (v, w) is called* free *if v and w are free and $(v, w) \notin F$.*

Definition 3 *A* free bridge *of a fault-free vertex v is the path $(v, \bar{v})$ of length one if $\bar{v}$ is free and $(v, \bar{v}) \notin F$; otherwise, it is a path $(v, w, \bar{w})$ of length two such that $w \neq \bar{v}$, $(v, w) \notin F$, and $(w, \bar{w})$ is a free edge.*

We denote by $H[v, w|G, F]$ a hamiltonian path in $G \backslash F$ joining a pair of fault-free vertices v and w in a graph G with a set F of faulty elements. When we find a hamiltonian path/cycle, sometimes we regard some fault-free vertices and/or edges as faulty elements. They are called *virtual* faults.

2.1. Fault-hamiltonicity of $G_0 \oplus G_1$

We denote by V_i and E_i the sets of vertices and edges in G_i, $i = 0, 1$, and by E_2 the set of edges joining vertices in G_0 and vertices in G_1. We let $n = |V_0| = |V_1|$. F_0 and F_1 denote the sets of faulty elements in G_0 and G_1, respectively, and F_2 denotes the set of faulty edges in E_2, so that $F = F_0 \cup F_1 \cup F_2$. Let $f_0 = |F_0|$, $f_1 = |F_1|$, and $f_2 = |F_2|$.

If G_i is f-fault hamiltonian-connected and $f+1$-fault hamiltonian, $i = 0, 1$, then

$$f \leq \delta(G_i) - 3, \text{ and thus } f + 4 \leq n.$$

Theorem 1 *Let a graph G_i be f-fault hamiltonian-connected and $f+1$-fault hamiltonian, $i = 0, 1$. Then,*
(a) for any $f \geq 2$, $G_0 \oplus G_1$ is $f+1$-fault hamiltonian-connected,
(b) for $f = 1$, $G_0 \oplus G_1$ with $2(= f+1)$ faulty elements has a hamiltonian path joining s and t unless s and t are contained in the same component and $\bar{s}$ and $\bar{t}$ are the faulty elements(vertices), and
(c) for $f = 0$, $G_0 \oplus G_1$ with $1(= f+1)$ faulty elements has a hamiltonian path joining s and t unless s and t are contained in the same component and the faulty element is contained in the other component.

Proof To prove (a), assuming $|F| \leq f+1$, we will construct a hamiltonian path P in $G_0 \oplus G_1 \backslash F$ joining any pair of vertices s and t.

Case 1: $f_0, f_1 \leq f$.
When s is in G_0 and t is in G_1, we first find a free edge $(x, \bar{x})$ in E_2. The existence of a free edge $(x, \bar{x})$ is due to the fact that there are $n(\geq f+4)$ candidates for free edges and at most $f+1$ faulty elements and two terminals s and t can "block" the candidates. Assuming $x \in V_0$, two hamiltonian paths $H[s, x|G_0, F_0]$ and $H[\bar{x}, t|G_1, F_1]$ are merged with the free edge $(x, \bar{x})$ into a desired hamiltonian path P. That is, $P = (H[s, x|G_0, F_0], H[\bar{x}, t|G_1, F_1])$.

Hereafter in this case, we assume that both s and t are in G_0. When $f_0 = f$, we first find $P_0 = H[s, t|G_0, F_0]$. Let x and y be the vertices next to s and t on P_0, respectively. Then, s, x, y, and t are distinct since P_0 has at least $n - f(\geq 4)$ vertices. Either $(s, \bar{s})$ and $(x, \bar{x})$ or $(t, \bar{t})$ and $(y, \bar{y})$ are a pair of fault-free edges whose endvertices are also fault-free since $f_1 + f_2 \leq 1$. Assuming $(s, \bar{s})$ and $(x, \bar{x})$ are such a pair, we have $P = (s, H[\bar{s}, \bar{x}|G_1, F_1], P_0 \backslash s)$.

When $f_0 = f - 1$, we first consider the subcase that $(s, \bar{s})$ is the free bridge of s. We pick a free edge $(x, \bar{x})$ in E_2, and assume $x \in V_0$. Then, we have $P = (s, H[\bar{s}, \bar{x}|G_1, F_1], H[x, t|G_0, F_0 \cup \{s\}])$. Here, in the construction of a hamiltonian path in G_0 between x and t, s plays a role of a virtual fault. Symmetrically, in the subcase that $(t, \bar{t})$ is the free bridge of t, we can construct P. Now, we assume that both free bridges of s and t are of length two. Observe that for any free vertex v in G_0, $(v, \bar{v})$ is a free edge since $f_1 + f_2 \leq 2$. We find a hamiltonian path $P_0 = H[s, t|G_0, F_0]$, and assume (x, y) is an edge on the path such that $x, y \neq s, t$. Letting $P_0 = (s, Q_1, x, y, Q_2, t)$, we have $P = (s, Q_1, x, H[\bar{x}, \bar{y}|G_1, F_1], y, Q_2, t)$.

When $f_0 \leq f - 2$, we first find a free edge $(x, \bar{x})$ in E_2 and then find a free bridge $B_s = (s, \ldots, s')$ of s disjoint from the free edge. The existence of such a B_s is due to that there are $\delta(G_0) + 1 (\geq f+4)$ candidates and at most $f+1$ faulty elements, one terminal t, and the free edge $(x, \bar{x})$ can block the candidates. Then, assuming $x \in V_0$, $P = (B_s, H[s', \bar{x}|G_1, F_1], H[x, t|G_0, F_0 \cup F'])$, where $F' = V(B_s) \cap V_0$. The hamiltonian path $H[x, t|G_0, F_0 \cup F']$ exists since $|F_0| + |F'| \leq |F_0| + 2 \leq f$.

Case 2: $f_0 = f + 1$ (or symmetrically, $f_1 = f+1$).
We have $f_1 = f_2 = 0$. First, we consider the subcase $s, t \in V_0$. We find a hamiltonian cycle C_0 in $G_0 \backslash F_0$, and then partition C_0 into two path segments, an s-x path Q_1 and an y-t path Q_2 for some x and y, so that $V(Q_1) \cap V(Q_2) = \emptyset$ and $V(Q_1) \cup V(Q_2) = V(C_0)$. Then, $P = (Q_1, H[\bar{x}, \bar{y}|G_1, \emptyset], Q_2)$. Secondly, we consider the subcase $s \in V_0$ and $t \in V_1$. We find a hamiltonian cycle C_0 in $G_0 \backslash F_0$, and let x and y be the two vertices on C_0 next to s, that is, $C_0 = (s, x, \ldots, y)$. We assume without loss of generality that $\bar{x} \neq t$. Then, $P = (C_0 \backslash (s, x), H[\bar{x}, t|G_1, \emptyset])$. Finally, we consider the subcase $s, t \in V_1$. We find a free bridge $B_s = (s, \ldots, s')$ of s, and then find a hamiltonian cycle C_0 in $G_0 \backslash F_0$. Let $C_0 = (s', x, \ldots, y)$, and assume $\bar{x} \neq t$. Then, $P = (B_s, C_0 \backslash (s', x), H[\bar{x}, t|G_1, F'])$, where $F' = V(B_s) \cap V_1$. The hamiltonian path in G_1 joining $\bar{x}$ and t exists since $|F'| \leq 2 \leq f$. This completes the proof of (a).

It immediately follows from Case 1 and the first and second subcases of Case 2, where the assumption $f \geq 2$ is never used, that for $f = 0, 1$, $G_0 \oplus G_1$ with $f+1$ faulty elements has a hamiltonian path joining s and t unless s and t are contained in the same component and all the faulty elements are contained in the other component. Thus, the proof of (c) is done. To prove (b), assuming without loss of generality $(s, \bar{s})$ is the free bridge of s, we employ the construction of the last subcase of Case 2. Note that in the construction, $|F'| = 1 = f$. This completes the proof. □

Remark 1 *It is unknown till now whether or not for $f = 0, 1$, Theorem 1(a) holds true. If one tries to construct a counter example, of course, he/she should concentrate on the case that s and t are contained in the same component and $f+1$ faulty elements are contained in the other component. It will be proved later in Lemma 1 that for any $f \geq 0$, $G_0 \oplus G_1$ is also f-fault hamiltonian-connected.*

Theorem 2 *Let a graph G_i be f-fault hamiltonian-connected and $f+1$-fault hamiltonian, $i = 0, 1$. Then,*

(a) for any $f \geq 1$, $G_0 \oplus G_1$ is $f+2$-fault hamiltonian, and
(b) for $f = 0$, $G_0 \oplus G_1$ with $2(= f+2)$ faulty elements has a hamiltonian cycle unless one faulty element is contained in G_0 and the other faulty element is contained in G_1.

Proof To prove (a), we let $|F| \leq f+2$. Assuming $f_0 \geq f_1$ without loss of generality, we will construct a hamiltonian cycle C in $G_0 \oplus G_1 \backslash F$. When $f_0 \leq f$, we find two free edges $(x, \bar{x})$ and $(y, \bar{y})$ in E_2. Such free edges exist since there are $n(\geq f+4)$ candidates and at most $f+2$ blocking elements. Note that there are no terminals. Then, assuming $x, y \in V_0$, $C = (H[x, y|G_0, F_0], H[\bar{y}, \bar{x}|G_1, F_1])$. When $f_0 = f+1$, we find a hamiltonian cycle C_0 in $G_0 \backslash F_0$, and let (x, y) be an edge on C_0 such that both $(x, \bar{x})$ and $(y, \bar{y})$ are free edges. The existence of such an edge is due to the fact that the length of C_0 is at least three and $f_1 + f_2 \leq 1$. Then, $C = (C_0 \backslash (x, y), H[\bar{y}, \bar{x}|G_1, F_1])$. The hamiltonian path in G_1 between $\bar{y}$ and $\bar{x}$ exists since $|F_1| \leq 1 \leq f$. Finally, when $f_0 = f+2$, we select an arbitrary faulty element α in G_0, regarding it as *a virtual fault-free element*, find a hamiltonian cycle C_0 in $G_0 \backslash F'$, where $F' = F_0 \backslash \alpha$. The existence of C_0 is due to $|F'| = f+1$. If α is a faulty vertex, let x and y be the two vertices on C_0 next to α and let $P_0 = C_0 \backslash \alpha$; else if C_0 passes through the faulty edge α, let x and y be the endvertices of α and let $P_0 = C_0 \backslash (x, y)$; else let (x, y) be an arbitrary edge on C_0 and let $P_0 = C_0 \backslash (x, y)$. Then, $C = (P_0, H[\bar{y}, \bar{x}|G_1, \emptyset])$. The proof of (b) follows immediately from the proof of (a), where the assumption $f \geq 1$ is used only when $f_1 = 1$ in the second case of $f_0 = f+1$. $\square$

Remark 2 *For $f = 0$, Theorem 2(a) does not hold true. We can construct a counter example using Petersen graph, which is a well-known nonhamiltonian graph. We let P be the Petersen graph. P has two disjoint induced cycles of length five, and they are merged with five pairwise nonadjacent edges. That is, P is isomorphic to $P_0 \oplus_M P_1$ for some permutation M, where P_0 and P_1 denote the induced cycles. We let $V(P_0) = \{v_1, v_2, v_3, v_4, v_5\}$ and $V(P_1) = \{w_1, w_2, w_3, w_4, w_5\}$. To the Petersen graph, we add two vertices v_0 and w_0, and eleven edges $\{(v_0, v_i), (w_0, w_i) | 1 \leq i \leq 5\} \cup \{(v_0, w_0)\}$. See Figure 2. Let the resulting graph be G. We claim G is a counter example. We let G_i be the induced subgraph $G\langle V_i \rangle$ by V_i, $i = 0, 1$, where $V_0 = V(P_0) \cup \{v_0\}$ and $V_1 = V(P_1) \cup \{w_0\}$. Then, $G = G_0 \oplus_{M'} G_1$ for some permutation M'. Observe that G_i is 0-fault hamiltonian-connected and 1-fault hamiltonian. However, G is not 2-fault hamiltonian since $G \backslash \{v_0, w_0\}$, which is isomorphic to Petersen graph P, has no hamiltonian cycle.*

Contrary to the previous two theorems, we can obtain

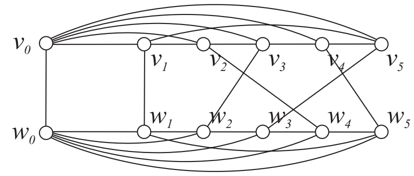


Figure 2. A graph which is not 2-fault hamiltonian.

fault-hamiltonicity of $G_0 \oplus G_1$ which holds true for any $f \geq 0$, if we reduce the bound on the number of faulty elements in $G_0 \oplus G_1$ by one (Lemma 1), or if we put a restriction on the structure of $G_0 \oplus G_1$ such as $G_0 \times K_2$ (Lemma 2).

Lemma 1 *Let a graph G_i be f-fault hamiltonian-connected and $f+1$-fault hamiltonian, $i = 0, 1$. Then,*
(a) for any $f \geq 0$, $G_0 \oplus G_1$ is f-fault hamiltonian-connected, and
(b) for any $f \geq 0$, $G_0 \oplus G_1$ is $f+1$-fault hamiltonian.

Proof To prove (a), it is sufficient to consider the case $f = 0, 1$ by Theorem 1(a). To obtain a hamiltonian path in $G \backslash F$ joining s and t, we can apply Theorem 1 (b) and (c) after we choose $f+1-|F|$ fault-free edges in E_2 and regard them as virtual faults. Similarly, we can prove (b). We assume $f = 0$ by Theorem 2(a). We can construct a hamiltonian cycle in $G \backslash F$ by Theorem 2(b) after we choose $2-|F|$ virtual faulty edges in E_2. $\square$

Lemma 2 *For $\delta \geq 3$, let G be a δ-regular graph such that G is $\delta-3$-fault hamiltonian-connected and $\delta-2$-fault hamiltonian. Then,*
(a) $G \times K_2$ is $\delta-2$-fault hamiltonian-connected, and
(b) $G \times K_2$ is $\delta-1$-fault hamiltonian.

Proof In $G \times K_2$, let G_0 be one copy of G and let G_1 be the other copy of G. To prove (a), we assume that $|F| = \delta-2$ by Lemma 1, and assume that $\delta = 3$ or 4, s and t are contained in G_1, and all the faulty elements are contained in G_0 by Theorem 1. We are to construct a hamiltonian path P joining s and t. When $\delta = 3$, we find a hamiltonian cycle C_0 in $G_0 \backslash F$ and a hamiltonian path $P_1 = H[s, t|G_1, \emptyset]$, and then we find a free edge $(x, \bar{x})$ in E_2, $x \in V_0$. There exists an edge (x, y) on C_0 such that $(\bar{x}, \bar{y})$ is an edge on P_1 since C_0 passes through two of the three edges incident to x and P_1 passes through two of the three edges incident to $\bar{x}$. Note that G_i is 3-regular. Then, letting $P_1 = (s, Q_1, \bar{x}, \bar{y}, Q_2, t)$, we have $P = (s, Q_1, \bar{x}, C_0 \backslash (x, y), \bar{y}, Q_2, t)$.

When $\delta = 4$, we further assume that $\bar{s}$ and $\bar{t}$ are faulty (and no other faulty elements exist) by Theorem 1(b). The construction for $\delta = 4$ is similar to the case $\delta = 3$. We first find a hamiltonian cycle C_0 in $G_0 \backslash F$ and choose a free edge

$(x, \bar{x})$, $x \in V_0$, such that x is adjacent to $\bar{s}$. And then, letting a virtual fault set $F' = \{(s, \bar{x})\}$, we find a hamiltonian path $P_1 = H[s, t|G_1, F']$. There exists an edge (x, y) on C_0 such that $(\bar{x}, \bar{y})$ is an edge on P_1 since C_0 passes through two of the three edges incident to x (excluding $(x, \bar{s})$) and P_1 passes through two of the three edges incident to $\bar{x}$ (excluding $(\bar{x}, s)$). Then, letting $P_1 = (s, Q_1, \bar{x}, \bar{y}, Q_2, t)$, we have $P = (s, Q_1, \bar{x}, C_0 \backslash (x, y), \bar{y}, Q_2, t)$.

To prove (b), we assume by Lemma 1 and Theorem 2 that $\delta = 3$ and there are two faulty elements, one per each G_i. Let C_i be a hamiltonian cycle in $G_i \backslash F_i$, $i = 0, 1$. Similar to the proof of (a) for $\delta = 3$, there exists an edge (x, y) on C_0 such that $(\bar{x}, \bar{y})$ is an edge on C_1. Then, we have a hamiltonian cycle $C = (C_0 \backslash (x, y), C_1 \backslash (\bar{x}, \bar{y}))$. □

2.2. Fault-hamiltonicity of $H_0 \oplus H_1$

In $H_0 \oplus H_1$, where $H_0 = G_0 \oplus G_1$ and $H_1 = G_2 \oplus G_3$, H_0 and H_1 are called components and all G_i, $0 \le i \le 3$, are called *subcomponents*. In this subsection, we are to derive fault-hamiltonicity of $H_0 \oplus H_1$ which holds true for any $f \ge 0$, provided we are given fault-hamiltonicity of subcomponents (we are not given fault-hamiltonicity of components).

Throughout this paper, when we are concerned with $H_0 \oplus H_1$, we denote by V_0 and V_1 the sets of vertices in H_0 and in H_1, respectively, and E_2 the set of edges joining vertices in H_0 and vertices in H_1. $V(G_i)$ denotes the set of vertices in G_i, and $E_{i,j}$ denotes the set of edges joining vertices in G_i and vertices in G_j, $i \ne j$. We denote by F_0 and F_1 the sets of faulty elements in H_0 and in H_1, respectively, F_2 the set of faulty edges in E_2, and let $f_i = |F_i|$, $i = 0, 1, 2$. We let $l_{i,j} = |E_{i,j}|$ and $n = |V(G_i)|$. Observe that $l_{0,1} = l_{2,3} = n$, $l_{0,2} + l_{0,3} = l_{1,2} + l_{1,3} = n$, $l_{0,2} = l_{1,3}$, and $l_{0,3} = l_{1,2}$.

For a vertex v in $H_0 \oplus H_1$, we denote by $\bar{v}$ the vertex adjacent to v which is in a component different from the component in which v is contained, and denote by $\hat{v}$ the vertex which is adjacent to v and contained in the same component with v and in a different subcomponent from v. We denote by P^R the reverse of a path P, that is, $P^R = (v_l, v_{l-1}, \ldots, v_1)$ for $P = (v_1, v_2, \ldots, v_l)$.

Lemma 3 *Let a graph G_i be f-fault hamiltonian-connected and $f+1$-fault hamiltonian, $i = 0, 1$. Then, for any faulty set F with $|F| \le f + 2$, $G_0 \oplus G_1 \backslash F$ has two hamiltonian paths Q_x and Q_y from an arbitrary vertex v to some vertices x and y, respectively. Furthermore, when $f = 0$ and each G_i has one faulty element, both x and y are contained in a component different from the component in which v is contained.*

Proof If there exists a hamiltonian cycle C in $G_0 \oplus G_1 \backslash F$, then letting $C = (v, x, \ldots, y)$, we have $Q_x = C \backslash (v, x)$ and $Q_y = C \backslash (v, y)$. Suppose otherwise, by Theorem 2 (a) and (b), we can see that $f = 0$ and one faulty element is in G_0 and the other faulty element is in G_1. We assume w.l.o.g. v is in G_0. We find a hamiltonian cycle $C_0 = (v, a, \ldots, b)$ in $G_0 \backslash F_0$, and then assuming w.l.o.g. $\hat{a}$ is fault-free, we find a hamiltonian cycle $C_1 = (\hat{a}, x, \ldots, y)$ in $G_1 \backslash F_1$. Then, we have $Q_x = (C_0 \backslash (v, a), C_1 \backslash (\hat{a}, x))$ and $Q_y = (C_0 \backslash (v, a), C_1 \backslash (\hat{a}, y))$. This completes the proof. □

Theorem 3 *Let a graph G_i be f-fault hamiltonian-connected and $f+1$-fault hamiltonian, $i = 0, 1, 2, 3$, and let $H_0 = G_0 \oplus G_1$ and $H_1 = G_2 \oplus G_3$. Then,*
(a) for any $f \ge 0$, $H_0 \oplus H_1$ is $f+2$-fault hamiltonian-connected, and
(b) for any $f \ge 0$, $H_0 \oplus H_1$ is $f+3$-fault hamiltonian.

Proof Since the theorem holds true for any $f \ge 2$ by Theorem 1 and 2, we assume that $f = 0, 1$. By Lemma 1, H_0 and H_1 are f-fault hamiltonian-connected and $f+1$-fault hamiltonian. To prove (a), assuming $|F| \le f + 2$, we will construct a hamiltonian path P joining an arbitrary pair of fault-free vertices s and t. Remember $f + 4 \le n$.

Case 1: $s \in V_0, t \in V_1$.
We assume without loss of generality that $f_0 \ge f_1$ and that s is in G_0 and t is in G_2.

Subcase 1.1: $f_0 \le f + 1$.
If there exists a free edge $(x, \bar{x}) \in E_{1,3}$, then assuming $x \in V(G_1)$, $P = (H[s, x|H_0, F_0], H[\bar{x}, t|H_1, F_1])$. Existence of the hamiltonian paths is due to Lemma 1 and Theorem 1 (b) and (c). Otherwise, there exists a free edge $(y, \bar{y}) \in E_{1,2}$. If $f_1 \le f$ or if $f_1 = f + 1$ and not all the faulty elements in H_1 are contained in G_3, then $H[\bar{y}, t|H_1, F_1]$ exists, and thus we have $P = (H[s, y|H_0, F_0], H[\bar{y}, t|H_1, F_1])$. We assume that $f_1 = f+1$ and all the faulty elements in H_1 are contained in G_3. It follows that $f = 0$, $f_0 = f_1 = 1$, and $f_2 = 0$ since $f+2 \ge f_0 + f_1 + f_2 \ge (f+1) + (f+1) + f_2$. If the faulty element in H_0 is not contained in G_1, for some free edge $(y', \bar{y'}) \in E_{0,3}$, $y' \in V(G_0)$, we have $P = (H[s, y'|H_0, F_0], H[\bar{y'}, t|H_1, F_1])$. Now, there are two faulty elements, one in G_1 and the other in G_3. Remember there are no free edges in $E_{1,3}$. Thus $l_{1,3} \le 2$, and $l_{1,3} = 2$ if and only if the two faulty elements, say α and β, are vertices such that $(\alpha, \bar{\alpha}), (\beta, \bar{\beta}) \in E_{1,3}$ and $(\alpha, \bar{\alpha}) \ne (\beta, \bar{\beta})$.

We first find a hamiltonian cycle C_1 in $G_1 \backslash F_0$. There exists an edge (a, b) on C_1 such that $(a, \hat{a})$, $(b, \hat{b})$ are free edges, that is, $\hat{a}, \hat{b} \ne s$. A hamiltonian path $H[\hat{a}, \hat{b}|G_0, \emptyset]$ and $C_1 \backslash (a, b)$ are merged with $(a, \hat{a})$ and $(b, \hat{b})$ into a cycle C. Let the cycle $C = (s, x, \ldots, y)$. The vertices x and y are contained in G_0. If $(x, \bar{x})$ is a free edge such that $\bar{x} \in V(G_3)$, we have $P = (C \backslash (s, x), H[\bar{x}, t|H_1, F_1])$. Similarly, we can also construct P if $(y, \bar{y})$ is a free edge in $E_{0,3}$. Suppose otherwise, it follows that $l_{1,3} \ge 1$ and one of the two conditions holds true:

A1: both $(x, \bar{x})$ and $(y, \bar{y})$ are edges in $E_{0,2}$, and thus $l_{1,3} = 2(= l_{0,2})$ and $E_{0,2} = \{(x, \bar{x}), (y, \bar{y})\}$;

A2: one of $\bar{x}$ and $\bar{y}$, say $\bar{y}$, is the faulty element in G_3, and thus $l_{1,3} = 1$ and $E_{0,2} = \{(x, \bar{x})\}$.

Symmetrically to the above construction of C, we can construct a hamiltonian cycle $C' = (t, x', \ldots y')$ in $H_1 \backslash F_1$ such that $x', y' \in V(G_2)$. If at least one of $(x', \bar{x'})$ or $(y', \bar{y'})$ are free edges in $E_{1,2}$, we can construct P similarly to the above construction and we are done. Suppose otherwise, either

B1: both $(x', \bar{x'})$ and $(y', \bar{y'})$ are edges in $E_{0,2}$, and thus $E_{0,2} = \{(x', \bar{x'}), (y', \bar{y'})\}$ ($l_{1,3} = 2$), or

B2: one of $\bar{x'}$ and $\bar{y'}$, say $\bar{y'}$, is the faulty element in G_1, and thus $E_{0,2} = \{(x', \bar{x'})\}$ ($l_{1,3} = 1$).

Observe that both A1 and B1 hold true, or both A2 and B2 hold true. In either case, we can assume without loss of generality that (x, x') is a free edge in $E_{0,2}$. Then, we can obtain P by merging C and C' with (x, x'), that is, $P = (C \backslash (s, x), C' \backslash (t, x'))$.

Subcase 1.2: $f_0 = f + 2$ ($f_1 = f_2 = 0$).
By Lemma 3, there exist two hamiltonian paths Q_x and Q_y in $H_0 \backslash F_0$ from s to some vertices x and y, respectively. Then, assuming w.l.o.g. $\bar{x} \neq t$, we have $P = (Q_x, H[\bar{x}, t | H_1, \emptyset])$.

Case 2: $s, t \in V_0$ (or symmetrically, $s, t \in V_1$).

Subcase 2.1: $f_0, f_1 \leq f + 1$.
When there is a hamiltonian path $P_0 = H[s, t | H_0, F_0]$, we will show a claim that there exist an edge (x, y) on P_0 such that

(i) $\bar{x}$ and $\bar{y}$ are fault-free,

(ii) $(x, \bar{x})$ and $(y, \bar{y})$ are also fault-free, and

(iii) a hamiltonian path $P_1 = H[\bar{x}, \bar{y} | H_1, F_1]$ exists.

Once the claim is proved, letting $P_0 = (s, Q_1, x, y, Q_2, t)$, we can construct a hamiltonian path $P = (s, Q_1, x, P_1, y, Q_2, t)$. Let us prove the claim. First, we are to show there exists an edge (x, y) on P_0 satisfying (i) and (ii). There are $|V(P_0)| - 1$ candidates for such an edge, and there are $f_1 + f_2$ faulty elements, each can block at most two candidates. We are sufficient to show $|V(P_0)| - 1 > 2(f_1 + f_2)$. Since $|V(P_0)| \geq 2n - f_0$, $n \geq f + 4$, and $f_0 + f_1 + f_2 \leq f + 2$, we have $|V(P_0)| - 1 - 2(f_1 + f_2) \geq 2n - f_0 - 2(f_1 + f_2) - 1 \geq 2n - 2(f_0 + f_1 + f_2) - 1 \geq 2(f + 4) - 2(f + 2) - 1 = 3 > 0$. If $f_1 \leq f$ or if $f_1 = f + 1$ and not all the faulty elements are contained in one subcomponent, then P_1 exists and thus we are done. Suppose that $f_1 = f + 1$ and all the faulty elements are contained in one subcomponent, say G_3. When $f = 1$ ($f_1 = 2$), there are at least three edges (x, y) on P_0 satisfying (i), (ii), and thus by Theorem 1(b), we can conclude that at least two of them also satisfy (iii). When $f = 0$ ($f_1 = 1$), we will show that there exists an edge (x, y) on P_0 satisfying (i), (ii), and (iii′) $\bar{x}$, $\bar{y}$, and the faulty element in G_3 are located so that we can see the existence of P_1 by Theorem 1(c). There are at least $2n - f_0 - 1$ candidate edges, there are $f_1 + f_2$ faulty elements and each of them can block at most two candidates, and there are at most $n - 1$ candidate edges (a, b) such that both $\bar{a}$ and $\bar{b}$ are contained in G_2. Unless $n = 4$, there exists such an edge (x, y) since $(2n - f_0 - 1) - 2(f_1 + f_2) - (n - 1) \geq n - 2(f_0 + f_1 + f_2) \geq n - 4$. If $n = 4$, G_i is isomorphic to a complete graph K_4 and H_1 is isomorphic to $K_4 \times K_2$. By Lemma 2, H_1 is 1-fault hamiltonian-connected, and thus every edge (x, y) on P_0 satisfying (i) and (ii) also satisfies (iii). This completes the proof of our claim.

Let us consider when there does not exist a hamiltonian path $H[s, t | H_0, F_0]$, that is, when $f_0 = f + 1$ ($f_1 + f_2 \leq 1$), s and t are contained in one subcomponent, say G_0, and all the faulty elements in H_0 are contained in G_1. Assuming $(s, \bar{s})$ is the free bridge of s, we find two hamiltonian paths Q_x and Q_y in $H_0 \backslash F_0 \cup \{s\}$ from t to some vertices x and y, respectively, by Lemma 3. We assume w.l.o.g. $(x, \bar{x})$ is a free edge. If $f = 1$ or if $f = 0$ and either $f_1 = 0$ or there is one faulty edge in $E_{2,3}$, then there exists a hamiltonian path $P_1 = H[\bar{s}, \bar{x} | H_1, F_1]$, and thus we have $P = (s, P_1, Q_x^R)$ and we are done. Hereafter, we assume w.l.o.g. that $f = 0$ and there is one faulty element in G_3. If $\bar{s}$ is in G_3, the above construction also gives P since P_1 exists. Or if $\bar{t}$ is fault-free and $\bar{t}$ is in G_3, we can construct P symmetrically. Thus, we also assume that $\bar{s}$ is contained in G_2 and that $\bar{t}$ is faulty or contained in G_2.

We first consider when the faulty element α in G_3 is an edge or is a vertex such that $\bar{\alpha} = t$ or $\bar{\alpha} \in V(G_1)$. If there exists a free edge $(x, \bar{x})$ in $E_{1,3}$ with $x \in V(G_1)$, then we find a hamiltonian cycle $C_1 = (x, a, \ldots, b)$ in $G_1 \backslash F_1$, and then assuming $\hat{a} \neq s$, find a hamiltonian path $P_0 = H[\hat{a}, s | G_0, \emptyset]$. Letting $P_0 = (\hat{a}, Q_1, t, y, Q_2, s)$, we have $P = (s, Q_2^R, y, H[\bar{y}, \bar{x} | H_1, F_1], C_1 \backslash (x, a), \hat{a}, Q_1, t)$. By assumption, $\bar{y}$ is fault-free. If there does not exist such a free edge $(x, \bar{x})$ in $E_{1,3}$, then we can see $l_{1,3} = l_{0,2} \leq 2$. Note that $l_{1,3} = 2$ if and only if both α and the faulty element β in G_1 are vertices such that $(\alpha, \bar{\alpha}), (\beta, \bar{\beta}) \in E_{1,3}$ and $(\alpha, \bar{\alpha}) \neq (\beta, \bar{\beta})$. At least one of $\hat{s}$ and $\hat{t}$ are fault-free. When $\hat{s}$ is fault-free, we find two hamiltonian cycles $C_0 = (t, x, \ldots, y)$ in $G_0 \backslash s$ and $C_1 = (\hat{s}, a, \ldots, b)$ in $G_1 \backslash F_1$. Observe that for at least one of x and y, say x, $(x, \bar{x})$ is a free edge with $\bar{x} \in V(G_3)$ since $l_{0,2} \leq 2$ and $(s, \bar{s})$ is the free bridge of s with $\bar{s} \in V(G_2)$. Assuming w.l.o.g. $(a, \bar{a})$ is a free edge, we have $P = (s, C_1 \backslash (\hat{s}, a), H[\bar{a}, \bar{x} | H_1, F_1], C_0 \backslash (t, x))$. When $\hat{t}$ is fault-free, we can also construct P symmetrically with the roles of s and t interchanged.

Finally, α is a faulty vertex in G_3 such that $\bar{\alpha} \in V(G_0)$

and $\bar{\alpha} \neq s, t$. Furthermore, $(t, \bar{t})$ is the free bridge of t with $\bar{t} \in V(G_2)$. There always exists a free edge $(x, \bar{x})$ in $E_{1,3}$ with $x \in V(G_1)$ since $l_{1,3} \geq 2$. We find a hamiltonian cycle $C_1 = (x, a, \ldots, b)$ in $G_1 \backslash F_1$ and a hamiltonian path $P_0 = H[s, t|G_0, \emptyset]$. Let $P_0 = (v_1, v_2, \ldots, v_n)$, where $s = v_1$ and $t = v_n$. We are to decompose P_0 into two nonempty path segments $Q_1 = (v_1, v_2, \ldots, v_k)$ and $Q_2 = (v_{k+1}, \ldots, v_n)$ such that either $v_k \in \{\hat{a}, \hat{b}\}$ and $v_{k+1} \neq \bar{\alpha}$ or $v_{k+1} \in \{\hat{a}, \hat{b}\}$ and $v_k \neq \bar{\alpha}$. Let $\bar{\alpha} = v_j$ for some $1 < j < n$, and let $\hat{a} = v_{i_a}$ and $\hat{b} = v_{i_b}$. We assume w.l.o.g. $1 \leq i_a < i_b \leq n$. It is straightforward to see that the decomposition is possible for k defined as follows:

$$k = \begin{cases} i_a, & \text{if } i_a + 1 \neq j; \\ i_b - 1, & \text{if } i_b - 1 \neq j; \\ i_a - 1, & \text{if } i_a + 1 = j = i_b - 1 \text{ and } i_a > 1; \\ i_b, & \text{if } i_a + 1 = j = i_b - 1 \text{ and } i_b < n. \end{cases}$$

Note that since $n \geq 4$, if $i_a + 1 = j = i_b - 1$ then $i_a > 1$ or $i_b < n$. Without loss of generality, assuming $v_k = \hat{a}$ and $v_{k+1} \neq \bar{\alpha}$, we have $P = (Q_1, C_1 \backslash (x, a), H[\bar{x}, \bar{v}_{k+1}|H_1, F_1], Q_2)$.

Subcase 2.2: $f_0 = f + 2$ ($f_1 = f_2 = 0$).
When there is a faulty element α such that if we regard α as a fault-free element, there is a hamiltonian path $P_0 = H[s, t|H_0, F_0 \backslash \alpha]$ by Theorem 1 (b) and (c), we can construct P as follows. If α is a faulty vertex, let $P_0 = (s, Q_1, x, \alpha, y, Q_2, t)$; else if α is a faulty edge which P_0 passes through, let $\alpha = (x, y)$ and $P_0 = (s, Q_1, x, y, Q_2, t)$; else for an arbitrary edge (x, y) on P_0, we let $P_0 = (s, Q_1, x, y, Q_2, t)$. Then, $P = (s, Q_1, x, H[\bar{x}, \bar{y}|H_1, \emptyset], y, Q_2, t)$. When there does not exist such a faulty element α, s and t are contained in one subcomponent, say G_0, and all the faulty elements are contained in the other subcomponent G_1. Since G_1 is $f+1$-fault hamiltonian and has $f+2$ faulty elements, there exists a hamiltonian path P_1 in $G_1 \backslash F_0$ from x to y for some pair of vertices x and y. Assuming w.l.o.g. $\hat{y} \neq s$, we find a hamiltonian path $P_0 = H[\hat{y}, s|G_0, \emptyset]$, and let $P_0 = (\hat{y}, Q_1, t, z, Q_2, s)$. Then, we have $P = (s, Q_2^R, z, H[\bar{z}, \bar{x}|H_1, \emptyset], P_1, \hat{y}, Q_1, t)$.

Subcase 2.3: $f_1 = f + 2$ ($f_0 = f_2 = 0$).
First, let us consider when both s and t are contained in G_0. When $(s, \bar{s})$ is the free bridge of s, we find two hamiltonian paths Q_x and Q_y in $H_1 \backslash F_1$ from $\bar{s}$ to some vertices x and y, respectively, by Lemma 3. Then, assuming w.l.o.g. $\bar{x} \neq t$, we have $P = (s, Q_x, H[\bar{x}, t|H_0, F'])$, where $F' = \{s\}$. Symmetrically we can construct P if $(t, \bar{t})$ is the free bridge of t. Now, we assume that both $(s, \bar{s})$ and $(t, \bar{t})$ are not the free bridges of s and t, respectively. It follows that $\bar{s}$ and $\bar{t}$ are faulty vertices. If $f = 0$, they are all the faulty elements; if $f = 1$, there is one more faulty element α in H_1 besides them. We find a hamiltonian path $P_0 = H[s, t|G_0, F']$, where $F' = \{(s, \bar{\alpha})\}$ if $f = 1$, α is a faulty vertex, and $\bar{\alpha} \in V(G_0)$; otherwise, $F' = \emptyset$. Let $P_0 = (s, x, z, Q, t)$. By the construction, $\bar{x}$ is fault-free. We construct two hamiltonian paths Q_y and $Q_{y'}$ in $H_1 \backslash F_1$ from $\bar{x}$ to some vertices y and y', respectively, by Lemma 3. If both $\bar{y}$ and $\bar{y'}$ are contained in G_1, assuming w.l.o.g. $\bar{y} \neq \hat{z}$, we have $P = (s, x, Q_y, H[\bar{y}, \hat{z}|G_1, \emptyset], z, Q, t)$. Otherwise, we assume that $\bar{y}$ is contained in G_0. Note that $\bar{y} \neq s, x, t$. Letting $P_0 = (s, x, z, Q_1, \bar{y}, u, Q_2, t)$, we have $P = (s, x, Q_y, \bar{y}, Q_1^R, z, H[\hat{z}, \hat{u}|G_1, \emptyset], u, Q_2, t)$.

Now, let us consider when s is in G_0 and t is in G_1. When there is a hamiltonian cycle C_1 in $H_1 \backslash F_1$, we will use the edges (x, y) on C_1 such that $\bar{x} \in V(G_0)$ and $\bar{y} \in V(G_1)$. Obviously, there are at least two such edges on C_1. If there exists an edge (x, y) on C_1 such that $\bar{x} \in V(G_0)$, $\bar{y} \in V(G_1)$, $\bar{x} \neq s$, and $\bar{y} \neq t$, then we have $P = (H[s, \bar{x}|G_0, \emptyset], C_1 \backslash (x, y), H[\bar{y}, t|G_1, \emptyset])$. Otherwise, there exist exactly two such edges (x, y) and (x', y'). Assuming $\bar{x} = s$, we let $C_1 = (x, y, \ldots, z)$. We can see that $\bar{z}$ is contained in G_0 since there are at least $|V(C_1)| - n$ ($\geq 2n - (f+2) - n \geq 2$) vertices v on C_1 such that $\bar{v} \in V(G_0)$. Then, $P = (s, C_1 \backslash (x, z), H[\bar{z}, t|H_0, F'])$, where $F' = \{s\}$. The existence of $H[\bar{z}, t|H_0, F']$ is due to Lemma 1 and Theorem 1(c).

When there is no hamiltonian cycle in $H_1 \backslash F_1$, it follows that $f = 0$ and each of G_2 and G_3 has one faulty element. We assume w.l.o.g. that $\bar{t}$ is in G_2. When there exists a free edge $(x, \bar{x}) \in E_{0,2}$ with $x \in V(G_0)$, we find a hamiltonian path $P_0 = H[s, x|G_0, \emptyset]$, and then find two hamiltonian paths Q_y and $Q_{y'}$ in $H_1 \backslash F_1$ from $\bar{x}$ to some vertices y and y', respectively. By the construction, both y and y' are in G_3. If at least one of $\bar{y}$ and $\bar{y'}$, say $\bar{y}$, are contained in G_1, then $P = (P_0, Q_y, H[\bar{y}, t|G_1, \emptyset])$. If both $\bar{y}$ and $\bar{y'}$ are contained in G_0, we let $P_0 = (s, Q_1, \bar{y}, z, Q_2, x)$. Note that $\bar{y}, \bar{y'} \neq x$. We can assume $\hat{z} \neq t$ since otherwise, we can interchange the roles of y and y'. Then, we have $P = (s, Q_1, \bar{y}, Q_y^R, x, Q_2^R, z, H[\hat{z}, t|G_1, \emptyset])$. When there does not exist such a free edge $(x, \bar{x}) \in E_{0,2}$, we can see that $l_{0,2} \leq 2$ and there exists a free edge $(x, \bar{x}) \in E_{0,3}$ with $x \in V(G_0)$. We find two hamiltonian paths Q_y and $Q_{y'}$ in $H_1 \backslash F_1$ from $\bar{x}$ and to some vertices y and y' in G_2, respectively. If one of y and y', say y, has $\bar{y}$ in G_1 with $\bar{y} \neq t$, then $P = (H[s, x|G_0, \emptyset], Q_y, H[\bar{y}, t|G_1, \emptyset])$. Otherwise, we can assume w.l.o.g. $\bar{y} = s$ and $\bar{y'} = t$ since both $\bar{y}$ and $\bar{y'}$ can not be contained in G_0 simultaneously. Then, we have $P = (s, Q_y^R, H[x, t|H_0, F'])$, where $F' = \{s\}$. This completes the proof of (a).

To prove (b), we will construct a hamiltonian cycle C in $H_0 \oplus H_1 \backslash F$. We can assume $|F| = f + 3$ since otherwise, we can construct C by utilizing (a). We assume w.l.o.g. $f_0 \geq f_1$. When $f_0 = f + 3$, we carefully select a faulty element α such that if we regard α as a fault-free element, then there exists a hamiltonian cycle C_0 in $H_0 \backslash F'$, where $F' = F_0 \backslash \alpha$. If $f = 0$ and there is a component with only one faulty element, then we are sufficient to select the faulty element; otherwise, we select an arbitrary faulty element. Using C_0, we construct a hamiltonian path P_0 in $H_0 \backslash F$ joining some vertices x and y. That is, $P_0 = C_0 \backslash \alpha$ if C_0 passes through α; otherwise, $P_0 = C_0 \backslash (x, y)$ for some edge (x, y) on C_0. Then, we have $C = (P_0, H[\bar{y}, \bar{x}|H_1, \emptyset])$.

When $f_0 = f + 2$ ($f_1 + f_2 = 1$), we assume w.l.o.g. that G_2 has no faulty element. We find a free edge $(x, \bar{x})$ with $x \in V(G_3)$, and find two hamiltonian paths Q_y and $Q_{y'}$ in $H_0 \backslash F_0$ from $\bar{x}$ to some vertices y and y', respectively. Assuming $(y, \bar{y})$ is a free edge, we have $C = (Q_y, H[\bar{y}, x|H_1, F_1])$.

When $f_0 \leq f + 1$, we find a pair of free edges $(x, \bar{x})$ and $(y, \bar{y})$ such that either $(x, \bar{x}) \in E_{0,2}$ and $(y, \bar{y}) \in E_{1,3}$ or $(x, \bar{x}) \in E_{0,3}$ and $(y, \bar{y}) \in E_{1,2}$. Such a pair always exist since there are n ($= l_{0,2} + l_{0,3}$) candidate pairs and there are $f + 3$ faulty elements, each of which can block at most one candidate. Remember $n \geq f + 4$. We assume $x, y \in V_0$. There exist hamiltonian paths $P_0 = H[x, y|H_0, F_0]$ and $P_1 = H[\bar{y}, \bar{x}|H_1, F_1]$ by Lemma 1 and Theorem 1 (b) and (c). Thus, we have $C = (P_0, P_1)$. This completes the proof of (b). □

3. Hypercube-Like Interconnection Networks

Vaidya *et al.*[29] introduced a class of hypercube-like interconnection networks, called *HL-graphs*, which can be defined by applying the $\oplus$ operation repeatedly as follows: $HL_0 = \{K_1\}$; for $m \geq 1$, $HL_m = \{G_0 \oplus G_1 | G_0, G_1 \in HL_{m-1}\}$. Then,

- $HL_1 = \{K_2\}$,
- $HL_2 = \{C_4\}$,
- $HL_3 = \{Q_3,\ G(8,4)\}$.

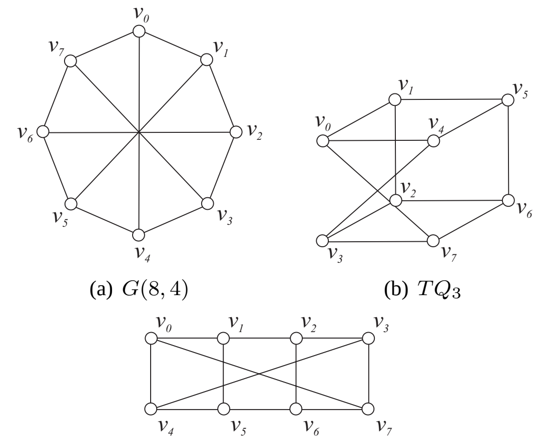
Here, C_4 is a cycle graph with 4 vertices, Q_3 is a 3-dimensional hypercube, and $G(8,4)$ is a recursive circulant which is isomorphic to twisted cube TQ_3 and Mӧbius ladder as shown in Figure 3. An arbitrary graph which belongs to HL_m is called an *m-dimensional HL-graph*. Recently, it was shown by Park and Chwa in [17] that every nonbipartite HL-graph is hamiltonian-connected and every bipartite HL-graph is hamiltonian-laceable.

Obviously, some m-dimensional HL-graphs such as an m-dimensional hypercube are bipartite. They are not f-fault hamiltonian-connected for any $f \geq 0$ and not f-fault hamiltonian for any $f \geq 1$. Thus, we are to define a subclass of HL-graphs which seems "highly" nonbipartite, and then consider their fault-hamiltonicity.

Definition 4 *A subclass of nonbipartite HL-graphs, called* restricted HL-graphs, *is defined recursively as follows:*

- $RHL_m = HL_m$ *for* $0 \leq m \leq 2$*;*
- $RHL_3 = HL_3 \backslash Q_3 = \{G(8,4)\}$*;*
- $RHL_m = \{G_0 \oplus G_1 | G_0, G_1 \in RHL_{m-1}\}$ *for* $m \geq 4$.

*A graph which belongs to RHL_m is called an m-*dimensional restricted HL-graph.



(a) $G(8,4)$ (b) TQ_3

(c) Mӧbius ladder

Figure 3. Isomorphic graphs.

Many of the nonbipartite hypercube-like interconnection networks such as twisted cube[10], crossed cube[7], multiply twisted cube[6], Mӧbius cube[5], Mcube[25], generalized twisted cube[3], etc. proposed in the literature are restricted HL-graphs, with the exception of recursive circulants $G(2^m, 4)$[19] and "near" bipartite interconnection networks such as twisted m-cube[8]. In order to see some hypercube-like interconnection networks are indeed restricted HL-graphs, it is sufficient to review their definitions.

3.1. Restricted HL-graphs

It is straightforward to verify that the 3-dimensional restricted HL-graph $G(8,4)$ is (0-fault) hamiltonian-connected and 1-fault hamiltonian. Furthermore, it holds true that every 4-dimensional restricted HL-graph $G(8,4) \oplus G(8,4)$ is 1-fault hamiltonian-connected and 2-fault hamiltonian. Its proof is omitted due to space limit. An arbitrary m-dimensional restricted HL-graph, $m \geq 5$, is isomorphic to $[G_0 \oplus_{M_1} G_1] \oplus_M [G_2 \oplus_{M_2} G_3]$ for some permutations M_1, M_2, and M, where G_0, G_1, G_2, and G_3 are $m-2$-dimensional HL-graphs. Therefore, by an inductive argument utilizing Theorem 3, we can get a theorem as follows.

Theorem 4 *Every m-dimensional restricted HL-graph, $m \geq 3$, is $m-3$-fault hamiltonian-connected and $m-2$-fault hamiltonian.*

Corollary 1 *(a) Twisted cube TQ_m, $m \geq 3$, is $m-3$-fault hamiltonian-connected and $m-2$-fault hamiltonian[15].*
(b) Crossed cube CQ_m, $m \geq 3$, is $m-3$-fault hamiltonian-connected and $m-2$-fault hamiltonian[14].
(c) Multiply twisted cube MQ_m, $m \geq 3$, is $m-3$-fault hamiltonian-connected and $m-2$-fault hamiltonian.
(d) Both 0-Möbius cube and 1-Möbius cube of dimension m, $m \geq 3$, are $m-3$-fault hamiltonian-connected and $m-2$-fault hamiltonian[4].
(e) The m-Mcube, $m \geq 3$, is $m-3$-fault hamiltonian-connected and $m-2$-fault hamiltonian.
(f) Generalized twisted cube GQ_m, $m \geq 3$, is $m-3$-fault hamiltonian-connected and $m-2$-fault hamiltonian.

Remark 3 Recursive circulant $G(2^m, 4)$ is a HL-graph. However, it is not a restricted HL-graph. Relying on Theorem 3, we can easily see that $G(2^m, 4)$, $m \geq 3$, is $m-3$-fault hamiltonian-connected and $m-2$-fault hamiltonian. Fault-hamiltonicity of recursive circulants was considered in [27] as well.

3.2. Twisted m-cube

Let (v_0, v_1) and (v_2, v_3) be two nonadjacent edges in an arbitrary cycle (v_0, v_1, v_2, v_3) of length four in hypercube Q_m. The twisted m-cube[8] is constructed as follows. Delete edges (v_0, v_1) and (v_2, v_3) from Q_m. Then connect, via an edge, v_0 to v_2 and v_1 to v_3. Obviously, twisted m-cube is an HL-graph. It can be represented in the form of $G_0 \oplus G_1$, where G_0 is isomorphic to hypercube Q_{m-1} and G_1 is isomorphic to twisted $m-1$-cube. Due to [17], we have the following lemma. However, we can not expect that twisted m-cube is good in fault-hamiltonicity since it is a "near" bipartite graph.

Lemma 4 *[17] Twisted m-cube, $m \geq 3$, is hamiltonian-connected.*

Theorem 5 *Twisted m-cube, $m \geq 3$, is not f-fault hamiltonian-connected for any $f \geq 1$.*

Proof If we adopt the coloring induced by proper bicoloring of hypercube Q_m, then there are exactly two edges joining vertices with the same color in twisted m-cube: one joining two black vertices b_1, b_2 and the other joining two white vertices w_1, w_2. Assuming the edge joining b_1 and b_2 is faulty, there can not exist a hamiltonian path between w_1 and w_2. Thus, the proof is completed. □

Remark 4 In a similar way, we can show that twisted m-cube, $m \geq 3$, is 1-fault hamiltonian and not f-fault hamiltonian for any $f \geq 2$.

4. Application to Disjoint Path Covers

Disjoint paths can be categorized as three types: one-to-one, one-to-many, and many-to-many. One-to-one type deals with the disjoint paths joining a single source s and a single sink t. One-to-many type considers the disjoint paths joining a single source s and k distinct sinks $t_1, t_2, \ldots, t_k$. Many-to-many type deals with the disjoint paths joining k distinct sources $s_1, s_2, \ldots, s_k$ and k distinct sinks $t_1, t_2, \ldots, t_k$. All of three types of disjoint paths in a graph G can be accommodated with the covering of vertices in G. A *disjoint path cover* in a graph G is a set of disjoint paths containing all the vertices in G. A disjoint path cover problem that originated from an interconnection network is concerned with the application where the full utilization of nodes is important.

Let F be a set of faulty elements (vertices and/or edges) in a graph G. Given a set of k sources $S = \{s_1, s_2, \ldots, s_k\}$ and a set of k sinks $T = \{t_1, t_2, \ldots, t_k\}$ in $G \backslash F$ such that $S \cap T = \emptyset$, a *many-to-many k-disjoint path cover* joining S and T is a set of k disjoint paths P_i joining s_i and t_i, $1 \leq i \leq k$, such that (a) $\bigcup_{1 \leq i \leq k} V(P_i) = V(G) \backslash F$, (b) $V(P_i) \cap V(P_j) = \emptyset$ for all $i \neq j$, and (c) every edge on each path P_i is fault-free. Such a many-to-many k-DPC is denoted by k-DPC$[\{(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)\}|G, F]$.

A graph G is called *f-fault many-to-many k-disjoint path coverable* if $f + 2k \leq |V(G)|$ and for any set F of faulty elements with $|F| \leq f$, G has k-DPC$[\{(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)\}|G, F]$ for any set of k sources $S = \{s_1, s_2, \ldots, s_k\}$ and any set of k sinks $T = \{t_1, t_2, \ldots, t_k\}$ in $G \backslash F$ such that $S \cap T = \emptyset$. In a similar way, we can also define *f-fault one-to-many k-disjoint path coverable graphs* and *f-fault one-to-one k-disjoint path coverable graphs.* To see the definitions, refer [18].

It was shown in [18] that if a graph G is f-fault one-to-many k-disjoint path coverable, then G is also f-fault one-to-one k-disjoint path coverable. Constructions of one-to-many disjoint path covers in $G_0 \oplus G_1$[18] and many-to-many disjoint path covers in $H_0 \oplus H_1$[22] were investigated as follows.

Theorem 6 *[18] For any $f \geq 0$ and $k \geq 2$, let G_i, $i = 0, 1$, be a graph with n vertices satisfying the following three conditions:*
(a) G_i is f-fault one-to-many k-disjoint path coverable.
(b) G_i is $f + k - 2$-fault hamiltonian-connected(2-disjoint path coverable).
(c) G_i is $f + k - 1$-fault hamiltonian.
Then, $G_0 \oplus G_1$ is f-fault one-to-many $k+1$-disjoint path coverable.

Theorem 7 *[22] For any $f \geq 0$ and $k \geq 1$, let G_i be a graph with n vertices satisfying the following conditions, $i = 0, 1, 2, 3$:*
(a) G_i is $f + 2j$-fault many-to-many $k - j$-disjoint path coverable for every j, $0 \leq j < k$.
(b) G_i is $f + 2k - 1$-fault hamiltonian.

Then, $H_0 \oplus H_1$ is $f + 2j$-fault many-to-many $k + 1 - j$-disjoint path coverable for every j, $0 \le j < k$, where $H_0 = G_0 \oplus G_1$ and $H_1 = G_2 \oplus G_3$.

Combining the above two theorems with Theorem 4, we can obtain interesting results on disjoint path coverability of restricted HL-graphs.

Theorem 8 *(a) Every m-dimensional restricted HL-graph, $m \ge 3$, is f-fault one-to-many k-disjoint path coverable for any $f \ge 0$ and $k \ge 2$ such that $f + k \le m - 1$.*
(b) Every m-dimensional restricted HL-graph, $m \ge 3$, is f-fault many-to-many k-disjoint path coverable for any $f \ge 0$ and $k \ge 1$ such that $f + 2k \le m - 1$.

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