Fault-Hamiltonicity of Product Graph of Path and Cycle*

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Abstract. We investigate hamiltonian properties of $P_m \times C_n$, $m \geq 2$ and even $n \geq 4$, which is bipartite, in the presence of faulty vertices and/or edges. We show that $P_m \times C_n$ with n even is strongly hamiltonian-laceable if the number of faulty elements is one or less. When the number of faulty elements is two, it has a fault-free cycle of length at least $mn-2$ unless both faulty elements are contained in the same partite vertex set; otherwise, it has a fault-free cycle of length $mn-4$. A sufficient condition is derived for the graph with two faulty edges to have a hamiltonian cycle. By applying fault-hamiltonicity of $P_m \times C_n$ to a two-dimensional torus $C_m \times C_n$, we obtain interesting hamiltonian properties of a faulty $C_m \times C_n$.

1 Introduction

Embedding of linear arrays and rings into a faulty interconnection graph is one of the central issues in parallel processing. The problem is modeled as finding as long fault-free paths and cycles as possible in the graph with some faulty vertices and/or edges. Fault-hamiltonicity of various interconnection graphs were investigated in the literature. Among them, hamiltonian properties of faulty $P_m \times C_n$ and $C_m \times C_n$ were considered in [4–6, 8]. Here, P_m is a path with m vertices and C_n is a cycle with n vertices. Many interconnection graphs such as tori, hypercubes, recursive circulants[7], and double loop networks have a spanning subgraph isomorphic to $P_m \times C_n$ for some m and n. Hamiltonian properties of $P_m \times C_n$ with faulty elements play an important role in discovering fault-hamiltonicity of such interconnection graphs.

A graph G is called *k-fault hamiltonian* (resp. *k-fault hamiltonian-connected*) if $G - F$ has a hamiltonian cycle (resp. a hamiltonian path joining every pair of vertices) for any set F of faulty elements such that $|F| \leq k$. It was proved in [4, 8] that $P_m \times C_n$, $n \geq 3$ odd, is hamiltonian-connected and 1-fault hamiltonian. Throughout this paper, a hamiltonian path (resp. cycle) in a graph G with faulty elements F means a hamiltonian path (resp. cycle) in $G - F$.

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We let G be a bipartite graph with N vertices such that $|B| = |W|$, where B and W are the sets of black and white vertices in G, respectively. We denote by F_v and F_e the sets of faulty vertices and edges in G, respectively. We let $F = F_v \cup F_e$, $f_v^w = |F_v \cap W|$, $f_v^b = |F_v \cap B|$, $f_e = |F_e|$, $f_v = f_v^w + f_v^b$, and $f = f_v + f_e$. When $f_v^b = f_v^w$, a fault-free path of length $N - 2f_v^b - 1$ joining a pair of vertices with different colors is called an L^{opt}*-path*. For a pair of vertices with the same color, a fault-free path of length $N - 2f_v^b - 2$ between them is called an L^{opt}*-path*. When $f_v^b < f_v^w$, fault-free paths of length $N - 2f_v^w$ for a pair of black vertices, of length $N - 2f_v^w - 1$ for a pair of vertices with different colors, and of length $N - 2f_v^w - 2$ for a pair of white vertices, are called L^{opt}*-paths*. Similarly, we can define an L^{opt}-path for a bipartite graph with $f_v^w < f_v^b$. A fault-free cycle of length $N - 2\max\{f_v^b, f_v^w\}$ is called an L^{opt}*-cycle*. The lengths of an L^{opt}-path and an L^{opt}-cycle are the longest possible. In other words, there are no fault-free path and cycle longer than an L^{opt}*-path* and an L^{opt}*-cycle*, respectively.

A bipartite graph with $|B| = |W|$ (resp. $|B| = |W| + 1$) is called *hamiltonian-laceable* if it has a hamiltonian path joining every pair of vertices with different colors (resp. joining every pair of black vertices). Strong hamiltonian-laceability of a bipartite graph with $|B| = |W|$ was defined in [2]. We extend the notion of strong hamiltonian-laceability to a bipartite graph with faulty elements as follows. For any faulty set F such that $|F| \leq k$, a bipartite graph G which has an L^{opt}-path between every pair of fault-free vertices is called *k-fault strongly hamiltonian-laceable*.

$P_m \times P_n$, $m, n \geq 4$, is hamiltonian-laceable[3], and $P_m \times P_n$ with $f = f_v \leq 2$ has an L^{opt}-cycle when both m and n are multiples of four[5]. It has been known in [6, 8] that $P_m \times C_n$, $n \geq 4$ even, with one or less faulty element is hamiltonian-laceable. We will show in Section 3 that $P_m \times C_n$, $n \geq 4$ even, is 1-fault strongly hamiltonian-laceable, which is an extension of the work in [6, 8]. Moreover, we will show that $P_m \times C_n$, $n \geq 4$ even, has an L^{opt}-cycle if $f = 2$ and $f_v \geq 1$. When $f = f_e = 2$, it has a fault-free cycle of length at least $mn - 2$, and has a hamiltonian cycle if $m \geq 3$, $n \geq 6$ even and two faulty edges are not incident to a common vertex of degree three.

It has been known in [4] that a non-bipartite $C_m \times C_n$ is 1-fault hamiltonian-connected and 2-fault hamiltonian, and that a bipartite $C_m \times C_n$ with one or less faulty element is hamiltonian-laceable. $C_m \times C_n$ with $f = f_v \leq 4$ has an L^{opt}-cycle when both m and n are multiples of four[5]. We will show in Section 4, by utilizing hamiltonian properties of faulty $P_m \times C_n$, that a bipartite $C_m \times C_n$ is 1-fault strongly hamiltonian-laceable and has an L^{opt}-cycle when $f \leq 2$.

2 Preliminaries

The vertex set V of $P_m \times C_n$ is $\{v_j^i | 1 \leq i \leq m, 1 \leq j \leq n\}$, and the edge set $E = E_r \cup E_c$, where $E_r = \{(v_j^i, v_{j+1}^i) | 1 \leq i \leq m, 1 \leq j < n\} \cup \{(v_n^i, v_1^i) | 1 \leq i \leq m\}$ and $E_c = \{(v_j^i, v_j^{i+1}) | 1 \leq i < m, 1 \leq j \leq n\}$. An edge contained in E_r is called a *row edge*, and an edge in E_c is called a *column edge*. We denote by $R(i)$ and $C(j)$ the vertices in row i and column j, respectively. That is, $R(i) = \{v_j^i | 1 \leq j \leq n\}$

and $C(j) = \{v_j^i | 1 \le i \le m\}$. We let $R(i, i') = \bigcup_{i \le k \le i'} R(k)$ if $i \le i'$; otherwise, $R(i, i') = \emptyset$. Similarly, we let $C(j, j') = \bigcup_{j \le k \le j'} C(k)$ if $j \le j'$; otherwise, $C(j, j') = \emptyset$. v_j^i is a *black* vertex if $i + j$ is even; otherwise, it is a *white* vertex.

In $P_m \times C_n$, every pair of vertices v and w in $R(i) \cup R(m - i + 1)$ for each i, $1 \le i \le m$, are *similar*, that is, there is an automorphism ϕ such that $\phi(v) = w$. A pair of edges (v, w) and (v', w') are called *similar* if there is an automorphism ψ such that $\psi(v) = v'$ and $\psi(w) = w'$. Any two row edges in $\{(v, w)|$ either $v, w \in R(i)$ or $v, w \in R(m-i+1)\}$ are similar for each i, $1 \le i \le m$, and any two column edges in $\{(v, w)|$ either $v \in R(i), w \in R(i+1)$ or $v \in R(m-i+1), w \in R(m-i)\}$ are also similar for each i, $1 \le i \le m$.

We employ lemmas on hamiltonian properties of $P_m \times P_n$ and $P_m \times C_n$. We call a vertex in $P_m \times P_n$ a *corner vertex* if it is of degree two.

Lemma 1. *[1] Let G be a rectangular grid $P_m \times P_n$, $m, n \ge 2$. (a) If mn is even, then G has a hamiltonian path from any corner vertex v to any other vertex with color different from v. (b) If mn is odd, then G has a hamiltonian path from any corner vertex v to any other vertex with the same color as v.*

Lemma 2. *[8] (a) $P_m \times C_n$, $n \ge 3$ odd, is hamiltonian-connected and 1-fault hamiltonian. (b) $P_m \times C_n$, $n \ge 4$ even, is 1-fault hamiltonian-laceable.*

We denote by $H[v, w|X]$ a hamiltonian path in $G\langle X \rangle - F$ joining a pair of vertices v and w, if any, where $G\langle X \rangle$ is the subgraph of G induced by a vertex subset X. A path is represented as a sequence of vertices. If $G\langle X \rangle - F$ is empty or has no hamiltonian path between v and w, $H[v, w|X]$ is an empty sequence.

We let P and Q be two vertex-disjoint paths $(a_1, a_2, \cdots, a_k)$ and $(b_1, b_2, \cdots, b_l)$ on a graph G, respectively, such that (a_i, b_1) and (a_{i+1}, b_l) are edges in G. If we replace (a_i, a_{i+1}) with (a_i, b_1) and (a_{i+1}, b_l), then P and Q are merged into a single path $(a_1, a_2, \cdots, a_i, b_1, b_2, \cdots, b_l, a_{i+1}, \cdots, a_k)$. We call such a replacement a *merge* of P and Q w.r.t. (a_i, b_1) and (a_{i+1}, b_l). If P is a closed path (that is, a cycle), the merge operation results in a single cycle. We denote by $V(P)$ the set of vertices on a path P.

3 $P_m \times C_n$ with Even $n \ge 4$

3.1 $P_m \times C_n$ with One or Less Faulty Element

We will show, in this section, that $P_m \times C_n$, $n \ge 4$ even, is 1-fault strongly hamiltonian-laceable. First of all, we are going to show that $P_m \times C_n$ with a single faulty vertex is strongly hamiltonian-laceable by constructing an L^{opt}-path P joining every pair of fault-free vertices s and t.

Lemma 3. *$P_2 \times C_n$, n even, with a single faulty vertex is strongly hamiltonian-laceable. Furthermore, there is an L^{opt}-path joining every pair of vertices which passes through both an edge in $G\langle R(1) \rangle$ and an edge in $G\langle R(2) \rangle$.*

Proof. W.l.o.g., we assume that the faulty vertex is v_n^1. We let $s = v_i^x$ and $t = v_j^y$, $1 \le x, y \le 2$, and assume w.l.o.g. that $i \le j$.

Case 1 $s, t \in B$ (see Fig. 1 (a)). $P = (H[s, v_1^2|C(1, j-1)], v_n^2, H[v_{n-1}^2, t|C(j, n-1)])$. Note that when $j = n$, $H[v_{n-1}^2, t|C(j, n-1)]$ is an empty sequence. The existence of a nonempty $H[s, v_1^2|C(1, j-1)]$ is due to Lemma 1 (a).

Case 2 $s, t \in W$. First, let us consider the case that $i \ne 1$. If $j = i+1$ (see Fig. 1 (b)), $P = (s, H[v_{i-1}^x, v_1^2|C(1, i-1)], v_n^2, H[v_{n-1}^2, v_{j+1}^y|C(j+1, n-1)], t)$; otherwise (see Fig. 1 (c)), $P = (H[s, v_i^{3-x}|C(i, j-1)], H[v_{i-1}^{3-x}, v_2^2|C(2, i-1)], v_1^2, v_n^2, H[v_{n-1}^2, v_{j+1}^y|C(j+1, n-1)], t)$. For the case that $i = 1$, $P = (s, H[v_2^2, t|C(2, n-1)])$.

Case 3 Either $s \in B$, $t \in W$ or $s \in W$, $t \in B$.

Case 3.1 $i \ne j$. Let us consider the case that $j \ne n$ (see Fig. 1 (d)). Let P' be a hamiltonian path joining s and t in $G\langle C(i, j)\rangle$. P' passes through both edges (v_i^1, v_i^2) and (v_j^1, v_j^2) since P' passes through one of the two vertices v_i^1 and v_i^2 (resp. v_j^1 and v_j^2) of degree 2 in $G\langle C(i, j)\rangle$ as an intermediate vertex. Let $Q' = H[v_{i-1}^1, v_{i-1}^2|C(1, i-1)]$ and $Q'' = H[v_{j+1}^1, v_{j+1}^2|C(j+1, n-1)]$. We let P'' be a resulting path by a merge of P' and Q' w.r.t. (v_i^1, v_{i-1}^1) and (v_i^2, v_{i-1}^2) if Q' is not empty; otherwise, let $P'' = P'$. If Q'' is empty, P'' is an L^{opt}-path; otherwise, by applying a merge of P'' and Q'' w.r.t. (v_j^1, v_{j+1}^1) and (v_j^2, v_{j+1}^2), we can get an L^{opt}-path. Now, we consider the case that $j = n$. If $i \ne n-1$ (see Fig. 1 (e)), $P = (H[s, v_{n-2}^2|C(1, n-2)], v_{n-1}^2, t)$; otherwise (see Fig. 1 (f)), $P = (H[s, v_2^2|C(2, n-1)], v_1^2, t)$.

Case 3.2 $i = j$. We first let i be odd. When $i \ne 1, n-1$ (see Fig. 1 (g)), $P = (v_i^1, H[v_{i-1}^1, v_2^2|C(2, i-1)], v_1^2, v_n^2, H[v_{n-1}^2, v_{i+1}^2|C(i+1, n-1)], v_i^2)$. When $i = 1$, $P = H[v_1^1, v_1^2|C(1, n-1)]$. When $i = n-1$, $P = H[v_{n-1}^1, v_{n-1}^2|C(1, n-1)]$. For the case that i is even (see Fig. 1 (h)), $P = (v_i^1, H[v_{i+1}^1, v_{n-1}^2|C(i+1, n-1)], v_n^2, v_1^2, H[v_2^2, v_{i-1}^2|C(2, i-1)], v_i^2)$.

Unless either (i) $n = 4$ and $s, t \in W$, or (ii) $n = 4$ and $s \in R(1)$ has a color different from $t \in R(1)$, any L^{opt}-path passes through a vertex in $R(1)$ and a vertex in $R(2)$ as intermediate vertices, and thus it passes through both an edge in $G\langle R(1)\rangle$ and an edge in $G\langle R(2)\rangle$. For the cases (i) and (ii), it is easy to see that the L^{opt}-paths constructed here (Case 2 and Case 3.1) always satisfy the condition. □

Lemma 4. *$P_m \times C_n$, n even, with a single faulty vertex is strongly hamiltonian-laceable. Furthermore, there is an L^{opt}-path joining every pair of vertices which passes through both an edge in $G\langle R(1)\rangle$ and an edge in $G\langle R(m)\rangle$.*

Proof. The proof is by induction on m. We consider the case $m \ge 3$ by Lemma 3. We assume w.l.o.g. that the faulty vertex v_f is white and contained in $G\langle R(1, m-1)\rangle$ due to the similarity of $P_m \times C_n$ discussed in Section 2.

Case 1 $s, t \in R(1, m-1)$. We let P' be an L^{opt}-path joining s and t in $G\langle R(1, m-1)\rangle$ which passes through both an edge in $G\langle R(1)\rangle$ and an edge (x, y) in $G\langle R(m-1)\rangle$. A merge of P' and $G\langle R(m)\rangle - (x', y')$ w.r.t. (x, x') and (y, y') results in an L^{opt}-path P, where x' and y' are the vertices in $R(m)$ adjacent to

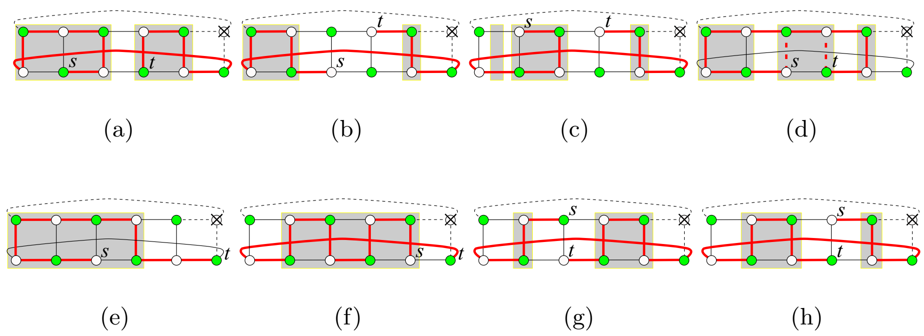


Fig. 1. Illustration of the proof of Lemma 3

x and y, respectively. Obviously, P passes through an edge in $G\langle R(m)\rangle$ as well as an edge in $G\langle R(1)\rangle$.

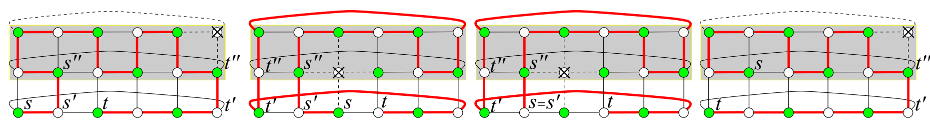
Case 2 $s, t \in R(m)$. When $s, t \in B$ (see Fig. 2 (a)) or s has a color different from t (see Fig. 2 (b)), we choose s' and t' in $R(m)$ which are not adjacent to v_f such that there are two paths P' joining s and s' and P'' joining t' and t which satisfy $V(P') \cap V(P'') = \emptyset$ and $V(P') \cup V(P'') = R(m)$. When $s, t \in W$ (see Fig. 2 (c)), we choose s' and t' in $R(m)$ which are not adjacent to v_f such that there are two paths P' joining s and s' and P'' joining t' and t which satisfy $V(P') \cap V(P'') = \emptyset$, $V(P') \cup V(P'') \subseteq R(m)$, and $|V(P') \cup V(P'')| = n-1$. We let s'' and t'' be the vertices in $R(m-1)$ which are adjacent to s' and t', respectively. Observe that $s'', t'' \in B$ if $s, t \in B$; otherwise, s'' has a color different from t''. $P = (P', Q, P'')$ is a desired L^{opt}-path, where Q is an L^{opt}-path in $G\langle R(1, m-1)\rangle$ joining s'' and t'' which satisfies the condition.

Case 3 $s \in R(1, m-1)$ and $t \in R(m)$. When $s, t \in B$ or s has a color different from t (see Fig. 2 (d)), we choose t' in $R(m)$ which is not adjacent to s and v_f such that there is a path P' joining t' and t which satisfies $V(P') = R(m)$. When $s, t \in W$, we choose t' in $R(m)$ such that there is a path P' joining t' and t which satisfies $V(P') \subseteq R(m)$ and $|V(P')| = n-1$. We let t'' be the vertex in $R(m-1)$ which is adjacent to t'. $P = (Q, P')$ is a desired L^{opt}-path, where Q is an L^{opt}-path in $G\langle R(1, m-1)\rangle$ between s and t'' which satisfies the condition. □

Strong hamiltonian-laceability of $P_m \times C_n$, $n \geq 4$ even, with a single faulty edge can be shown by utilizing Lemma 4 as follows.

Lemma 5. *$P_m \times C_n$, n even, with a single faulty edge is strongly hamiltonian-laceable.*

Proof. By Lemma 2 (b), $P_m \times C_n$ has a hamiltonian path between any two vertices with different colors. It remains to show that there is an L^{opt}-path (of length $mn-2$) joining every pair of vertices s and t with the same color. Let



(a) (b) (c) (d)

Fig. 2. Illustration of the proof of Lemma 4

(x, y) be the faulty edge. We assume w.l.o.g. that x is black and y is white. When s and t are black, we find an L^{opt}-path P between s and t regarding y as a faulty vertex by using Lemma 4. P does not pass through (x, y) as well as y, and the length of P is $mn - 2$. Thus, P is a desired L^{opt}-path. In a similar way, we can construct an L^{opt}-path for a pair of white vertices. □

We know, by Lemma 4 and Lemma 5, that $P_m \times C_n$, $n \geq 4$ even, with a single faulty element is strongly hamiltonian-laceable, which implies that $P_m \times C_n$ without faulty elements is also strongly hamiltonian-laceable. Thus, we have the following theorem.

Theorem 1. *$P_m \times C_n$, $n \geq 4$ even, is 1-fault strongly hamiltonian-laceable.*

Corollary 1. *$P_m \times C_n$, $n \geq 4$ even, has a hamiltonian cycle passing through any arbitrary edge when $f = f_e \leq 1$.*

An m-dimensional hypercube Q_m has a spanning subgraph isomorphic to $P_2 \times C_{2^{m-1}}$. A recursive circulant $G(cd^m, d)$ with degree four or more has a spanning subgraph isomorphic to $P_d \times C_{cd^{m-1}}$. $G(cd^m, d)$ with degree four or more is bipartite if and only if c is even and d is odd[7].

Corollary 2. *(a) An m-dimensional hypercube Q_m, $m \geq 3$, is 1-fault strongly hamiltonian-laceable. (b) A bipartite recursive circulant $G(cd^m, d)$ with degree four or more is 1-fault strongly hamiltonian-laceable.*

3.2 $P_m \times C_n$ with Two Faulty Elements

A bipartite graph is called 2-*vertex-fault L^{opt}-cyclic* if it has an L^{opt}-cycle when $f = f_v \leq 2$.

Lemma 6. *$P_2 \times C_n$, $n \geq 4$ even, is 2-vertex-fault L^{opt}-cyclic.*

Proof. It is sufficient to show that $P_2 \times C_n$ has an L^{opt}-cycle C when $f = f_v = 2$ by Theorem 1. We assume w.l.o.g. that v_n^1 is faulty, and let v_f be the faulty vertex other than v_n^1. Let us consider the case that $v_f \in B$ first. When $v_f = v_i^1$ and $i \neq 1$ (see Fig. 3 (a)), $C = (H[v_1^2, v_{i-1}^2 | C(1, i-1)], v_i^2, H[v_{i+1}^2, v_{n-1}^2 | C(i+1, n-1)], v_n^2)$.

When $v_f = v_1^1$, $C = (v_1^2, H[v_2^2, v_{n-1}^2|C(2, n-1)], v_n^2)$. When $v_f = v_i^2$ and $i \neq n$ (see Fig. 3 (b)), $C = (H[v_1^2, v_{i-1}^1|C(1, i-1)], v_i^1, H[v_{i+1}^1, v_{n-1}^2|C(i+1, n-1)], v_n^2)$. When $v_f = v_n^2$, $C = H[v_1^1, v_1^2|C(1, n-1)] + (v_1^1, v_1^2)$. Now, we consider the case that $v_f \in W$. When $v_f = v_i^1$ (see Fig. 3 (c)), $C = (v_1^2, H[v_2^2, v_{i-1}^2|C(2, i-1)], v_i^2, H[v_{i+1}^2, v_{n-2}^2|C(i+1, n-2)], v_{n-1}^2, v_n^2)$. When $v_f = v_i^2$ and $i \neq 1, n-1$ (see Fig. 3 (d)), $C = (v_1^2, H[v_2^2, v_{i-1}^1|C(2, i-1)], v_i^1, H[v_{i+1}^1, v_{n-2}^2|C(i+1, n-2)], v_{n-1}^2, v_n^2)$. When $v_f = v_1^2$, $C = H[v_2^1, v_2^2|C(2, n-1)] + (v_2^1, v_2^2)$. When $v_f = v_{n-1}^2$, $C = H[v_1^1, v_1^2|C(1, n-2)] + (v_1^1, v_1^2)$. □

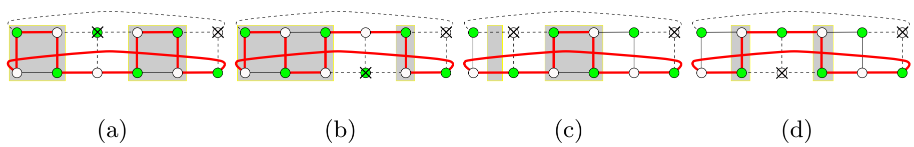


Fig. 3. Illustration of the proof of Lemma 6

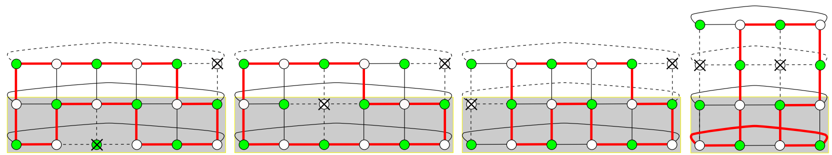
Theorem 2. *$P_m \times C_n$, $n \geq 4$ even, is 2-vertex-fault L^{opt}-cyclic.*

Proof. The proof is by induction on m. We are sufficient to construct an L^{opt}-cycle C for the case that $m \geq 3$ and $f = f_v = 2$. We assume w.l.o.g. that at most one faulty vertex is contained in $R(1)$ due to the similarity of $P_m \times C_n$.

Case 1 There is one faulty vertex in $R(1)$. We assume w.l.o.g. that v_n^1 is faulty, and let v_f be the faulty vertex other than v_n^1. When $v_f \in B$ (see Fig. 4 (a)), $C = (v_1^1, v_2^1, \cdots, v_{n-1}^1, P')$, where P' is an L^{opt}-path between v_{n-1}^2 and v_1^2 in $G\langle R(2, m)\rangle$. The existence of P' is due to Theorem 1. When $v_f \in W$ and $v_f \neq v_1^2$ (see Fig. 4 (b)), $C = (v_1^1, v_2^1, \cdots, v_{n-2}^1, P')$, where P' is an L^{opt}-path between v_{n-2}^2 and v_1^2 in $G\langle R(2, m)\rangle$. When $v_f = v_1^2$ (see Fig. 4 (c)), $C = (v_2^1, v_3^1, \cdots, v_{n-1}^1, P')$, where P' is an L^{opt}-path between v_{n-1}^2 and v_2^2 in $G\langle R(2, m)\rangle$.

Case 2 There is no faulty vertex in $R(1)$. We let C' be an L^{opt}-cycle in $G\langle R(2, m)\rangle$. If C' passes through an edge (x, y) in $G\langle R(2)\rangle$, a merge of C' and $G\langle R(1)\rangle - (x', y')$ w.r.t. (x, x') and (y, y') results in an L^{opt}-cycle, where x' and y' are the vertices in $R(1)$ adjacent to x and y, respectively. No such an edge (x, y) exists only when $n = 4$ and a pair of vertices with the same color in $R(2)$ are faulty. We consider the case that $n = 4$ and two white vertices are faulty. For $m \geq 4$ (see Fig. 4 (d)), $C = (v_2^2, v_2^1, v_3^1, v_4^1, v_4^2, P')$, where P' is an L^{opt}-path in $G\langle R(3, m)\rangle$ between v_4^3 and v_2^3. For $m = 3$, $C = (v_2^2, v_2^1, v_3^1, v_4^1, v_4^2, v_4^3, v_3^3, v_2^3)$. Similarly, we can construct an L^{opt}-cycle for the case that two black vertices are faulty. □

A bipartite graph is called 1*-vertex and* 1*-edge-fault L^{opt}-cyclic* if it has an L^{opt}-cycle when $f_v \leq 1$ and $f_e \leq 1$.



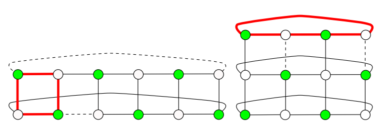
(a) (b) (c) (d)

Fig. 4. Illustration of the proof of Theorem 2

Theorem 3. *$P_m \times C_n$, $n \geq 4$ even, is 1-vertex and 1-edge-fault $L^{\rm opt}$-cyclic.*

Proof. We consider the case that $f_v = 1$ and $f_e = 1$ by Theorem 1. We let v_f and (x, y) be the faulty vertex and edge, respectively. We assume w.l.o.g. that x has a color different from v_f. We find an $L^{\rm opt}$-cycle C regarding x (as well as v_f) as a faulty vertex by using Theorem 2. C does not pass through (x, y) and v_f, and the length of C is $mn - 2$. Thus, C is a desired $L^{\rm opt}$-cycle. □

Contrary to Theorem 2 and 3, $P_m \times C_n$, $n \geq 4$ even, with two faulty edges does not always have an $L^{\rm opt}$-cycle since both faulty edges may be incident to a common vertex of degree 3. Note that, when there are no faulty vertices, an $L^{\rm opt}$-cycle means a hamiltonian cycle. There are some other fault patterns which prevent $P_m \times C_n$ from having a hamiltonian cycle. For example, $P_2 \times C_n$ has no hamiltonian cycle if (v_n^1, v_1^1) and (v_2^2, v_3^2) are faulty (see Fig. 5 (a)): supposing that there is a hamiltonian cycle, a faulty edge (v_n^1, v_1^1) (resp. (v_2^2, v_3^2)) forces (v_1^1, v_2^1) and (v_1^1, v_1^2) (resp. (v_2^2, v_1^2) and (v_2^2, v_2^1)) to be included in the hamiltonian cycle, which is impossible. Similarly, we can see that $P_m \times C_4$ has no hamiltonian cycle if (v_2^1, v_2^2) and (v_4^1, v_4^2) are faulty (see Fig. 5 (b)).



(a) (b)

Fig. 5. Nonhamiltonian $P_m \times C_n$ with $f = f_e = 2$

Theorem 4. *$P_m \times C_n$, $n \geq 4$ even, with two faulty edges has a fault-free cycle of length at least $mn - 2$.*

Proof. We let (x, y) and (x', y') be the faulty edges, and assume w.l.o.g. that x and x' are black. We find an L^{opt}-cycle C regarding x and y' as faulty vertices by using Theorem 2. C does not pass through (x, y) and (x', y') as well as x and y', and the length of C is $mn - 2$, as required by the theorem. □

Theorem 5. *$P_m \times C_n$, $m \geq 3$ and even $n \geq 6$, with two faulty edges has a hamiltonian cycle if both faulty edges are not incident to a common vertex of degree* 3.

Proof. The proof is by induction on m. We let e_f and e'_f be the faulty edges, and will construct a hamiltonian cycle C.

Case 1 $e_f, e'_f \in G\langle R(1)\rangle$. We assume w.l.o.g. that $e_f = (v_n^1, v_1^1)$ and $e'_f = (v_i^1, v_{i+1}^1)$. By assumption, we have $i \neq 1, n-1$. When i is even, $C = (H[v_1^m, v_i^m|C(1,i)], H[v_{i+1}^m, v_n^m|C(i+1,n)])$. When both i and m are odd, $C = (H[v_1^m, v_i^m|C(1,i)], H[v_{i+1}^m, v_n^m|C(i+1,n)])$. The existence of hamiltonian paths in $G\langle C(1,i)\rangle$ and in $G\langle C(i+1,n)\rangle$ is due to Lemma 1 (b). When i is odd and m is even, $C = (H[v_1^m, v_i^{m-1}|C(1,i)], H[v_{i+1}^{m-1}, v_n^m|C(i+1,n)])$.

Case 2 $e_f \in G\langle R(1)\rangle$ and $e'_f \notin G\langle R(1)\rangle$. W.l.o.g., we let $e_f = (v_n^1, v_1^1)$. By assumption, both (v_1^1, v_1^2) and (v_n^1, v_n^2) are fault-free. $C = (v_1^1, v_2^1, \cdots, v_n^1, P')$, where P' is a hamiltonian path in $G\langle R(2,m)\rangle$ between v_n^2 and v_1^2 due to Theorem 1.

Case 3 $e_f, e'_f \notin G\langle R(1)\rangle$.

Case 3.1 There is a faulty column edge joining a vertex in $R(1)$ and a vertex in $R(2)$. There exists i such that both (v_i^1, v_i^2) and (v_{i+1}^1, v_{i+1}^2) are fault-free since $f_e = 2$ and $n \geq 6$. $C = (H[v_i^1, v_{i+1}^1|R(1)], P')$, where P' is a hamiltonian path in $G\langle R(2,m)\rangle$ between v_{i+1}^2 and v_i^2.

Case 3.2 There is no faulty column edge joining a vertex in $R(1)$ and a vertex in $R(2)$. First, we consider the case that there is a faulty edge in $G\langle R(2)\rangle$. We assume w.l.o.g. that $e_f = (v_n^2, v_1^2)$. We find a hamiltonian path P' in $G\langle R(2,m)\rangle$ between v_n^2 and v_1^2 regarding e_f as a fault-free edge by using Theorem 1. Obviously, P' does not pass through e_f. Thus, we have a hamiltonian cycle $C = (H[v_1^1, v_n^1|R(1)], P')$. The construction of a hamiltonian cycle for the base case $m = 3$ is completed since the case that there is a faulty edge in $G\langle R(3)\rangle$ is reduced to Case 1 and 2, and the case that there is a faulty edge joining a vertex in $R(2)$ and a vertex in $R(3)$ is reduced to Case 3.1.

The remaining case is that $m \geq 4$ and there is no faulty edge in $G\langle R(1,2)\rangle$. The faulty edges are contained in $G\langle R(2,m)\rangle$, and both of them are not incident to a common vertex in $R(2)$. That is, we have $G\langle R(2,m)\rangle$ isomorphic to $P_{m-1} \times C_n$ such that both faulty edges are not incident to a common vertex of degree 3. Thus, we have a hamiltonian cycle C' in $G\langle R(2,m)\rangle$ by the induction hypothesis. A merge of C' and $G\langle R(1)\rangle - (x', y')$ w.r.t. (x, x') and (y, y') results in a hamiltonian cycle in $P_m \times C_n$, where x and y are the vertices in $R(2)$ such that (x, y) is an edge on C', and x' and y' are the vertices in $R(1)$ adjacent to x and y, respectively. This completes the proof. □

4 $C_m \times C_n$ with Even m and n

Let us consider fault-hamiltonicity of a bipartite $C_m \times C_n$ with $m, n \geq 4$. $C_m \times C_n$ is bipartite if and only if both m and n are even.

Theorem 6. *(a) $C_m \times C_n$, m and n even, is 1-fault strongly hamiltonian-laceable. (b) $C_m \times C_n$, m and n even, is 2-fault L^{opt}-cyclic.*

Proof. The statement (a) is due to Theorem 1. It is sufficient to show that $C_m \times C_n$ with two faulty edges has a hamiltonian cycle by Theorem 2 and Theorem 3. $C_m \times C_n$ has a spanning subgraph isomorphic to $P_m \times C_n$ or $P_n \times C_m$ which has at most one faulty edge. Thus, $C_m \times C_n$ has a hamiltonian cycle by Corollary 1. □

5 Concluding Remarks

We proved that $P_m \times C_n$, $n \geq 4$ even, is 1-fault strongly hamiltonian-laceable, 2-vertex-fault L^{opt}-cyclic, 1-vertex and 1-edge-fault L^{opt}-cyclic. If there are two faulty edges, $P_m \times C_n$ has a fault-free cycle of length at least $mn - 2$. It was also proved that $P_m \times C_n$, $m \geq 3$ and even $n \geq 6$, is hamiltonian if both faulty edges are not incident to a common vertex of degree 3. By employing fault-hamiltonicity of $P_m \times C_n$, we found that a bipartite $C_m \times C_n$ is 1-fault strongly hamiltonian-laceable and 2-fault L^{opt}-cyclic.

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