Longest Paths and Cycles in Faulty Star Graphs*

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Abstract

In this paper, we investigate the star graph S_n with faulty vertices and/or edges from the graph theoretic point of view. We show that between every pair of vertices with different colors in a bicoloring of S_n, $n \geq 4$, there is a fault-free path of length at least $n! - 2f_v - 1$, and there is a path of length at least $n! - 2f_v - 2$ joining a pair of vertices with the same color, when the number of faulty elements is $n - 3$ or less. Here, f_v is the number of faulty vertices. S_n, $n \geq 4$, with at most $n-2$ faulty elements has a fault-free cycle of length at least $n! - 2f_v$ unless the number of faulty elements are $n - 2$ and all the faulty elements are edges incident to a common vertex. It is also shown that S_n, $n \geq 4$, is strongly hamiltonian-laceable if the number of faulty elements is $n - 3$ or less and the number of faulty vertices is one or less.

Key Words: Fault-hamiltonicity, star graph, hamiltonian-laceability, fault tolerance

1 Introduction

Embedding of linear arrays and rings into a faulty interconnection graph is one of the central issues in parallel processing. The problem is modeled as finding as long fault-free paths and cycles as possible in the graph with some faulty vertices and/or edges. Fault-hamiltonicity of various interconnection graphs was investigated in the literature [4, 6, 7, 9–13].

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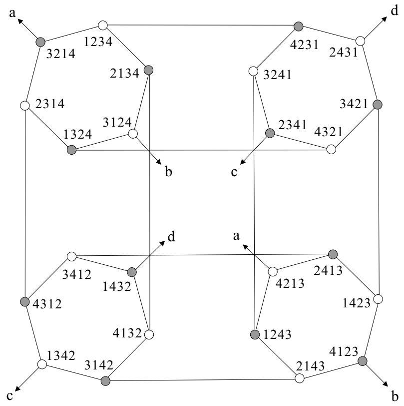

Figure 1: 4-dimensional star graph S_4

The star graph has been recognized as an attractive alternative to the hypercube graph. The vertices of an n-dimensional *star graph* S_n are all the permutations of $\{1, 2, \cdots, n\}$. A permutation $a_1a_2\cdots a_k\cdots a_n$ is connected to $a_ka_2\cdots a_{k-1}a_1a_{k+1}\cdots a_n$ via an edge for every k, $2 \leq k \leq n$. An edge which joins a permutation and another permutation obtained by interchanging the first and kth symbol is called a *k-dimensional edge*. A 4-dimensional star graph S_4 is shown in Figure 1. The star graph S_n is vertex symmetric and edge symmetric [2]. The degree and diameter of S_n are $n-1$ and $\lfloor 3(n-1)/2 \rfloor$, respectively [2].

The n-dimensional star graph S_n is bipartite, that is, the vertices can be colored with white and black in such a way that endvertices of every edge have different colors. Moreover, the number of black vertices is equal to that of white vertices [1]. S_n is strongly hierarchical, that is, for every k, $2 \leq k \leq n$, S_n can be decomposed into n components which are isomorphic to S_{n-1} if all the k-dimensional edges are deleted [2]. All vertices contained in a component have the same kth symbol. If we delete all 4-dimensional edges in S_4 shown in Figure 1, we have four components isomorphic to S_3.

S_n has a hamiltonian cycle [8]. Furthermore, S_n has a hamiltonian path between every pair of vertices with different colors, and has a path of length $n! - 2$ between every pair of vertices with the same color [5]. Here, the length of a path is the number of edges in the path. However, even S_n with a single faulty vertex is not hamiltonian, that is, it has no cycle passing through all the fault-free vertices. We have an interest in fault-hamiltonicity of star graphs, that is, hamiltonian properties of star graphs with faulty vertices and/or edges.

We need some definitions on the longest fault-free paths and cycles in bipartite graphs. We let G be a bipartite graph with N vertices such that $|B| = |W|$, where B and W are the sets of black and white vertices in G, respectively. We denote by F_v and F_e the sets of faulty vertices and edges in G, respectively. We let $F = F_v \cup F_e$, $f_v^w = |F_v \cap W|$, $f_v^b = |F_v \cap B|$, $f_e = |F_e|$, $f_v = f_v^w + f_v^b$, and $f = f_v + f_e$.

Definition 1 *L^{opt}-path and L^{opt}-cycle*
*When $f_v^b = f_v^w$, a fault-free path of length $N - 2f_v^b - 1$ joining a pair of black and white vertices is called an L^{opt}-*path. *For a pair of vertices with the same color, a fault-free path of length $N - 2f_v^b - 2$ between them is called an L^{opt}-path. When $f_v^b < f_v^w$, the length of an L^{opt}-path is $N - 2f_v^w$ for a pair of black vertices, $N - 2f_v^w - 1$ for a pair of black and white vertices, and $N - 2f_v^w - 2$ for a pair of white vertices. Similarly, we can define an L^{opt}-path for a bipartite graph with $f_v^w < f_v^b$. A cycle of length $N - 2\max\{f_v^b, f_v^w\}$ is called an L^{opt}-*cycle.

Definition 2 *L-path and L-cycle*
*A fault-free path of length $N - 2f_v - 1$ or more between a pair of vertices with different colors is called an L-*path. *Between a pair of vertices with the same color, a path of length $N - 2f_v - 2$ or more is called an L-path. A cycle of length $N - 2f_v$ or more is called an L-*cycle.

The lengths of an L^{opt}-path and an L^{opt}-cycle are the maximum possible. In other words, there are no fault-free path and cycle longer than an L^{opt}-*path* and an L^{opt}-*cycle*, respectively. The length of an L-path (resp. L-cycle) is the maximum in a sense of worst case. In the following propositions, we will discuss about relationships among the notions of L^{opt}-path, L-path, L^{opt}-cycle, and L-cycle, and will give a necessary condition for a bipartite graph to have such a path (or cycle).

Proposition 1 *Let G be a bipartite graph with $|B| = |W|$.*
(a) Every L^{opt}-path in G is an L-path.
(b) Every L^{opt}-cycle in G is an L-cycle.
(c) When $f_v = 0$, every L-path is an L^{opt}-path.
(d) When $f_v^b \geq 1$ and $f_v^w = 0$ (resp. $f_v^w \geq 1$ and $f_v^b = 0$), every L-path joining a pair of black (resp. white) vertices or a pair of black and white vertices is an L^{opt}-path.
(e) When $f_v^b = 0$ or $f_v^w = 0$, every L-cycle is an L^{opt}-cycle.
(f) When $f_v = 0$, every L-cycle is a hamiltonian cycle.

Proposition 2 *Let G be a bipartite graph with $|B| = |W|$. Every cycle consisting of a fault-free edge (v, w) and an L^{opt}-path (resp. L-path) between v and w is an L^{opt}-cycle (resp. L-cycle).*

Proposition 3 *Let G be a bipartite graph with $|B| = |W|$.*
(a) It is necessary that $f \leq \delta(G) - 2$ for G to have an L-cycle (or L^{opt}-cycle) for any set F of faulty elements such that $|F| \leq f$, where $\delta(G)$ is the minimum degree of G.
(b) It is necessary that $f \leq \delta(G) - 2$ for G to have an L-path (or L^{opt}-path) between every-pair of fault-free vertices for any set F of faulty elements such that $|F| \leq f$.

Note that in Proposition 3, all the faulty elements may be edges incident to a common vertex.

A bipartite graph with $|B| = |W|$ is called *hamiltonian-laceable* if it has a hamiltonian path joining every pair of black and white vertices. Strong hamiltonian-laceability of a bipartite graph with $|B| = |W|$ was defined in [5]. We extend the notion of strong hamiltonian-laceability to a bipartite graph with faulty elements as follows.

Definition 3 *f-fault strong hamiltonian-laceability*
A bipartite graph G with $|B| = |W|$ is called f-fault strongly hamiltonian-laceable *if for any set F of faulty elements such that $|F| \leq f$, G has an $L^{\rm opt}$-path between every pair of fault-free vertices.*

Under various fault patterns, long fault-free cycles and paths in a faulty star graph S_n have been constructed in the literature [4, 6, 7, 9, 10, 13]. They are discussed in Section 2.

In this paper, we will show that for every pair of fault-free vertices in S_n, $n \geq 4$, with $f \leq n-3$, there is an L-path joining them (Theorem 1). The bound $n-3$ on the number of faulty elements is optimal due to Proposition 3. This result implies that S_n, $n \geq 4$, with $f \leq n-3$, has an L-cycle passing through an arbitrary fault-free edge (Corollary 1) due to Proposition 2. Beyond the bound $n-3$ on the number of faulty elements for S_n to have an L-cycle for any faulty set, we will consider S_n, $n \geq 4$, with $n-2$ faulty elements and show that it has an L-cycle except for the case that all the faulty elements are edges incident to a common vertex (Theorem 2). For the exceptional case, it has a cycle of length $n!-2$, which is the longest possible. Simple and recursive construction schemes of an L-path and an L-cycle in faulty S_n will be given, based on the construction of an $L^{\rm opt}$-path and an $L^{\rm opt}$-cycle in faulty S_4. We will utilize the fact that S_n is strongly hierarchical and a technique of fault distribution that not all faulty elements are contained in a single component if $f \geq 2$. Additionally, it will be also shown that S_n, $n \geq 4$, with $f \leq n-3$ and $f_v \leq 1$ is strongly hamiltonian-laceable.

Graph theoretic terms not defined here can be found in [3]. This paper is organized as follows. In Section 2, we discuss about previous works on the construction of long fault-free cycles and paths in faulty S_n. L-paths and L-cycles in faulty S_n are constructed recursively in Section 3. Finally, we give a summary and further remarks in Section 4.

2 Previous Works

Under various fault types such as vertex faults only, edge faults only, and hybrid faults, and under various bounds on the number of faulty elements, long fault-free cycles [4, 9, 13] and long fault-free paths [6, 7, 10] in a faulty star graph S_n have been constructed. They are summarized in Table 1 and 2. The item (a) in Table 1, for example, says that Latifi *et.*

Table 1: Previous works on long cycles in S_n, $n \geq 4$

	Who	Fault pattern	How long	Remarks
(a)	Latifi *et. al.* 1997 [9]	$f_v \leq n-2$ and $f_e = 0$	S_n has an L-cycle.	vertex faults
(b)	Tseng *et. al.* 1997 [13]	$f \leq n-3$	S_n has a cycle of length at least $n! - 4f_v$.	hybrid faults, shorter than an L-cycle
(c)	Chang *et. al.* 1999 [4]	$f \leq n-3$	S_n has an L-cycle.	hybrid faults

Table 2: Previous works on long paths in S_n, $n \geq 4$

	Who	Fault pattern	How long	Remarks
(a)	Hsieh *et. al.* 2001 [6]	$f_v \leq n-5$ and $f_e = 0$	S_n has an L-path between an arbitrary pair of vertices.	vertex faults, $n \geq 6$
(b)	Hsieh *et. al.* 2001 [7]	$f_v = 0$ and $f_e \leq n-3$	S_n has an L^{opt}-path between almost every pair of vertices.	edge faults, $n \geq 6$
(c)		$f_v = 0$ and $f_e \leq n-4$	S_n is strongly hamiltonian-laceable.	edge faults, $n \geq 6$
(d)	Li *et. al.* 2002 [10]	$f_v = 0$ and $f_e \leq n-3$	S_n is strongly hamiltonian-laceable.	edge faults
(e)		$f_v = 1$ and $f_e \leq n-4$	S_n has an L^{opt}-path between an arbitrary pair of vertices with colors different from the faulty vertex.	hybrid faults, restricted to $f_v = 1$

al. proved in [9] that S_n, $n \geq 4$, with $f_v \leq n-2$ and $f_e = 0$ has an L-cycle. Table 1 is concerned with fault-free cycles in S_n and Table 2 is concerned with fault-free paths in S_n.

Theorem 1 which states that for every pair of fault-free vertices in S_n, $n \geq 4$, with $f \leq n-3$, there is an L-path joining them, is an extension of every item except (e) in Table 2. Recall Proposition 1(c) which says that when $f_v = 0$, every L-path is an L^{opt}-path. Corollary 1 which states that S_n, $n \geq 4$, with $f \leq n-3$, has an L-cycle passing through an arbitrary fault-free edge, is an extension of items (b) and (c) in Table 1. Theorem 2 which states that S_n, $n \geq 4$, with $f = n-2$ has an L-cycle except for the case that all the faulty elements are edges incident to a common vertex, is an extension of item (a) in Table 1.

3 L-paths and L-cycles in Faulty S_n

We will show, in this section, that S_n, $n \geq 4$, with $f \leq n-3$ has an L-path between any pair of vertices, and that S_n, $n \geq 4$, with $f = n-2$ has an L-cycle unless all the faulty elements are edges incident to a common vertex. Finally, we will show that S_n, $n \geq 4$, with $f \leq n-3$ and $f_v \leq 1$ is strongly hamiltonian-laceable. First, we will discuss about the base case of $n = 4$ and decompositions of S_n into n components isomorphic to S_{n-1}, and then give simple and recursive constructions of an L-path and an L-cycle in faulty S_n.

3.1 L^{opt}-paths and L^{opt}-cycles in faulty S_4

In this subsection, we are concerned with L^{opt}-paths and L^{opt}-cycles in S_4 rather than L-paths and L-cycles. We show that there exists an L^{opt}-path between any pair of vertices in S_4 with a single faulty element. This implies that S_4 is 1-fault strongly hamiltonian-laceable since an L^{opt}-path in S_4 with a single faulty edge is indeed an L^{opt}-path in S_4 without faulty elements. Proofs of Lemma 1 and 2 are given in Appendix.

Lemma 1 *For any pair of vertices v and w in S_4 with $f_v = 1$ and $f_e = 0$, there exists an L^{opt}-path joining them.*

Lemma 2 *For any pair of vertices v and w in S_4 with $f_v = 0$ and $f_e = 1$, there exists an L^{opt}-path joining them.*

We show that S_4 with two or less faulty elements has an L^{opt}-cycle unless there are two faulty edges incident to a common vertex. S_4 with two faulty edges incident to a common vertex is shown to have a cycle of length $4! - 2$, which is the longest possible. By Lemma 1, 2 and Proposition 2, we are sufficient to consider S_4 with two faulty elements. Proofs of Lemma 3, 4, and 5 are given in Appendix.

Lemma 3 *S_4 with $f_v = 2$ and $f_e = 0$ has an L^{opt}-cycle.*

Lemma 4 *S_4 with $f_v = 1$ and $f_e = 1$ has an L^{opt}-cycle.*

Contrary to Lemma 3 and 4, S_4 with two faulty edges does not always have an L^{opt}-cycle since both faulty edges may be incident to a common vertex of degree three. Recall Proposition 1(f) which implies that, when there are no faulty vertices, an L^{opt}-cycle means a hamiltonian cycle.

Lemma 5 *S_4 with $f_v = 0$ and $f_e = 2$ has a fault-free hamiltonian cycle except for the case that the two faulty edges are incident to a common vertex. For the exceptional case, it has a fault-free cycle of length $4! - 2$.*

3.2 Decompositions of S_n into n components

We are going to discuss about some preliminary properties on decomposition of S_n into n components S^1_{n-1}, S^2_{n-1}, $\cdots$, S^n_{n-1} which are isomorphic to S_{n-1}. The decomposition can be achieved if we delete all the k-dimensional edges for any k, $2 \leq k \leq n$. There are $n-1$ vertices adjacent to each vertex in S^i_{n-1}. Among them, exactly one vertex is not contained in S^i_{n-1} and the other vertices are contained in S^i_{n-1}.

Lemma 6 *We let x and y be adjacent vertices in a component S^i_{n-1}, and let x' and y' be the vertices adjacent to x and y not contained in S^i_{n-1}, respectively. Then, x' and y' are contained in different components.*

Proof We let S^i_{n-1} be a component obtained from the decomposition by k-dimensional edges, and let $x = a_1a_2\cdots a_n$ and $y = b_1b_2\cdots b_n$. It holds true that $a_1 \neq b_1$ and $a_k = b_k$. Thus, $x' = a_ka_2\cdots a_{k-1}a_1a_{k+1}\cdots a_n$ and $y' = b_kb_2\cdots b_{k-1}b_1b_{k+1}\cdots b_n$ are contained in different components since they have different kth symbols. □

Lemma 7 *We let $n \geq 4$. There are $(n-2)!$ pairwise non-adjacent edges which join vertices in S^i_{n-1} and vertices in S^j_{n-1} for any pair of i and j such that $i \neq j$. Half of the edges have black endvertices in S^i_{n-1}, and the other half have white endvertices in S^i_{n-1}.*

Proof We let S^i_{n-1} be a component obtained from the decomposition by k-dimensional edges, and let b be the kth symbol of vertices in S^i_{n-1}. For $a \neq b$, we let V^i_a be the set of vertices in S^i_{n-1} whose first symbol is a. Obviously, $|V^i_a| = (n-2)!$. Furthermore, we can observe that V^i_a has the same number of black and white vertices, or equivalently V^i_a has the same number of odd and even permutations. Note that every even (resp. odd) permutation can be interpreted as a white (resp. black) vertex, as in Figure 1. Thus, we have the lemma. □

Lemma 8 *S_n with $f \geq 2$ faulty elements can be decomposed into n components S^1_{n-1}, S^2_{n-1}, $\cdots$, S^n_{n-1} isomorphic to S_{n-1} such that every S^i_{n-1} contains at most $f-1$ faulty elements.*

Proof If S_n has a faulty edge, say a k-dimensional edge, the decomposition of S_n by k-dimensional edges is sufficient. If S_n has no faulty edge, S_n has two faulty vertices x and y such that $x = a_1a_2\cdots a_n$, $y = b_1b_2\cdots b_n$. There exists k, $2 \leq k \leq n$, such that $a_k \neq b_k$. If we decompose S_n by k-dimensional edges, two vertices x and y are contained in different components. This completes the proof. □

3.3 L-paths in S_n with $n-3$ or less faulty elements

We will give a recursive construction of an L-path between an arbitrary pair of vertices v and w in S_n with $f \leq n-3$. First of all, S_n, $n \geq 5$, is decomposed into n components S^1_{n-1},

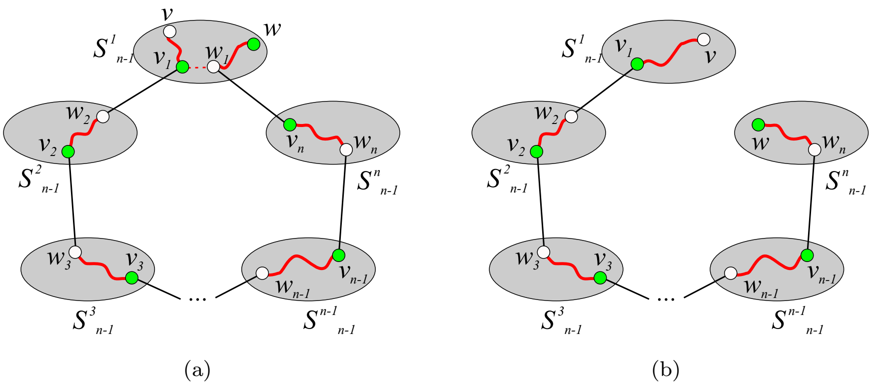

Figure 2: Illustration of Theorem 1

S^2_{n-1}, $\cdots$, S^n_{n-1} isomorphic to S_{n-1} such that every S^i_{n-1} contains at most $n-4$ faulty elements, and then n or $n-1$ inter-component edges are found depending on whether or not v and w are contained in the same component. In each component, an L-path is found. Finally, all the L-paths in the components are merged into an L-path between v and w with the inter-component edges.

Theorem 1 *For any pair of vertices v and w in S_n, $n \geq 4$, with $f \leq n-3$, there exists an L-path joining them.*

Proof We prove the theorem by induction on n. When $n = 4$, the theorem holds true by Lemma 1 and 2. We assume that $n \geq 5$. We first decompose S_n into n components S^1_{n-1}, S^2_{n-1}, $\cdots$, S^n_{n-1} isomorphic to S_{n-1} such that every S^i_{n-1} contains at most $n-4$ faulty elements. The decomposition is trivial if $f \leq n-4$. When $f = n-3$, it is also possible by Lemma 8. Since S^i_{n-1} has at most $n-4$ faulty elements, S^i_{n-1} has an L-path joining any pair of vertices. Now, we are going to construct an L-path joining v and w in S_n. We have two cases.

Case 1 Both v and w are contained in some S^p_{n-1} (see Figure 2 (a)).
Without loss of generality, we assume that $p = 1$. Let P_1 be an L-path joining v and w in S^1_{n-1}. We are going to choose an edge (v_1, w_1) on P_1 such that (i) the vertices w_2 and v_n not contained in S^1_{n-1} which are adjacent to v_1 and w_1, respectively, are fault-free, and (ii) the edges (v_1, w_2) and (w_1, v_n) are fault-free. This process is always possible since the length $l(P_1)$ of P_1 is sufficiently larger than two times the number of faulty elements. That is, $l(P_1) \geq (n-1)! - 2(n-4) - 2 > 2(n-3) \geq 2f$ for all $n \geq 5$. Deleting the edge (v_1, w_1) decomposes P_1 into two path segments: the path P'_1 from v to v_1 and the path P''_1 from w_1 to w. We can assume that, by Lemma 6, w_2 and v_n are contained in S^2_{n-1} and S^n_{n-1}, respectively.

Now, we choose pairs of vertices v_i, w_{i+1} for all $2 \leq i < n$, which satisfy that (i) v_i and w_{i+1} are fault-free vertices contained in S^i_{n-1} and S^{i+1}_{n-1}, respectively, (ii) (v_i, w_{i+1}) is an edge and fault-free, and (iii) the colors of v_i and w_{i+1} are the same as v_1 and w_1, respectively. This process is possible since there are $(n-2)!/2$ candidates for a pair of vertices v_i and w_{i+1} by Lemma 7 and the number of candidates is greater than that of faulty elements. That is, $(n-2)!/2 > n-3 \geq f$ for all $n \geq 5$.

Let P_i be an L-path in S^i_{n-1} between w_i and v_i for $2 \leq i \leq n$. The paths P'_1, P''_1, P_2, $\cdots$, P_n, and edges (v_1, w_2), (v_2, w_3), $\cdots$, (v_{n-1}, w_n), and (v_n, w_1) constitute a path P between v and w. Let us consider the length $l(P)$ of P. We let f^i_v be the number of faulty vertices in S^i_{n-1} so that $f_v = \sum_{1 \leq i \leq n} f^i_v$. The length $l(P_i)$ of P_i is at least $(n-1)! - 2f^i_v - 1$ for all $2 \leq i \leq n$. We have that $l(P'_1) + l(P''_1) = l(P_1) - 1$ and $l(P_1) \geq (n-1)! - 2f^1_v - \Delta$, where $\Delta = 1$ if v and w have different colors; otherwise, $\Delta = 2$. Thus, $l(P) = l(P'_1) + l(P''_1) + \sum_{2 \leq i \leq n} l(P_i) + n \geq n! - 2f_v - \Delta$ and P is an L-path.

Case 2 v is in S^p_{n-1} and w is in S^q_{n-1} such that $p \neq q$ (see Figure 2 (b)).
We assume that $p = 1$ and $q = n$, and that no vertex in S^{n-1}_{n-1} is adjacent to w. Similarly to Case 1, we choose pairs of vertices v_i and w_{i+1} for all $1 \leq i < n$ such that (i) v_i and w_{i+1} are fault-free and contained in S^i_{n-1} and S^{i+1}_{n-1}, respectively, (ii) (v_i, w_{i+1}) is a fault-free edge, and (iii) v_i has a different color from v (and w_{i+1} has the same color as v). It holds true that $w_n \neq w$ by our assumption. We let P_1 be an L-path from v to v_1 in S^1_{n-1}, let P_i, $2 \leq i < n$, be an L-path from w_i to v_i in S^i_{n-1}, and let P_n be an L-path from w_n to w in S^n_{n-1}. An L-path P from v to w can be constructed by merging paths P_1, P_2, $\cdots$, P_n and edges (v_1, w_2), (v_2, w_3), $\cdots$, (v_{n-1}, w_n). $\Box$

Recalling Proposition 2 that a fault-free edge (v, w) and an L-path joining v and w form an L-cycle leads to the following corollary.

Corollary 1 *S_n, $n \geq 4$, with $f \leq n-3$ has an L-cycle passing through an arbitrary fault-free edge.*

Remember Proposition 1 that when there are no faulty vertices, every L-path is an L^{opt}-path and every L-cycle is a hamiltonian cycle.

Corollary 2 *[10] S_n, $n \geq 4$, with $f = f_e \leq n-3$ is strongly hamiltonian-laceable. That is, it has an L^{opt}-path joining an arbitrary pair of vertices.*

Corollary 3 *S_n, $n \geq 4$, with $f = f_e \leq n-3$ is edge-hamiltonian. That is, it has a fault-free hamiltonian cycle passing through an arbitrary edge.*

3.4 L-cycles in S_n with $n-2$ faulty elements

S_n, $n \geq 4$, with $n-2$ faulty elements can not have an L-cycle when all the faulty elements are edges incident to a common vertex. We will give a recursive construction of an L-cycle in

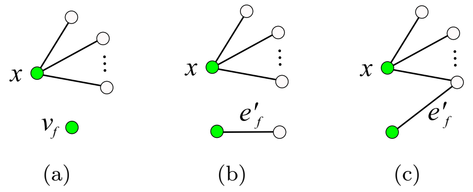

Figure 3: Three possible fault patterns

S_n with $f = n-2$ unless all the faulty elements are edges incident to a common vertex. The construction is similar to the construction of an L-path in Theorem 1. We first decompose S_n, $n \geq 5$, into n components isomorphic to S_{n-1} such that each component has at most $n-3$ faulty elements and does not have $n-3$ faulty edges incident to a common vertex in the component, and then we find n inter-component edges. In one component, an L-cycle is found, and in each of the other components, an L-path is found. Finally, they are merged into an L-cycle with the inter-component edges.

Lemma 9 *If not all the faulty elements are edges incident to a common vertex in S_n, $n \geq 5$, with $f = n-2$, then S_n can be decomposed into n components S^1_{n-1}, S^2_{n-1}, $\cdots$, S^n_{n-1} isomorphic to S_{n-1} such that for every i, (a) S^i_{n-1} has at most $n-3$ faulty elements and (b) no S^i_{n-1} has $n-3$ faulty edges incident to a common vertex in it.*

Proof If S_n has no $n-3$ faulty edges incident to a common vertex, the decomposition can be achieved by Lemma 8. We assume that S_n has $n-3$ faulty edges incident to a common vertex x. There are three possible types of fault pattern: the faulty element other than the $n-3$ faulty edges incident to x is a vertex v_f (Figure 3 (a)), an edge e'_f which is not adjacent to other faulty edges (Figure 3 (b)), or an edge e'_f which is adjacent to some faulty edges(Figure 3 (c)). Observe that e'_f can not be adjacent to two or more faulty edges since e'_f is, by assumption, not incident to x and S_n has no cycle of length three. If the fault pattern is of type (c) in Figure 3, we let e_f be one of the $n-3$ faulty edges incident to x such that e_f is adjacent to e'_f; otherwise, we let e_f be an arbitrary faulty edge incident to x. Assuming that e_f is a k-dimensional edge, we decompose S_n into n components by the k-dimensional edges. Obviously, e_f is an inter-component edge in the decomposition, and thus the decomposition satisfies the two conditions of the lemma. $\square$

Theorem 2 *S_n, $n \geq 4$, with $n-2$ faulty elements has an L-cycle except for the case that all the faulty elements are edges incident to a common vertex. For the exceptional case, it has a fault-free cycle of length $n!-2$.*

Proof When $n = 4$, the theorem holds true by Lemma 3, 4, and 5. We assume that $n \geq 5$. If all the faulty elements are edges incident to a common vertex x, we can construct a cycle

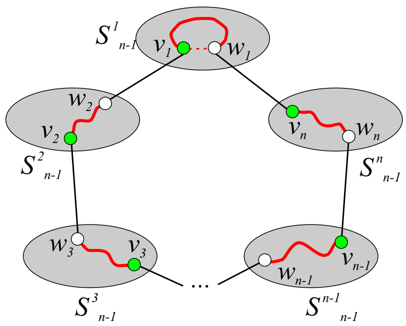

Figure 4: Illustration of the proof of Theorem 2

of length $n! - 2$ by regarding x as a faulty vertex and employing Corollary 1. From now on, we assume that not all the faulty elements are edges incident to a common vertex. First, by utilizing Lemma 9, we decompose S_n into n components S^1_{n-1}, S^2_{n-1}, $\cdots$, S^n_{n-1} isomorphic to S_{n-1} such that each S^i_{n-1} has at most $n-3$ faulty elements and does not have $n-3$ faulty edges incident to a common vertex in it.

Among the n components, we assume that S^1_{n-1} is a component with the maximum number of faulty elements. That is, $f^1 \geq f^i$ for all i, where f^i is the number of faulty elements in S^i_{n-1}. Moreover, when all the faulty elements are edges joining vertices in S^p_{n-1} and vertices in S^q_{n-1}, we assume that S^1_{n-1} is one of the two components. S^1_{n-1} has an L-cycle by an induction hypothesis. We claim that for every $2 \leq i \leq n$, S^i_{n-1} has an L-path between an arbitrary pair of vertices. If $f^1 \leq 1$, then $f^i \leq 1$; otherwise, $f^i \leq f - f^1 \leq (n-2) - 2 = n - 4$ for all $2 \leq i \leq n$. Thus, the claim holds true by Theorem 1.

Now, we are going to construct an L-cycle in a similar way to Case 1 in the proof of Theorem 1. See Figure 4. Let C_1 be an L-cycle in S^1_{n-1}. We find an edge (v_1, w_1) on C_1 such that (i) the vertices w_2 and v_n not contained in S^1_{n-1} which are adjacent to v_1 and w_1, respectively, are fault-free, and (ii) the edges (v_1, w_2) and (w_1, v_n) are fault-free. If we delete the edge (v_1, w_1) on C_1, we have a path P_1 between v_1 and w_1. We assume that w_2 and v_n are contained in S^2_{n-1} and S^n_{n-1}, respectively.

We choose pairs of vertices v_i, w_{i+1} for all $2 \leq i < n$, which satisfy that (i) v_i and w_{i+1} are fault-free vertices contained in S^i_{n-1} and S^{i+1}_{n-1}, respectively, (ii) (v_i, w_{i+1}) is a fault-free edge, and (iii) the colors of v_i and w_{i+1} are the same as v_1 and w_1, respectively. This process is possible since there are $(n-2)!/2$ candidates for a pair of vertices v_i and w_{i+1} and, by the choice of S^1_{n-1}, there are at most $n-3$ faulty edges joining vertices in S^i_{n-1} and vertices S^{i+1}_{n-1} for all $2 \leq i < n$.

Let P_i be an L-path in S^i_{n-1} between w_i and v_i for $2 \leq i \leq n$. The paths P_1, P_2, $\cdots$, P_n, and edges (v_1, w_2), (v_2, w_3), $\cdots$, (v_{n-1}, w_n), and (v_n, w_1) constitute an L-cycle C. Note that the number of fault-free vertices in S_{n-1}^i which are not on C is less than or equal to that of faulty vertices in S_{n-1}^i for all i. □

Corollary 4 *S_n, $n \geq 4$, with $f = f_e = n - 2$ has a fault-free hamiltonian cycle unless all the faulty edges are incident to a common vertex.*

3.5 Strong hamiltonian-laceability of faulty S_n

We will show that S_n, $n \geq 4$, with $f \leq n - 3$ and $f_v \leq 1$ is strongly hamiltonian-laceable, that is, it has an L^{opt}-path joining any pair of vertices v and w.

Theorem 3 *S_n, $n \geq 4$, with $f \leq n - 3$ and $f_v \leq 1$ is strongly hamiltonian-laceable.*

Proof When $f_v = 0$, by Corollary 2, the theorem holds true. We assume that a vertex v_f is faulty and that v_f is black. For a pair of white vertices v and w, we employ a result in [10] which states that S_n, $n \geq 4$, with $f_v = 1$ and $f_e \leq n - 4$ has an L^{opt}-path between them. Excluding the case that both v and w are white, an L-path between them, which can be found by Theorem 1, implies an L^{opt}-path due to Proposition 1(d). □

4 Concluding Remarks

We proved that S_n, $n \geq 4$, with $f \leq n-3$ has an L-path joining any pair of fault-free vertices, and that S_n, $n \geq 4$, with $f \leq n - 2$ has an L-cycle unless the number of faulty elements are $n - 2$ and all the faulty elements are edges incident to a common vertex. According to the constructions of an L-path and an L-cycle presented in this paper, we can design without difficulty efficient recursive algorithms for finding an L-path between an arbitrary pair of vertices and an L-cycle. It was also shown that S_n, $n \geq 4$, with $f \leq n - 3$ and $f_v \leq 1$ is strongly hamiltonian-laceable. It is open whether or not S_n, $n \geq 4$, is $n - 3$-fault strongly hamiltonian-laceable, that is, there exists an L^{opt}-path joining an arbitrary pair of fault-free vertices in S_n with $f \leq n - 3$.

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References

[1] S.B. Akers, D. Harel, and B. Krishnamurthy, "The star graph: an attractive alternative to the n-cube," in *Proc. International Conference on Parallel Processing ICPP'87*, pp. 393-400, 1987.

[2] S.B. Akers and B. Krishnamurthy, "A group-theoretic model for symmetric interconnection networks," *IEEE Trans. Computers* **38(4)**, pp. 555-566, 1989.

[3] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, 5th printing, American Elsevier Publishing Co., Inc., 1976.

[4] J.-H. Chang, C.-S. Shin, and K.Y. Chwa, "Ring embedding in faulty star graphs," *IEICE Trans. Fundamentals* **E82-A(9)**, pp. 1953-1964, 1999.

[5] S.-Y. Hsieh, G.-H. Chen, and C.-W. Ho, "Hamiltonian-laceability of star graphs," *Networks* **36(4)**, pp. 225-232, 2000.

[6] S.-Y. Hsieh, G.-H. Chen, and C.-W. Ho, "Longest fault-free paths in star graphs with vertex faults," *Theoretical Computer Science* **262(1-2)**, pp. 215-227, 2001.

[7] S.-Y. Hsieh, G.-H. Chen, and C.-W. Ho, "Longest fault-free paths in star graphs with edge faults," *IEEE Trans. Computers* **50(9)**, pp. 960-971, Sep. 2001.

[8] J.-S. Jwo, S. Lakshmivarahan, and S.K. Dhall, "Embedding of cycles and grids in star graphs," in *Proc. Second Symposium on Parallel and Distributed Processing SPDP'90*, pp. 540-547, 1990.

[9] S. Latifi, N. Bagherzadeh, and R.R. Gajjala, "Fault-tolerant embedding of linear arrays and rings in the star graph," *Computers Elect. Engng.* **23(2)**, pp. 95-107, 1997.

[10] T.-K. Li, J.J.M. Tan, and L.-H. Hsu, "Hamiltonian laceability on edge fault star graph," in *Proc. Ninth International Conference on Parallel and Distributed Systems ICPADS'02*, pp. 23-28, 2002.

[11] A. Sengupta, "On ring embedding in hypercubes with faulty nodes and links", *Inform. Proc. Lett.* **68**, pp. 207-214, 1998.

[12] T.-Y. Sung, C.-Y. Lin, Y.-C. Chuang, and L.-H. Hsu, "Fault tolerant token ring embedding in double loop networks," *Inform. Proc. Lett.* **66**, pp. 201-207, 1998.

[13] Y.-C. Tseng, S.-H. Chang, and J.-P. Sheu, "Fault-tolerant ring embedding in a star graph with both link and node failures," *IEEE Trans. Parallel and Distributed Systems* **8(12)**, pp. 1185-1195, 1997.

Appendix

A L^{opt}-paths and L^{opt}-cycles in faulty S_4

In this Appendix, we will prove Lemma 1 through 5. We denote by V and E the vertex and edge set of S_4, respectively.

A.1 S_4 with a single faulty vertex

To show that for any pair of fault-free vertices in S_4 with $f = f_v = 1$, there exists an L^{opt}-path joining them, assuming without loss of generality that an arbitrary vertex is faulty due to vertex symmetry of S_4, we are sufficient to find all L^{opt}-paths joining every pair of fault-free vertices. However, the number of such vertex pairs, $(4! - 1) * (4! - 2)/2 = 253$, is very large. In order to reduce the number of vertex pairs, we need to think of pairs of vertices being nondistinguishable. Thus, we are to define an equivalence relation on the set of unordered pairs of distinct vertices in $V \backslash v_f$, where v_f is the designated vertex 3214, the upper-left corner vertex of S_4 in Figure 1. We assume that v_f is black. Later, v_f will serve as a faulty vertex. Let us consider automorphisms of S_4 and a relation on $V \backslash v_f$ first.

Lemma 10 *There exist six automorphisms ϕ of S_4 such that $\phi(v_f) = v_f$.*

Proof S_4 can be drawn in two ways different from Figure 1 as given in Figure 5 (a) and (b), where the vertex v_f is restricted to be located at the upper-left corner position. For each of the three drawings, we can find a mirror drawing with respect to the straight line passing through the upper-left corner vertex v_f and the lower-right corner vertex, say 4123 in Figure 1. Thus, we have six drawings in total. Obviously, each of them induces an automorphism of S_4. We show that there are no more automorphisms. In 3! ways, we can label three vertices adjacent to v_f. For each labeling, we can observe that the three vertices from which there are two (disjoint) shortest paths of length 3 to v_f are labeled as 3124, 3412, and 3241 in a unique way, and then the vertices which are located on the cycles of length 6 passing through v_f and all the other vertices are labeled uniquely. □

Definition 4 *Let R_1 be a relation on $V \backslash v_f$ such that xR_1y if there exists an automorphism ϕ of S_4 satisfying $\phi(v_f) = v_f$ and $\phi(x) = y$.*

Lemma 11 *R_1 is an equivalence relation. There are six equivalence classes relative to R_1 as follows. Three of them W_1, W_2, and W_3 are sets of white vertices and the other three B_1, B_2, and B_3 are sets of black vertices.*

- $W_1 = \{\underline{2431}, 1342, 4321, 2143, 4132, 1423\}$
- $W_2 = \{\underline{3124}, 3412, 3241\}$
- $W_3 = \{\underline{1234}, 2314, 4213\}$
- $B_1 = \{\underline{4231}, 4312, 1324, 1243, 2134, 2413\}$

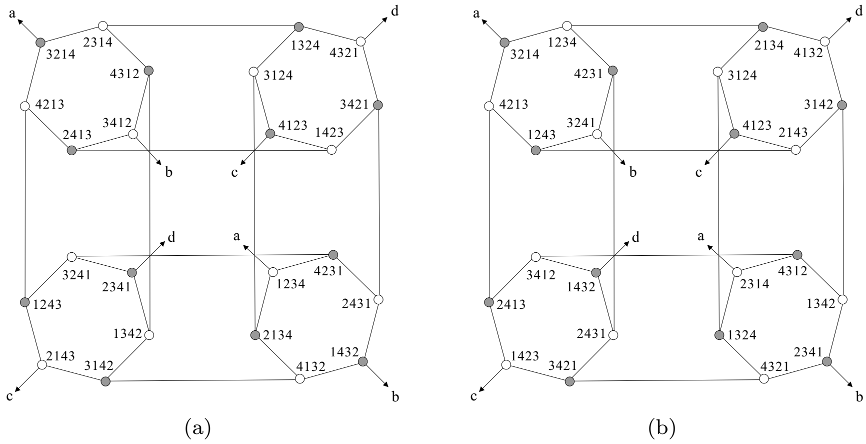

Figure 5: Other representations of S_4

- $B_2 = \{\underline{2341}, 1432, 4123\}$
- $B_3 = \{\underline{3421}, 3142\}$

Proof It is trivial to show that R_1 is an equivalence relation. By carefully observing six automorphisms given in Lemma 10, we can find out pairs of vertices that are R_1-related. For example, the upper-right corner vertices 2431 in Figure 1, 4321 and 4132 in Figure 5 as well as 1342, 2143 and 1423 in their mirror drawings are R_1-related, and thus they are contained in the same set W_1. Continuing this process, we can construct the above six equivalence classes. □

Two shortest paths from a vertex in W_2 to v_f of length three are disjoint, and thus they form a cycle of length six. For each vertex in S_4, there are three distinct cycles of length six passing through the vertex, one per a pair of edges incident to the vertex. We call a vertex underlined in each equivalence class of Lemma 11 the *representative vertex* of the class.

Definition 5 *Let R_2 be a relation on the set of unordered pairs of vertices in $V \backslash v_f$ such that $(x, y)R_2(x', y')$ if there exists an automorphism ϕ of S_4 satisfying $\phi(v_f) = v_f$ and either $\phi(x) = x'$ and $\phi(y) = y'$ or $\phi(x) = y'$ and $\phi(y) = x'$.*

Lemma 12 *R_2 is an equivalence relation. There are 49 equivalence classes relative to R_2 as follows. Among at most six vertex pairs in each class, one representative pair is shown.*

- $C_1 = \{(2431, 1342)\}$, $C_2 = \{(2431, 4321)\}$, $C_3 = \{(2431, 2143)\}$, $C_4 = \{(2431, 4132)\}$ $\subseteq W_1 \times W_1$
- $C_5 = \{(2431, 3124)\}$, $C_6 = \{(2431, 3412)\}$, $C_7 = \{(2431, 3241)\} \subseteq W_1 \times W_2$
- $C_8 = \{(2431, 1234)\}$, $C_9 = \{(2431, 2314)\}$, $C_{10} = \{(2431, 4213)\} \subseteq W_1 \times W_3$
- $C_{11} = \{(3124, 3412)\} \subseteq W_2 \times W_2$
- $C_{12} = \{(3124, 1234)\}$, $C_{13} = \{(3124, 4213)\} \subseteq W_2 \times W_3$

- $C_{14} = \{(1234, 2314)\} \subseteq W_3 \times W_3$
- $C_{15} = \{(2431, 4231)\}$, $C_{16} = \{(2431, 4312)\}$, $C_{17} = \{(2431, 1324)\}$, $C_{18} = \{(2431, 1243)\}$, $C_{19} = \{(2431, 2134)\}$, $C_{20} = \{(2431, 2413)\} \subseteq W_1 \times B_1$
- $C_{21} = \{(2431, 2341)\}$, $C_{22} = \{(2431, 1432)\}$, $C_{23} = \{(2431, 4123)\} \subseteq W_1 \times B_2$
- $C_{24} = \{(2431, 3421)\}$, $C_{25} = \{(2431, 3142)\} \subseteq W_1 \times B_3$
- $C_{26} = \{(3124, 4231)\}$, $C_{27} = \{(3124, 1324)\}$, $C_{28} = \{(3124, 1243)\} \subseteq W_2 \times B_1$
- $C_{29} = \{(3124, 2341)\}$, $C_{30} = \{(3124, 4123)\} \subseteq W_2 \times B_2$
- $C_{31} = \{(3124, 3421)\} \subseteq W_2 \times B_3$
- $C_{32} = \{(1234, 4231)\}$, $C_{33} = \{(1234, 4312)\}$, $C_{34} = \{(1234, 1324)\} \subseteq W_3 \times B_1$
- $C_{35} = \{(1234, 2341)\}$, $C_{36} = \{(1234, 1432)\} \subseteq W_3 \times B_2$
- $C_{37} = \{(1234, 3421)\} \subseteq W_3 \times B_3$
- $C_{38} = \{(4231, 4312)\}$, $C_{39} = \{(4231, 1324)\}$, $C_{40} = \{(4231, 1243)\}$, $C_{41} = \{(4231, 2134)\} \subseteq B_1 \times B_1$
- $C_{42} = \{(4231, 2341)\}$, $C_{43} = \{(4231, 1432)\}$, $C_{44} = \{(4231, 4123)\} \subseteq B_1 \times B_2$
- $C_{45} = \{(4231, 3421)\}$, $C_{46} = \{(4231, 3142)\} \subseteq B_1 \times B_3$
- $C_{47} = \{(2341, 1432)\} \subseteq B_2 \times B_2$
- $C_{48} = \{(2341, 3421)\} \subseteq B_2 \times B_3$
- $C_{49} = \{(3421, 3142)\} \subseteq B_3 \times B_3$

Proof Obviously, R_2 is an equivalence relation. Similar to the proof of Lemma 11, by observing six automorphisms given in Lemma 10, we can obtain the above 49 equivalence classes. □

A path in S_4 can be represented as a starting vertex followed by a sequence of edges, where an edge is represented as its dimension number. Note that every vertex in S_4 is incident to one 2-dimensional edge, one 3-dimensional edge, and one 4-dimensional edge.

Proof of Lemma 1 We assume that the vertex $v_f = 3214$ is faulty. By Lemma 12, it is sufficient to construct $L^{\rm opt}$-paths between 49 pairs of vertices, one pair for each equivalence class. They are shown in Table 3. Note that the lengths of $L^{\rm opt}$-paths are 22, 21, and 20, respectively, for a pair of white vertices, for a pair of white and black vertices, and for a pair of black vertices. □

It was shown in [6] that between every pair of adjacent vertices in S_4 with $f = f_v = 1$, there exists an $L^{\rm opt}$-path. Among the equivalence classes given in Lemma 12, six classes C_{15}, C_{22}, C_{24}, C_{27}, C_{30}, and C_{32} are sets of pairs of adjacent vertices. Although we need not construct $L^{\rm opt}$-paths between these pairs in the proof of Lemma 1, they are included here for completeness.

Table 3: L^{opt}-paths between v and w in S_4 with $f = f_v = 1$

$v = 2431$	$w = 1342$	4 3 2 4 3 2 4 3 4 3 4 2 4 3 4 3 2 3 4 2 3 4
	$w = 4321$	4 3 4 2 3 2 4 3 4 2 4 3 2 4 3 4 3 2 4 2 4 2
	$w = 2143$	4 2 3 2 4 2 3 4 2 3 4 2 4 2 4 3 4 2 4 2 3 2
	$w = 4132$	4 3 2 4 3 4 2 4 3 2 3 2 3 4 3 2 4 3 2 4 2 3
	$w = 3124$	4 3 4 2 3 2 4 3 4 2 4 3 2 4 3 4 3 4 2 4 2 4
	$w = 3412$	4 2 3 2 3 4 3 2 3 2 4 3 2 3 2 4 2 3 2 3 2 4
	$w = 3241$	4 3 2 4 3 2 4 3 2 3 2 3 4 2 3 4 2 3 4 2 4 3
	$w = 1234$	4 3 2 4 3 4 2 4 3 2 3 2 3 4 3 4 3 2 4 2 3 4
	$w = 2314$	4 3 2 3 2 3 4 2 4 3 2 3 2 4 3 2 3 2 3 4 2 3
	$w = 4213$	4 3 2 4 3 2 4 3 2 4 3 4 2 4 3 2 4 3 2 4 3 2
$v = 3124$	$w = 3412$	4 3 4 2 3 4 3 4 3 2 4 3 2 3 4 3 2 4 2 4 2 3
	$w = 1234$	4 3 2 3 2 3 4 2 4 3 4 2 3 4 2 3 2 4 2 3 4 2
	$w = 4213$	4 3 4 2 4 3 4 3 4 2 4 3 4 3 4 2 4 2 4 3 4 3
$v = 1234$	$w = 2314$	4 2 4 2 4 3 2 4 2 4 2 3 4 2 4 2 4 3 2 4 2 4

$v = 2431$	$w = 4231$	4 3 2 3 4 2 4 3 2 3 4 2 4 2 4 3 4 3 4 2 4
	$w = 4312$	4 3 4 2 3 2 4 2 4 2 3 4 2 3 4 2 4 2 4 3 4
	$w = 1324$	4 3 2 3 2 3 4 2 4 3 2 3 2 4 3 2 3 2 3 4 2
	$w = 1243$	4 3 2 3 4 2 3 4 2 4 3 4 3 4 2 4 2 4 3 2 3
	$w = 2134$	4 3 2 4 3 2 4 3 4 2 4 3 2 4 3 2 3 4 3 4 2
	$w = 2413$	4 3 2 4 3 4 2 4 2 4 3 2 4 3 2 4 2 4 2 3 2
	$w = 2341$	4 3 2 4 3 4 2 4 3 2 3 4 3 4 2 3 4 3 4 2 4
	$w = 1432$	3 4 3 2 3 2 3 4 3 2 4 3 2 3 4 3 4 3 2 3 2
	$w = 4123$	4 3 2 3 4 2 3 4 2 4 3 4 2 3 2 3 4 2 4 2 4
	$w = 3421$	4 3 2 3 2 3 4 2 4 3 2 3 4 2 4 3 2 3 2 3 4
	$w = 3142$	4 3 2 4 3 4 2 4 3 2 3 2 3 4 3 2 4 3 2 4 2
$v = 3124$	$w = 4231$	4 3 4 2 3 4 3 4 3 2 4 3 2 3 4 3 4 2 4 2 4
	$w = 1324$	4 3 4 2 3 2 3 2 4 2 4 2 3 4 3 2 3 4 2 3 4
	$w = 1243$	4 3 4 2 3 4 3 4 3 2 3 4 2 4 2 4 3 4 3 2 3
	$w = 2341$	4 3 4 2 3 4 3 4 2 3 2 4 2 4 2 3 4 2 3 4 2
	$w = 4123$	3 4 2 3 2 4 3 4 2 3 2 3 2 4 2 4 2 3 2 3 2
	$w = 3421$	4 3 4 2 3 4 3 4 3 2 4 3 2 4 3 2 4 2 4 2 3
$v = 1234$	$w = 4231$	2 4 2 3 2 3 2 4 3 4 2 4 3 2 4 3 2 3 4 3 2
	$w = 4312$	4 2 4 3 4 2 3 2 3 2 4 2 3 4 2 3 4 3 2 3 4
	$w = 1324$	4 2 4 3 4 2 3 2 4 3 4 3 4 2 4 2 3 4 3 4 3
	$w = 2341$	4 2 4 3 2 4 3 4 2 4 3 2 3 2 3 4 3 4 3 2 4
	$w = 1432$	4 2 3 4 3 2 3 2 3 4 3 4 3 2 4 3 4 3 4 2 3
	$w = 3421$	4 2 4 3 2 3 4 2 4 3 2 3 2 3 4 3 4 3 2 4 2

$v = 4231$	$w = 4312$	2 4 3 4 2 3 2 3 2 4 2 3 4 2 3 4 3 2 3 4
	$w = 1324$	2 4 3 4 2 3 2 4 3 4 3 4 2 4 2 3 4 3 4 3
	$w = 1243$	2 3 4 3 4 3 2 4 3 4 3 4 2 3 4 3 4 3 2 4
	$w = 2134$	2 4 3 2 3 4 2 4 3 2 3 4 2 4 2 4 3 4 3 4
	$w = 2341$	2 4 3 2 4 3 4 2 4 3 2 3 2 3 4 3 4 3 2 4
	$w = 1432$	2 3 4 3 2 3 2 3 4 3 4 3 2 4 3 4 3 4 2 3
	$w = 4123$	2 4 3 2 3 4 2 4 3 2 3 4 2 4 2 3 4 3 4 3
	$w = 3421$	2 4 3 2 3 4 2 4 3 2 3 2 3 4 3 4 3 2 4 2
	$w = 3142$	2 4 3 4 2 3 2 3 2 4 2 3 4 3 4 3 2 3 4 3
$v = 2341$	$w = 1432$	4 3 2 4 2 3 4 3 4 2 4 3 4 3 4 2 4 2 3 4
	$w = 3421$	4 3 2 3 2 3 4 3 2 3 2 3 4 3 4 2 3 2 4 2
$v = 3421$	$w = 3142$	4 3 4 2 4 3 4 3 2 4 2 3 4 3 2 4 2 4 2 3

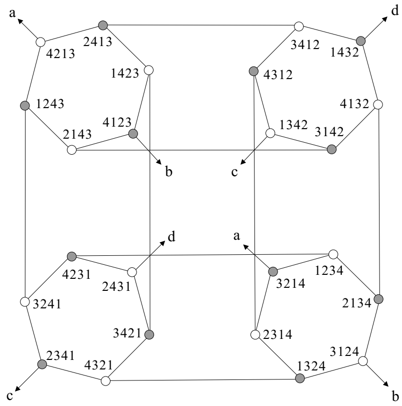

Figure 6: Another representation of S_4

A.2 S_4 with a single faulty edge

In a similar approach taken in Section A.1, we are going to show that for any pair of vertices in S_4 with $f = f_e = 1$, there exists an L^{opt}-path joining them. We let e_f be the designated edge $(3214, 4213)$. Let us discuss about automorphisms ψ of S_4 first. For a pair of vertices (x, y), we say that $\psi(x, y) = (x', y')$ if either $\psi(x) = x'$ and $\psi(y) = y'$ or $\psi(x) = y'$ and $\psi(y) = x'$.

Lemma 13 *There exist four automorphisms ψ of S_4 such that $\psi(e_f) = e_f$.*

Proof We can draw S_4 in a different way from Figure 1 without altering the position of the edge e_f, as shown in Figure 6. The two drawings and their mirrors with respect to the straight line passing through the upper-left corner vertex and the lower-right corner vertex induce four automorphisms. We show that there are no more automorphisms. We can label the upper-left corner vertex as 3214 or 4213 in two ways. For each labeling, we can also label in two ways the vertices on the unique cycle of length six which passes through the upper-left corner vertex and does not pass through the other endvertex of e_f. Note that for each vertex, there are three cycles of length six passing through the vertex, one per a pair of edges incident to the vertex. Once labels of the cycle are fixed, we can observe that all the other vertices are labeled in a unique way. □

Definition 6 *Let R'_1 be a relation on V such that xR'_1y if there exists an automorphism ψ of S_4 satisfying $\psi(e_f) = e_f$ and $\psi(x) = y$.*

Lemma 14 *R'_1 is an equivalence relation. There are seven equivalence classes relative to R'_1 as follows.*

- $X_1 = \{\underline{1234}, 2314, 2413, 1243\}$
- $X_2 = \{2134, 1324, \underline{1423}, 2143\}$

- $X_3 = \{4231, 4312, 3412, \underline{3241}\}$
- $X_4 = \{\underline{2431}, 1342, 1432, 2341\}$
- $X_5 = \{3421, 3142, 4132, \underline{4321}\}$
- $X_6 = \{\underline{3124}, 4123\}$
- $X_7 = \{\underline{4213}, 3214\}$

Proof Obviously, R_1' is an equivalence relation. By using the four automorphisms of S_4 given in Lemma 13, we can obtain the above seven equivalence classes. □

Definition 7 *Let R_2' be a relation on the set of unordered pairs of white and black vertices in V such that $(x, y)R_2'(x', y')$ if there exists an automorphism ψ of S_4 satisfying $\psi(e_f) = e_f$ and $\psi(x, y) = (x', y')$.*

Lemma 15 *R_2' is an equivalence relation. There are 43 equivalence classes relative to R_2' as follows. Among at most four vertex pairs in each class, one representative pair is shown.*

- $Y_1 = \{(1234, 2413)\}$, $Y_2 = \{(1234, 1243)\} \subseteq X_1 \times X_1$
- $Y_3 = \{(1234, 2134)\}$, $Y_4 = \{(1234, 1324)\} \subseteq X_1 \times X_2$
- $Y_5 = \{(1234, 4231)\}$, $Y_6 = \{(1234, 4312)\} \subseteq X_1 \times X_3$
- $Y_7 = \{(1234, 1432)\}$, $Y_8 = \{(1234, 2341)\} \subseteq X_1 \times X_4$
- $Y_9 = \{(1234, 3421)\}$, $Y_{10} = \{(1234, 3142)\} \subseteq X_1 \times X_5$
- $Y_{11} = \{(1234, 4123)\} \subseteq X_1 \times X_6$
- $Y_{12} = \{(1234, 3214)\} \subseteq X_1 \times X_7$
- $Y_{13} = \{(1423, 2134)\}$, $Y_{14} = \{(1423, 1324)\} \subseteq X_2 \times X_2$
- $Y_{15} = \{(1423, 4231)\}$, $Y_{16} = \{(1423, 4312)\} \subseteq X_2 \times X_3$
- $Y_{17} = \{(1423, 1432)\}$, $Y_{18} = \{(1423, 2341)\} \subseteq X_2 \times X_4$
- $Y_{19} = \{(1423, 3421)\}$, $Y_{20} = \{(1423, 3142)\} \subseteq X_2 \times X_5$
- $Y_{21} = \{(1423, 4123)\} \subseteq X_2 \times X_6$
- $Y_{22} = \{(1423, 3214)\} \subseteq X_2 \times X_7$
- $Y_{23} = \{(3241, 4231)\}$, $Y_{24} = \{(3241, 4312)\} \subseteq X_3 \times X_3$
- $Y_{25} = \{(3241, 1432)\}$, $Y_{26} = \{(3241, 2341)\} \subseteq X_3 \times X_4$
- $Y_{27} = \{(3241, 3421)\}$, $Y_{28} = \{(3241, 3142)\} \subseteq X_3 \times X_5$
- $Y_{29} = \{(3241, 4123)\} \subseteq X_3 \times X_6$
- $Y_{30} = \{(3241, 3214)\} \subseteq X_3 \times X_7$
- $Y_{31} = \{(2431, 1432)\}$, $Y_{32} = \{(2431, 2341)\} \subseteq X_4 \times X_4$

- $Y_{33} = \{(2431, 3421)\}$, $Y_{34} = \{(2431, 3142)\} \subseteq X_4 \times X_5$
- $Y_{35} = \{(2431, 4123)\} \subseteq X_4 \times X_6$
- $Y_{36} = \{(2431, 3214)\} \subseteq X_4 \times X_7$
- $Y_{37} = \{(4321, 3421)\}$, $Y_{38} = \{(4321, 3142)\} \subseteq X_5 \times X_5$
- $Y_{39} = \{(4321, 4123)\} \subseteq X_5 \times X_6$
- $Y_{40} = \{(4321, 3214)\} \subseteq X_5 \times X_7$
- $Y_{41} = \{(3124, 4123)\} \subseteq X_6 \times X_6$
- $Y_{42} = \{(3124, 3214)\} \subseteq X_6 \times X_7$
- $Y_{43} = \{(4213, 3214)\} \subseteq X_7 \times X_7$

Proof Obviously, R'_2 is an equivalence relation. Similarly to the proof of Lemma 14, we can obtain the above equivalence classes. □

Proof of Lemma 2 Due to edge symmetry of S_n, we assume that the edge e_f is faulty. For a pair of white (resp. black) vertices, we can find an $L^{\rm opt}$-path of length 22 joining them by regarding the black (resp. white) endvertex of e_f as a virtual vertex fault and employing Lemma 1. The path does not pass through the virtual fault, and thus does not pass through e_f. Table 4 shows $L^{\rm opt}$-paths joining 43 pairs of white and black vertices, one per each equivalence class given in Lemma 15. □

A.3 S_4 with two faulty elements

To represent a cycle, a starting vertex is followed by a sequence of edges, represented by their dimension numbers.

Proof of Lemma 3 We assume that $v_f = 3214$ is a faulty vertex. Moreover, we assume the other faulty vertex v'_f is a representative vertex of the equivalence classes given in Lemma 11. The lengths of $L^{\rm opt}$-cycles are 22 (resp. 20) if v'_f is white (resp. black). $L^{\rm opt}$-cycles in faulty S_4 are constructed and shown in Table 5. □

Proof of Lemma 4 We assume that $v_f = 3214$ is a faulty vertex. Let $e_f = (v, w)$ be an arbitrary faulty edge, and let w be a white vertex. We find an $L^{\rm opt}$-cycle of length 22 by regarding w (as well as v_f) as a faulty vertex and employing Lemma 3. The cycle does not pass through e_f (as well as v_f), and thus it is an $L^{\rm opt}$-cycle in the presence of two faulty elements v_f and e_f. □

Proof of Lemma 5 We assume that one of the two faulty edges is $e_f = (3214, 4213)$. Among the 43 equivalence classes given in Lemma 15, twelve are sets of pairs of adjacent vertices, that is, edges. They are Y_3, Y_5, Y_{12}, Y_{19}, Y_{21}, Y_{23}, Y_{26}, Y_{31}, Y_{33}, Y_{37}, Y_{41}, and Y_{43}. Thus, we can assume the other faulty edge e'_f is a representative edge of the equivalence

Table 4: L^{opt}-paths between pairs of white and black vertices in S_4 with $f = f_e = 1$

$v = 1234$	$w = 2413$	3 2 4 2 3 2 4 3 2 4 3 4 2 4 3 2 4 3 2 3 4 3 2
	$w = 1243$	3 2 4 3 2 4 3 2 4 3 2 3 2 3 4 2 3 4 2 3 4 2 3
	$w = 2134$	3 2 4 3 2 3 2 3 4 2 3 2 3 2 4 3 2 3 2 3 4 2 3
	$w = 1324$	3 2 4 3 2 4 2 3 2 4 3 2 4 3 4 2 4 3 2 3 2 3 4
	$w = 4231$	3 2 4 3 4 3 4 2 3 4 3 4 3 2 4 3 4 3 4 2 3 4 3
	$w = 4312$	3 2 3 4 2 4 3 2 3 2 3 4 3 4 3 2 4 2 3 2 4 3 2
	$w = 1432$	3 2 4 3 2 3 4 3 2 4 3 2 3 2 3 4 2 3 2 3 2 4 3
	$w = 2341$	3 2 4 3 2 3 4 3 2 4 2 4 2 3 2 3 2 4 3 4 2 3 2
	$w = 3421$	3 2 4 3 2 3 4 3 2 4 3 2 3 2 4 3 4 2 3 2 3 2 4
	$w = 3142$	3 2 4 3 4 3 4 2 3 4 2 3 4 2 3 4 3 2 3 4 2 3 4
	$w = 4123$	3 2 4 3 2 4 2 3 2 4 3 2 4 3 2 4 3 2 3 2 3 4 2
	$w = 3214$	4 2 4 2 4 3 2 4 2 4 2 3 4 2 4 2 4 3 2 4 2 4 2
$v = 1423$	$w = 2134$	3 2 3 2 3 4 2 4 2 3 4 3 2 4 2 3 4 3 2 4 2 3 4
	$w = 1324$	3 2 3 2 3 4 3 4 3 2 3 2 3 4 3 2 3 2 3 4 3 2 3
	$w = 4231$	3 2 3 2 3 4 2 4 2 3 4 3 2 4 2 3 2 4 3 2 4 2 3
	$w = 4312$	3 2 3 2 3 4 3 4 3 2 4 2 3 4 3 2 3 4 2 3 4 3 2
	$w = 1432$	3 2 3 2 3 4 2 4 2 3 2 3 2 4 2 3 4 2 3 2 4 2 3
	$w = 2341$	3 2 3 2 3 4 3 4 3 2 3 2 3 4 3 2 4 3 2 3 4 3 2
	$w = 3421$	3 2 3 2 3 4 2 3 2 3 2 4 3 2 3 2 3 4 2 3 2 3 2
	$w = 3142$	3 2 3 2 3 4 3 2 3 2 3 4 2 3 2 3 2 4 3 2 3 2 3
	$w = 4123$	3 2 3 2 4 3 2 3 2 3 4 2 3 2 3 2 4 3 2 3 2 3 4
	$w = 3214$	3 2 3 4 2 4 3 2 3 2 3 4 3 4 3 2 4 2 3 2 4 3 2
$v = 3241$	$w = 4231$	4 3 2 4 3 4 3 4 2 3 4 3 4 3 2 4 3 4 3 4 2 3 4
	$w = 4312$	4 3 2 4 3 2 4 3 2 4 3 4 2 4 3 2 4 3 2 4 3 2 4
	$w = 1432$	4 3 2 4 2 3 4 3 2 4 2 3 4 3 4 3 2 3 2 3 4 2 4
	$w = 2341$	4 3 2 4 2 4 2 3 4 2 4 2 4 3 2 4 2 4 2 3 4 2 4
	$w = 3421$	4 3 2 4 2 4 2 3 4 2 4 2 4 3 2 4 3 4 2 4 3 2 4
	$w = 3142$	4 3 2 4 3 2 4 3 2 4 3 4 3 4 2 3 4 2 3 4 2 3 4
	$w = 4123$	4 3 2 3 4 3 2 4 3 2 4 2 3 2 4 3 2 4 3 4 2 4 3
	$w = 3214$	4 3 2 4 2 3 4 3 2 4 2 3 4 3 2 4 2 4 2 3 2 3 2
$v = 2431$	$w = 1432$	3 4 2 3 4 3 4 3 2 4 3 4 3 4 2 3 4 3 4 3 2 4 3
	$w = 2341$	4 3 2 3 2 3 4 3 4 3 2 3 2 3 4 2 4 3 2 3 4 3 2
	$w = 3421$	4 3 4 2 3 4 3 4 3 2 4 3 4 3 4 2 3 4 3 4 3 2 4
	$w = 3142$	4 3 2 3 4 2 3 4 3 2 3 4 2 4 3 2 3 2 3 4 3 4 3
	$w = 4123$	4 3 2 3 2 3 4 3 2 3 2 3 4 3 2 3 2 4 3 2 3 2 3
	$w = 3214$	4 3 2 3 2 3 4 2 4 3 2 3 2 4 3 2 3 2 3 4 2 3 2
$v = 4321$	$w = 3421$	3 2 4 3 2 3 2 3 4 2 3 2 3 2 4 3 2 3 2 3 4 2 3
	$w = 3142$	2 4 2 4 2 3 2 3 2 4 2 4 2 3 2 4 3 2 4 2 3 2 4
	$w = 4123$	2 4 3 2 3 4 2 4 3 2 3 4 2 4 3 2 3 2 3 4 3 4 3
	$w = 3214$	2 4 3 2 3 2 3 4 2 3 4 2 3 4 2 3 2 4 2 3 4 2 3
$v = 3124$	$w = 4123$	3 2 3 2 3 4 2 3 2 3 2 4 3 2 3 2 3 4 2 3 2 3 2
	$w = 3214$	4 3 4 2 3 2 3 2 4 2 4 2 3 4 3 2 3 4 2 3 4 3 2
$v = 4213$	$w = 3214$	2 4 2 3 4 2 4 2 4 3 2 4 2 4 2 3 4 2 4 2 4 3 2

Table 5: L^{opt}-cycles in S_4 with two faulty vertices $v_f = 3214$ and v'_f

$v'_f = 2431$	1234 : 4 3 4 3 2 4 3 2 3 4 3 2 4 2 3 4 3 4 3 2 3 2
$v'_f = 3124$	1234 : 4 2 4 3 4 2 3 4 2 4 3 4 3 4 2 4 2 3 4 3 4 2
$v'_f = 1234$	2314 : 4 3 2 4 2 3 2 4 3 2 4 3 4 2 4 3 2 3 2 3 4 3
$v'_f = 4231$	2314 : 4 3 2 3 4 3 4 2 4 3 4 3 4 2 3 4 2 3 4 3
$v'_f = 2341$	1234 : 4 2 4 3 4 2 3 2 4 2 3 4 3 4 2 4 2 4 3 2
$v'_f = 3421$	1234 : 4 2 4 3 2 3 2 4 3 2 3 2 3 4 2 3 4 2 3 2

Table 6: Hamiltonian cycles in S_4 with two faulty edges $e_f = (3214, 4213)$ and e'_f

$e'_f = (1234, 2134)$	1234 : 4 2 4 2 4 3 2 4 2 4 2 3 4 2 4 2 4 3 2 4 2 4 2 3
$e'_f = (1234, 4231)$	1234 : 2 4 3 2 3 2 3 4 2 3 2 3 2 4 3 2 3 2 3 4 2 3 2 3
$e'_f = (1423, 3421)$	1234 : 2 3 4 2 3 2 3 2 4 3 2 3 2 3 4 2 3 2 3 2 4 3 2 3
$e'_f = (1423, 4123)$	1234 : 2 4 3 2 3 2 3 4 2 3 2 3 2 4 3 2 3 2 3 4 2 3 2 3
$e'_f = (3241, 4231)$	1234 : 4 2 4 2 4 3 2 4 2 4 2 3 4 2 4 2 4 3 2 4 2 4 2 3
$e'_f = (3241, 2341)$	1234 : 4 3 4 3 2 4 3 4 3 4 2 3 4 3 4 3 2 4 3 4 3 4 2 3
$e'_f = (2431, 1432)$	1234 : 4 2 3 2 3 2 4 3 2 3 2 3 4 2 3 2 3 2 4 3 2 3 2 3
$e'_f = (2431, 3421)$	1234 : 4 2 4 2 4 3 2 4 2 4 2 3 4 2 4 2 4 3 2 4 2 4 2 3
$e'_f = (4321, 3421)$	1234 : 4 3 4 3 2 4 3 4 3 4 2 3 4 3 4 3 2 4 3 4 3 4 2 3
$e'_f = (3124, 4123)$	1234 : 4 2 4 2 4 3 2 4 2 4 2 3 4 2 4 2 4 3 2 4 2 4 2 3

classes. We need not consider Y_{43} since Y_{43} has only one member e_f. When e'_f is the representative edge of Y_{12}, both e_f and e'_f are incident to vertex 3214, and thus we can construct a cycle of length 22 by regarding 3214 and an arbitrary white vertex, say 4213, as two faulty vertices and utilizing Lemma 3. For the other cases, hamiltonian cycles in S_4 with two faulty edges e_f and e'_f are listed in Table 6. □