International Journal of Computer Mathematics
Vol. xx, No. xx, Month 20xx, 1–12

RESEARCH ARTICLE

General-Demand Disjoint Path Covers in a Graph with Faulty Elements

Jae-Ha Lee[a] and Jung-Heum Park[b*]

[a] *Computer Engineering Department, Konkuk University, Seoul 143-701, Korea*; [b]*School of Computer Science and Information Engineering, The Catholic University of Korea, Bucheon 420-743, Korea*

(*revised 24 September 2011*)

A k-disjoint path cover of a graph is a set of k internally vertex-disjoint paths which cover the vertex set with k paths and each of which runs between a source and a sink. Given that each source and sink v is associated with an integer-valued demand $d(v) \geq 1$, we are concerned with *general-demand k-disjoint path cover* in which every source and sink v is contained in the $d(v)$ paths. In this paper, we present a reduction of a general-demand disjoint path cover problem to an unpaired many-to-many disjoint path cover problem, and obtain some results on disjoint path covers of restricted HL-graphs and proper interval graphs with faulty vertices and/or edges.

Keywords: Disjoint paths, Menger's theorem, Fan lemma, hypercube-like graphs, proper interval graphs.

AMS Subject Classification: 05C38; 05C40; 05C45; 68R10; 90B18.

1. Introduction

An interconnection network is often modeled as a graph, in which vertices and edges correspond to nodes and communication links, respectively. One of the central issues in interconnection networks is finding (vertex-)disjoint paths concerned with the routing among nodes and the embedding of linear arrays. Vertex-disjoint paths can be used as parallel paths for an efficient data routing among nodes. Disjoint paths can be categorized as three types: one-to-one type deals with the disjoint paths joining a single source and a single sink, one-to-many type considers the disjoint paths joining a single source and k distinct sinks, and many-to-many type deals with the disjoint paths joining k distinct sources and k distinct sinks. A path means a simple path in this paper.

The connectivity of an interconnection network corresponds to its reliability (or fault-tolerance) which is subject to node failures. According to Menger's theorem, a graph G is k-connected if and only if every pair of source s and sink t are joined by k internally disjoint paths of type one-to-one. So-called Fan Lemma states that a graph G is k-connected if and only if G has k internally disjoint paths of type one-to-many joining every source s and k distinct sinks $t_1, t_2, \ldots, t_k$ such that $t_i \neq s$ for all i [1]. Moreover, a graph G is k-connected if and only if G has k disjoint paths of type many-to-many joining any k distinct sources $s_1, s_2, \ldots, s_k$

*Corresponding author. Email: j.h.park@catholic.ac.kr

ISSN: 0020-7160 print/ISSN 1029-0265 online

DOI: 0020716YYxxxxxxxx
http://www.informaworld.com

and k distinct sinks t_1, t_2, …, t_k, where if a source s_i coincides with a sink t_j, s_i (or t_j) itself is considered as a path. In other words, letting $S = \{s_1, s_2, \ldots, s_k\}$ and $T = \{t_1, t_2, \ldots, t_k\}$, the condition says that $S = T$ or $G \setminus X$ has k' disjoint paths of type many-to-many joining $S \setminus X$ and $T \setminus X$, where $X = S \cap T$ and $k' = k - |X|$.

All of the three types of disjoint paths can be accommodated with the covering of vertices in the graph. A *k-disjoint path cover* (*k-DPC* for short) of a graph G is a set of k internally disjoint paths containing all the vertices in G. The k-DPC problem that originated from an interconnection network is concerned with the application where the full utilization of nodes is important [14]. The k-DPC problem can also be categorized into three types: one-to-one, one-to-many, and many-to-many. The many-to-many type is further subdivided into two subtypes: paired and unpaired. In the *paired* type problem, each source s_i is required to be paired to a designated sink t_i. In the *unpaired* type problem, on the other hand, the sources and sinks are allowed to be freely mapped. In other words, source s_i can be freely matched to sink t_{σ_i} under an arbitrary permutation σ on $\{1, 2, \ldots, k\}$.

Several types of graphs have been studied on their disjoint path covers. One-to-one DPC's in recursive circulants [10, 17] and hypercubes with faulty edges [2] were investigated. A one-to-one k-DPC is also known as a k^*-container [2, 17]. In hypercube-like interconnection networks, called *restricted HL-graphs*, with faulty vertices and/or edges, one-to-many DPC's [11], paired DPC's [14, 15], and unpaired DPC's [12] were constructed. The paired 2-DPC consisting of two paths of equal length was suggested in [8]. The disjoint path cover problem has also been studied for some bipartite graphs: paired DPC's for hypercubes [7] and for hypercubes with faulty vertices [6]; unpaired DPC's for hypercubes [3, 9] and for bipartite graphs obtained by adding some edges to hypercubes [4].

In this paper, we present the following framework, which generalizes the above-mentioned three DPC problems: one-to-one, one-to-many, and unpaired many-to-many DPC problems (excluding the paired one). We formally define our general-demand k-DPC problem in a graph G. Let $S = \{s_1, s_2, \ldots, s_{k'}\}$ denote a nonempty set of sources and let $T = \{t_1, t_2, \ldots, t_{k''}\}$ denote a nonempty set of sinks such that $S, T \subset V(G)$ and $S \cap T = \emptyset$. Each source and sink is associated with an integer-valued demand $d(\cdot) \geq 1$ such that $\sum_{s_j \in S} d(s_j) = \sum_{t_j \in T} d(t_j) = k$. A *general-demand k-DPC* joining S and T is a set of k internally disjoint paths P_i joining a source in S and a sink in T, $1 \leq i \leq k$, such that (a) $\bigcup_{1 \leq i \leq k} V(P_i) = V(G)$ and (b) $\sum_{1 \leq i \leq k} I(s_j, P_i) = d(s_j)$ and $\sum_{1 \leq i \leq k} I(t_j, P_i) = d(t_j)$, where $I(x, P_i)$ is a 0/1-variable indicating whether $x \in V(P_i)$. Here $V(P_i)$ denotes the vertex set of P_i. In a graph G with a set F of faulty elements, where $F \subset V(G) \cup E(G)$, a general-demand k-DPC joining S and T such that $S, T \subset V(G) \setminus F$ is defined as a general-demand k-DPC of $G \setminus F$ joining S and T. Such a disjoint path cover is denoted by k-DPC$[S, T|G, F]$.

Given S and T in a graph, the problem of determining whether there exists a general-demand k-DPC between S and T is NP-complete for any fixed $k \geq 1$, since the problem of determining one-to-one k-DPC, one-to-many k-DPC, and many-to-many k-DPC, regardless of paired or unpaired type, are all NP-complete for any fixed $k \geq 1$ [14, 15]. In this paper, we consider a graph which has a general-demand k-DPC for arbitrary set of sources and arbitrary set of sinks rather than fixed sources and sinks, which is called a general-demand k-disjoint path coverable graph.

Definition 1.1 *A graph G is called* f-fault general-demand k-disjoint path coverable *if $f + 2k \leq |V(G)|$ and for any fault set F with $|F| \leq f$, G has a k-DPC$[S, T|G, F]$ for any set S of sources and any set T of sinks contained in $V(G) \setminus F$ such that $S \cap T = \emptyset$ and $\sum_{s_j \in S} d(s_j) = \sum_{t_j \in T} d(t_j) = k$.*

In this paper, we will develop a reduction of an f-fault general-demand k-DPC problem into an f'-fault unpaired many-to-many k'-DPC problem for some $f' \geq f$ and $k' \leq k$ with $f' + k' = f + k$. In case when there exists a unique source (or symmetrically, a unique sink), the general-demand k-DPC problem is referred to as the *single-source k-DPC* problem. Reduction of an f-fault single-source k-DPC problem to an f-fault one-to-many k-DPC problem will also be addressed. By applying our reductions to restricted HL-graphs, we obtain that every m-dimensional restricted HL-graph, $m \geq 3$, is f-fault general-demand k-disjoint path coverable for any f and $k \geq 1$ with $f + k \leq m - 2$. Furthermore, the graph is f-fault single-source k-disjoint path coverable for any f and $k \geq 2$ with $f + k \leq m - 1$. For proper interval graphs, a necessary and sufficient condition for the graphs to be unpaired k-disjoint path coverable is derived. Using the characterization, we show that for an integer $B \geq 2$, a proper interval graph is f-vertex-fault general-demand k-disjoint path coverable for any f and $k \geq 2$ with $f + k \leq B$ if any only if it is $(B + 1)$-connected.

This paper is organized as follows. We will discuss about transformation of a general-demand DPC problem in Section 2 and transformation of a single-source DPC problem in Section 3. The general-demand/single-source DPC problems in restricted HL-graphs and proper interval graphs will be considered in Sections 4 and 5, respectively. Finally, in Section 6, concluding remarks will be mentioned.

2. Reduction of the General-Demand DPC Problem

A graph G is called *f-fault unpaired (many-to-many) k-disjoint path coverable* if $f + 2k \leq |V(G)|$ and for any set F of faulty elements with $|F| \leq f$, G has an unpaired k-DPC for any set S of k sources and any set T of k sinks in $G \setminus F$ such that $S \cap T = \emptyset$. The sources and sinks are called *terminals* in general. A vertex v is called *free* if v is fault-free and not a terminal. An edge (v, w) is called *free* if v and w are free and $(v, w) \notin F$. A path in a graph is represented as a sequence of vertices. A v-w path refers to a path from vertex v to w, and a v-path refers to a path whose starting vertex is v.

Suppose a graph G has a general-demand k-DPC between S and T. Let $\mathcal{P}$ be a disjoint path cover in $G \setminus F$ joining S and T. For any terminal, say source s_i, there are $d(s_i)$ paths in $\mathcal{P}$ joining s_i and some sinks. Suppose $d(s_i) \geq 2$. Among the paths, let $(s_i, w, \ldots, t_j)$ be an arbitrary s_i-path. Since the vertex w is neither a faulty vertex nor a source, w must be either a free vertex or a sink. Moreover, (s_i, w) must be fault-free, and whenever w is a sink, w must be equal to t_j and the path must be (s_i, t_j).

Let $D(G)$ denote $\sum_{s_i \in S}\{d(s_i) - 1\} + \sum_{t_j \in T}\{d(t_j) - 1\}$, that is, the sum of surplus demands over all terminals. Based on the above observation, we may construct a general-demand DPC in a graph G from a general-demand DPC in the graph with smaller sum of surplus demands as follows. Let w be "some" vertex adjacent to s_i via a fault-free edge such that w is a free vertex or a sink.

(1) If w is a free vertex, we regard it as a *virtual* source with unit demand and reduce the demand of s_i by one. And then, find a general-demand k-DPC, if any, and replace the w-path with (s_i, w-path). For example, a 3-DPC of the graph in Figure 1(a) can be obtained from the 3-DPC of Figure 1(b) by replacing (v_7, v_6, v_5) with (v_0, v_7, v_6, v_5). The symbol $\times$ on a vertex or on an edge in the figures indicates that the corresponding element is faulty. The demand of a terminal is in parenthesis.

(2) If w is a sink with unit demand, we regard it as a *virtual* fault and reduce

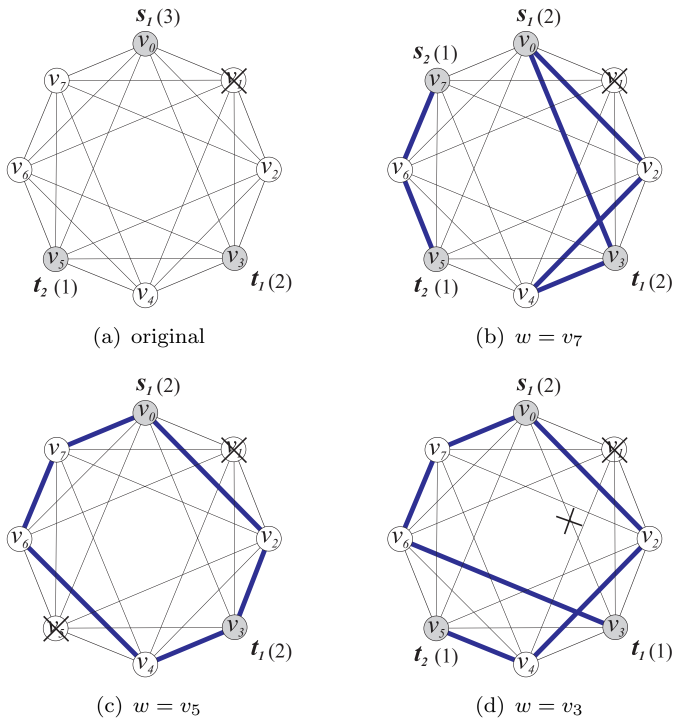

Figure 1. Reduction to a DPC problem with smaller sum of surplus demands. (a) $F = \{v_1\}$; (b) $F' = \{v_1\}$ and $\mathcal{P}' = \{(v_0, v_3), (v_0, v_2, v_4, v_3), (v_7, v_6, v_5)\}$; (c) $F' = \{v_1, v_5\}$ and $\mathcal{P}' = \{(v_0, v_2, v_3), (v_0, v_7, v_6, v_4, v_3)\}$; (d) $F' = \{v_1, (v_0, v_3)\}$ and $\mathcal{P}' = \{(v_0, v_2, v_4, v_5), (v_0, v_7, v_6, v_3)\}$.

$d(s_i)$ by one. And then, find a general-demand $(k-1)$-DPC, if any, and add a path (s_i, w) to the $(k-1)$-DPC. For example, a 3-DPC of Figure 1(a) can be obtained from the 2-DPC of Figure 1(c).

(3) If w is a sink with demand two or greater, we regard the edge (s_i, w) as a *virtual* fault and reduce both $d(s_i)$ and $d(w)$ by one. And then, find a general-demand $(k-1)$-DPC, if any, and add a path (s_i, w) to the $(k-1)$-DPC. For example, a 3-DPC of Figure 1(a) can be obtained from the 2-DPC of Figure 1(d).

It is obvious that at least one of the above three are always applicable provided G has a general-demand k-DPC and w is chosen carefully. The difficulty here is how to pick up such a "proper" vertex w. It might not always be sufficient to pick up w in an arbitrary manner.

On the other hand, we are concerned with the problem of determining whether a graph G is f-fault general-demand k-disjoint path coverable for any f and $k \geq 1$ with $f + k \leq B$ for some bound B. For the graph G to have a positive answer, it is necessary that for any f and $k \geq 1$ with $f + k \leq B$, G has an f-fault unit-demand k-DPC, or equivalently, G is f-fault unpaired many-to-many k-disjoint path coverable. A necessary condition for a graph to have an f-fault unpaired k-DPC was given in [15] as follows.

Lemma 2.1 *[15] If a graph G is f-fault unpaired many-to-many $k(\geq 2)$-disjoint path coverable, then $\delta(G) \geq \kappa(G) \geq f + k$, where $\delta(G)$ is the minimum degree in G and $\kappa(G)$ is the connectivity of G. Furthermore, if G has $f + 2k + 1$ or more vertices, then $\delta(G) \geq f + k + 1$.*

Let us revisit the above transformation of a general-demand DPC problem into a general-demand DPC problem with smaller sum of surplus demands. For the first case of w being a free vertex, the number f of faults and the total demand k of sources remain unchanged. For the second and third cases of w being a sink, the number of faults (including virtual faults) is increased from f to $f+1$ and the total demand of sources are decreased from k to $k-1$. In all cases, the number of faults plus the total demand of sources remains unchanged.

Let G be f-fault unpaired k-disjoint path coverable for any f and $k \geq 1$ with $f+k \leq B$ for some B. It holds $B \leq \delta(G)$ by Lemma 2.1. To reach a conclusion that G is also f-fault general-demand k-disjoint path coverable for any f and $k \geq 1$ with $f+k \leq B$, we will show that an f-fault general-demand k-DPC problem with $f+k \leq B$ reduces to an f'-fault unit-demand k'-DPC problem for some $f' \geq f$ and $k' \leq k$ such that $f'+k' = f+k \leq B$. For this reduction, we present a general-demand DPC algorithm employing an unpaired DPC algorithm. Under the condition of unpaired disjoint path coverability of G, the aforementioned difficulty of picking up a proper vertex w is resolved and it suffices to pick up an arbitrary vertex as follows.

Algorithm for the general-demand DPC problem
/* It is assumed that G is f-fault unpaired k-disjoint path coverable for any f and $k \geq 1$ with $f+k \leq B$ for some $B \leq \delta(G)$. */

(1) If $D(G) = 0$, then find an unpaired k-DPC$[S, T|G, F]$ and return the set $\mathcal{P}$ of disjoint paths.
(2) Otherwise, pick up any terminal, say source s_i, with demand two or greater. Let w be an arbitrary vertex adjacent to s_i such that $(s_i, w) \notin F$ and $w \in V \setminus (S \cup F)$.
  a) Case when w is a free vertex: Decrement $d(s_i)$ by one. Let $S' := S \cup \{w\}$ and $d(w) := 1$. Find $\mathcal{P} := k$-DPC$[S', T|G, F]$ and return $\mathcal{P} \cup \{(s_i, P_w)\} \setminus P_w$, where P_w is the w-path in $\mathcal{P}$.
  b) Case when w is a sink with $d(w) = 1$: Decrement $d(s_i)$ by one. Let $T' := T \setminus w$ and $F' := F \cup \{w\}$. Find $\mathcal{P} := (k-1)$-DPC$[S, T'|G, F']$ and return $\mathcal{P} \cup \{(s_i, w)\}$.
  c) Case when w is a sink with $d(w) \geq 2$: Decrement both $d(s_i)$ and $d(w)$ by one. Let $F' := F \cup \{(s_i, w)\}$. Find $\mathcal{P} := (k-1)$-DPC$[S, T|G, F']$ and return $\mathcal{P} \cup \{(s_i, w)\}$.

LEMMA 2.2 *There exists a vertex w adjacent to a terminal, say a source s_i, with demand two or greater such that $(s_i, w) \notin F$ and $w \in V \setminus (S \cup F)$.*

Proof Suppose, for a contradiction, that such vertex w does not exist, that is, for any vertex v adjacent to s_i, (i) $v \in S$, (ii) $v \in F$, or (iii) $(s_i, v) \in F$. The number of sources adjacent to s_i (which satisfies condition (i)) is at most $k-2$ since $d(s_i) \geq 2$. Furthermore, the number of vertices v adjacent to s_i which satisfies conditions (ii) or (iii) is at most f. Thus, the total number of vertices adjacent to s_i satisfying (i), (ii), or (iii) is at most $f+k-2$, which implies the degree $\delta(s_i)$ of s_i is at most $f+k-2$ and $\delta(G) \leq f+k-2$. This contradicts to the necessity of $f+k \leq \delta(G)$. ■

LEMMA 2.3 *Let G be an f-fault unpaired k-disjoint path coverable graph for any f and $k \geq 1$ with $f+k \leq B$ for some $B \leq \delta(G)$. Then, the f-fault general-demand k-DPC problem with $f+k \leq B$ reduces to the f'-fault unpaired k'-DPC problem for some $f' \geq f$ and $k' \leq k$ with $f'+k' = f+k$.*

Proof The proof is by induction on the sum of surplus demands over all terminals, $D(G)$. Due to Lemma 2.2, at least one of the three Cases 2(a), 2(b), and 2(c) in Step 2 of the algorithm is applicable. Notice that in Step 2, the number of faults plus the total demand of sources remains unchanged. Moreover, the f-fault general-demand k-DPC problem reduces to either the f-fault general-demand k-DPC problem or the $(f+1)$-fault general-demand $(k-1)$-DPC problem which have smaller sum of surplus demands, and eventually reduces to the f'-fault unpaired k'-DPC problem for some f' and k' with $f'+k' = f+k$. This completes the proof. ■

THEOREM 2.4 *Let G be an f-fault unpaired k-disjoint path coverable graph for any f and $k \geq 1$ with $f+k \leq B$ for some $B \leq \delta(G)$. Then, G is f-fault general-demand k-disjoint path coverable for any f and $k \geq 1$ with $f+k \leq B$.*

3. Reduction of the Single-Source DPC Problem

A graph G is called *f-fault one-to-many k-disjoint path coverable* if $f+k+1 \leq |V(G)|$ and for any set F of faulty elements with $|F| \leq f$, G has a one-to-many k-DPC for any source s and any set T of k sinks in $G \setminus F$ such that $s \notin T$. We begin with a necessary condition for a graph to be f-fault one-to-many k-disjoint path coverable.

THEOREM 3.1 *If a graph G is f-fault one-to-many k-disjoint path coverable, then $\kappa(G) \geq f+k$. Furthermore, if G has $f+k+2$ or more vertices, then $\kappa(G) \geq f+k+1$.*

Proof Suppose $\kappa(G) \leq f+k-1$. We will show that for some source s, set T of k sinks, and fault set F with $|F| \leq f$, even disjoint paths of type one-to-many joining s and T do not exist (irrespective of covering $G \setminus F$). We claim G is not a complete graph; suppose otherwise, then $|V(G)| \geq f+k+1$ by definition and thus $\kappa(G) \geq f+k$, which is a contradiction. Thus, there exists a vertex cut C of size $f+k-1$ or less. Let X and Y be the vertex sets of two distinct connected components of $G \setminus C$. In case when $s \in X$, $C \subsetneq F \cup T$, and $T \cap Y \neq \emptyset$, there exists no path joining s and a sink t_j of Y in $G \setminus (F \cup T \setminus t_j)$. This implies that $G \setminus F$ does not have disjoint paths between s and T of type one-to-many. Thus, we have $\kappa(G) \geq f+k$. Now, let $|V(G)| \geq f+k+2$. Suppose, for a contradiction, $\kappa(G) \leq f+k$. It follows that $\kappa(G) = f+k$ and G is not a complete graph. Thus, G has a vertex cut C of size $f+k$ and $G \setminus C$ has at least two connected components. Let s be contained in one component, and let x be an arbitrary vertex contained in the other component. In case when $C = F \cup T$, no fault-free path joining s and a sink can pass through x. Hence, the proof is completed. ■

Let $D'(G)$ denote $\sum_{t_j \in T}\{d(t_j)-1\}$, the sum of surplus demands over all sinks. Similar to the reduction addressed in Section 2, we assume that G is an f-fault one-to-many k-disjoint path coverable graph, and show that the f-fault single-source k-DPC problem in G reduces to the f-fault single-source k-DPC problem with smaller sum of surplus demands over all sinks, and eventually reduces to the f-fault one-to-many k-DPC problem. For our purpose, it is sufficient to pick up an arbitrary free vertex w adjacent via a fault-free edge to a sink t_j with demand two or greater and regard it as a *virtual* sink with unit demand.

Algorithm for the single-source DPC problem
/* It is assumed that G is f-fault one-to-many k-disjoint path coverable. */

(1) If $D'(G) = 0$, find a one-to-many k-DPC$[s, T|G, F]$ and return the set $\mathcal{P}$ of disjoint paths.
(2) Otherwise, let t_j be any sink with demand two or greater. Pick up an arbitrary free vertex w adjacent to t_j via a fault-free edge.

(3) Decrement $d(t_j)$ by one. Let $T' := T \cup \{w\}$ and $d(w) := 1$. Find $\mathcal{P} := k$-DPC$[s, T'|G, F]$ and return $\mathcal{P} \cup \{(P_w, t_j)\} \setminus P_w$, where P_w is the s-w path in $\mathcal{P}$.

Lemma 3.2 *Let G be an f-fault one-to-many k-disjoint path coverable graph. Then, the f-fault single-source k-DPC problem reduces to the f-fault one-to-many k-DPC problem.*

Proof We claim that there exists a free vertex w adjacent to a sink t_j with $d(t_j) \geq 2$ such that $(t_j, w) \notin F$. Suppose, for a contradiction, that for every vertex v adjacent to t_j, (i) v is a terminal, (ii) $v \in F$, or (iii) $(t_j, v) \in F$. The number of terminals adjacent to t_j is at most $k-1$ (one source and $k-2$ sinks), and the number of vertices v adjacent to t_j which satisfies conditions (ii) or (iii) is at most f. Thus, the total number of vertices adjacent to t_j satisfying (i), (ii), or (iii) is at most $f + k - 1$, which implies $\delta(t_j) \leq f + k - 1$. This contradicts to the necessary condition of $f + k \leq \delta(G)$ given in Theorem 3.1. Thus, the claim is proved. It is straightforward to see by induction on the sum of surplus demands $D'(G)$ over all sinks that the f-fault single-source k-DPC problem eventually reduces to the f-fault one-to-many k-DPC problem. The proof is completed. ■

Theorem 3.3 *An f-fault one-to-many k-disjoint path coverable graph is f-fault single-source k-disjoint path coverable.*

4. Disjoint Path Covers in Restricted HL-Graphs

For given two graphs G_0 and G_1 having the same number of vertices, we denote by $G_0 \oplus G_1$ an arbitrary graph whose vertex set is $V(G_0) \cup V(G_1)$ and edge set is $E(G_0) \cup E(G_1) \cup E_2$, where $E_2 = \{(v, \phi(v)) : v \in V(G_0)$ and $\phi : V(G_0) \to V(G_1)$ is a bijection$\}$. The classes of *hypercube-like graphs* (HL-graphs for short), introduced by Vaidya *et al.* [18], are recursively defined as follows: $HL_0 = \{K_1\}$ and $HL_m = \{G_0 \oplus G_1 : G_0, G_1 \in HL_{m-1}\}$ for $m \geq 1$. Then, $HL_1 = \{K_2\}$; $HL_2 = \{C_4\}$; $HL_3 = \{Q_3, G(8,4)\}$, where C_4 is a cycle graph with four vertices, Q_3 is the 3-dimensional hypercube, and $G(8,4)$ is a recursive circulant whose vertex set is $\{v_0, v_1, \ldots, v_7\}$ and edge set is $\{(v_i, v_j) : i+1 \text{ or } i+4 \equiv j \pmod 8\}$.

The *restricted HL-graphs* is a subclass of nonbipartite HL-graphs, which is defined recursively as follows [13]: $RHL_3 = HL_3 \setminus Q_3 = \{G(8,4)\}$; $RHL_m = \{G_0 \oplus G_1 : G_0, G_1 \in RHL_{m-1}\}$ for $m \geq 4$. A graph in RHL_m is called an m-dimensional restricted HL-graph. Many of the non-bipartite hypercube-like networks such as crossed cube, Möbius cube, twisted cube, multiply twisted cube, Mcube, generalized twisted cube, locally twisted cube, etc. proposed in the literature are indeed restricted HL-graphs. Fault-hamiltonicity of restricted HL-graphs was studied in [13] as follows. A graph G is called *f-fault hamiltonian* (resp. *f-fault hamiltonian-connected*) if there exists a hamiltonian cycle (resp. if each pair of vertices are joined by a hamiltonian path) in $G \setminus F$ for any set F of faulty elements with $|F| \leq f$.

Lemma 4.1 *[13] Every m-dimensional restricted HL-graph, $m \geq 3$, is $(m-3)$-fault hamiltonian-connected and $(m-2)$-fault hamiltonian.*

General-demand disjoint path coverability of restricted HL-graphs is a direct consequence of Theorem 2.4 and the following theorem on unpaired disjoint path coverability.

Theorem 4.2 *[12] Every m-dimensional restricted HL-graph, $m \geq 3$, is f-fault unpaired many-to-many k-disjoint path coverable for any f and $k \geq 1$ with $f + k \leq m - 2$.*

Corollary 4.3 *Every m-dimensional restricted HL-graph, $m \geq 3$, is f-fault general-demand k-disjoint path coverable for any f and $k \geq 1$ with $f + k \leq m - 2$.*

Now, we consider the problem of constructing single-source disjoint path covers in restricted HL-graphs. Due to Theorem 3.3, it suffices to construct one-to-many disjoint path covers. We begin by pointing out the fact that a graph is f-fault one-to-many 2-disjoint path coverable if and only if it is f-fault one-to-many 1-disjoint path coverable. Utilizing fault-hamiltonicity of m-dimensional restricted HL-graphs given in Lemma 4.1, an f-fault one-to-many k-DPC for $k = 1, 2$ can be constructed when $f \leq m - 3$. It was shown in [11] that an f-fault one-to-many k-DPC in $G_0 \oplus G_1$ can be constructed by using f-fault one-to-many $(k - 1)$-DPC and fault-hamiltonicity of G_i, $i = 0, 1$, as follows.

Lemma 4.4 *[11] For $f \geq 0$ and $k \geq 3$, let G_i be a graph with n vertices satisfying the following three conditions, $i = 0, 1$: (a) G_i is f-fault one-to-many $(k-1)$-disjoint path coverable, (b) G_i is $(f + k - 3)$-fault hamiltonian-connected (one-to-many 2-disjoint path coverable), and (c) G_i is $(f + k - 2)$-fault hamiltonian. Then, $G_0 \oplus G_1$ is f-fault one-to-many k-disjoint path coverable.*

Lemmas 4.1 and 4.4 lead to one-to-many disjoint path coverability of restricted HL-graphs.

Theorem 4.5 *Every m-dimensional restricted HL-graph, $m \geq 3$, is f-fault one-to-many k-disjoint path coverable for any f and $k \geq 2$ with $f + k \leq m - 1$.*

Proof The proof is by induction on m. For $m = 3$, the theorem holds true by Lemma 4.1. Let $m \geq 4$ and let $G = G_0 \oplus G_1$ be an m-dimensional restricted HL-graph, where G_0 and G_1 are $(m - 1)$-dimensional restricted HL-graphs. If $k = 2$, then $f \leq m - 3$ and by Lemma 4.1, G is f-fault one-to-many 2-disjoint path coverable. Assume $k \geq 3$. Since $f + k \leq m - 1$, each G_i is (i) f-fault one-to-many $(k - 1)$-disjoint path coverable by induction hypothesis, (ii) $(f + k - 3)$-fault hamiltonian-connected by Lemma 4.1, and (iii) $(f + k - 2)$-fault hamiltonian by Lemma 4.1. Thus, by Lemma 4.4, G is f-fault one-to-many k-disjoint path coverable. This completes the proof. ∎

It is worthy of remark that the bound of $f + k \leq m - 1$ achieved in Theorem 4.5 is optimal due to the necessary condition of Theorem 3.1. Note that the connectivity of an m-dimensional restricted HL-graph is m.

Corollary 4.6 *Every m-dimensional restricted HL-graph, $m \geq 3$, is f-fault single-source k-disjoint path coverable for any f and $k \geq 2$ with $f + k \leq m - 1$.*

5. Disjoint Path Covers in Proper Interval Graphs

An *interval graph* is the intersection graph of a family of intervals on the real line, where two vertices are connected with an edge if and only if their corresponding intervals intersect. It is a *proper interval graph* if no interval in the family properly contains another. Due to [16], proper interval graphs are also referred to in the literature as *unit interval graphs*, the intersection graphs of unit-length intervals on the real line.

An ordering $(v_1, v_2, \ldots, v_n)$ of the vertices of a graph G is a *consecutive ordering* if for any vertex v_i, its closed neighbor $N[v_i]$ is consecutive, i.e., $N[v_i] = \{v_j : l_i \leq j \leq r_i\}$ for some l_i and r_i, where $N[v_i]$ is the set of vertices adjacent to v_i plus v_i itself. A graph G is said to be *k-connected* if $\kappa(G) \geq k$.

Lemma 5.1 *[5] (a) A graph G is a proper interval graph if and only if G has a* *consecutive ordering. (b) For any positive integer k and any proper interval graph G of $n \geq k+1$ vertices with a consecutive ordering $(v_1, v_2, \ldots, v_n)$, G is k-connected if and only if $(v_i, v_j) \in E(G)$ whenever $1 \leq |i-j| \leq k$.*

We are to characterize unpaired k-disjoint path coverable proper interval graphs. Recall the necessary condition of Lemma 2.1 saying that when there are no faults, a graph G should be k-connected and if $|V(G)| \geq 2k+1$, then $\delta(G) \geq k+1$.

Lemma 5.2 *Let G be a proper interval graph with a consecutive ordering $(v_1, v_2, \ldots, v_n)$. If $n \geq 2k$ and G is k-connected, then for any set S of k sources and set T of k sinks such that $S \cap T = \emptyset$ and $v_1, v_n \in S \cup T$, there exists an unpaired k-DPC joining S and T.*

Proof The proof is by induction on n. In the base case of $n = 2k$, every vertex is a terminal. The consecutive ordering can be seen as a shuffle of source sequence $(s_1, s_2, \ldots, s_k)$ and sink sequence $(t_1, t_2, \ldots, t_k)$. That is, we can assume wlog that $p < q$ if $s_i = v_p$ and $s_{i+1} = v_q$ for $1 \leq i < k$ and that $p < q$ if $t_i = v_p$ and $t_{i+1} = v_q$ for $1 \leq i < k$. Then, $(s_i, t_i) \in E(G)$ for every i since, assuming wlog $s_i = v_p$ and $t_i = v_q$ with $p < q$, the set of vertices $\{v_j : p < j < q\}$ to the right of v_p and to the left of v_q should be a subset of $\{s_j : j > i\} \cup \{t_j : j < i\}$ of cardinality $k-1$. Therefore, we have an unpaired k-DPC $\mathcal{P} = \{(s_i, t_i) : 1 \leq i \leq k\}$.

Now, let $n \geq 2k+1$. We claim that there exists a nonterminal v_j such that (v_1, v_j) or $(v_j, v_n) \in E(G)$. Suppose, for a contradiction, that such nonterminal v_j does not exist. Then, there are k contiguous terminals $v_2, v_3, \ldots, v_{k+1}$ adjacent to v_1 and there are k contiguous terminals $v_{n-k}, v_{n-k+1}, \ldots, v_{n-1}$ adjacent to v_n. Furthermore, $k+1 < n-k$ due to the existence of a nonterminal. This implies there are at least $2k+2$ terminals (including v_1 and v_n), which is a contradiction. Assume wlog that v_j is a nonterminal adjacent to v_1 and v_1 is a source. Regarding v_j as a *virtual* source, we find an unpaired k-DPC $\mathcal{P}$ in the subgraph induced by $V(G) \setminus v_1$. Then, an unpaired k-DPC of G can be obtained from $\mathcal{P}$ by replacing the v_j-path with $(v_1, v_j$-path). The proof is completed. ■

Lemma 5.3 *Let G be a proper interval graph with a consecutive ordering $(v_1, v_2, \ldots, v_n)$. Then, G has a v_1-v_2 hamiltonian path if either $n = 2$ and G is connected or $n \geq 3$ and G is 2-connected.*

Proof We proceed by induction on n. For the base case of $n = 2$, G has an obvious v_1-v_2 hamiltonian path. Let $n \geq 3$ and G be 2-connected. The subgraph induced by $V(G) \setminus v_1$ satisfies the condition of this lemma, and thus there exists v_2-v_3 hamiltonian path P_h in the subgraph. Since (v_1, v_3) is an edge of G, (v_1, P_h^R) is a desired path, where P_h^R is the reverse of P_h, i.e., v_3-v_2 hamiltonian path of the subgraph. ■

Theorem 5.4 *Let G be a proper interval graph with a consecutive ordering $(v_1, v_2, \ldots, v_n)$. Then, G is unpaired k-disjoint path coverable for $k \geq 2$ if and only if G is k-connected and either $n = 2k$ or $n \geq 2k+1$ and $(v_i, v_{i+k+1}) \in E(G)$ for every i, $1 \leq i \leq n-2k$ or $k \leq i \leq n-k-1$.*

Proof Sufficiency. Let v_l and v_r respectively be the leftmost and rightmost terminals so that $S \cup T \subseteq X$ where $X := \{v_j : l \leq j \leq r\}$. Since the subgraph of G induced by X is k-connected, there exists a minimally k-connected subgraph G' whose vertex set is X and whose edge set is $\{(v_i, v_j) : 1 \leq |i-j| \leq k\}$. By Lemma 5.2, there exists an unpaired k-DPC $\mathcal{P}'$ of G' joining S and T. Now, we will extend v_l-path in $\mathcal{P}'$ to pass through all the vertices in $L := \{v_j : j < l\}$. Assume $L \neq \emptyset$; otherwise we are done. Let the v_l-path be (v_l, v_p, P') for some subpath P', where $p \leq l+k$. Then (v_{l-1}, v_p) is also an edge of G by the condition of this theorem. (Note that $p \leq (l-1)+k$ or $p = (l-1)+k+1$ and $1 \leq l-1 \leq n-2k$.) If $|L| = 1$, we replace the v_l-path with (v_l, v_{l-1}, v_p, P'). If $|L| \geq 2$, we replace the v_l-path with (v_l, P_L, v_p, P'), where P_L is v_{l-2}-v_{l-1} hamiltonian path in the subgraph induced by L from Lemma 5.3. Similarly, we can also extend the v_r-path to pass through all the vertices in $R := \{v_j : j > r\}$.

Necessity. The k-connectivity condition is from Lemma 2.1. For $n \geq 2k+1$, suppose there exists i, $1 \leq i \leq n-2k$ or $k \leq i \leq n-k-1$, such that $(v_i, v_{i+k+1}) \notin E(G)$. We show there exists S and T such that G has no unpaired k-DPC joining S and T. Let $S := \{v_j : i+1 \leq j \leq i+k\}$. Let $T := \{v_j : i+k+1 \leq j \leq i+2k\}$ if $1 \leq i \leq n-2k$; let $T := \{v_j : i-k+1 \leq j \leq i\}$ if $k \leq i \leq n-k-1$. Then, $G \setminus S$ has a connected component which contains no sink. This implies there exists no path joining a source s_j and a sink in $G \setminus (S \setminus s_j)$ that passes through some vertices of the connected component as intermediate vertices. The proof is completed. ■

Remark 1 For any $k \geq 2$, a proper interval graph having $n > 2k$ vertices is unpaired k-disjoint path coverable if the graph is $(k+1)$-connected. Theorem 5.4 means that the converse is true if and only if $n \geq 3k-1$.

The class of proper interval graphs is *hereditary*, i.e., every induced subgraph of a graph in the class is contained in the same class. Or equivalently, for a vertex fault set F_v in a proper interval graph G, $G \setminus F_v$ is also a proper interval graph. From Theorem 5.4, we can derive a necessary and sufficient condition for a proper interval graph to be *f-vertex-fault* unpaired k-disjoint path coverable as follows.

Theorem 5.5 *Let G be a proper interval graph with a consecutive ordering $(v_1, v_2, \ldots, v_n)$. Then, G is f-vertex-fault unpaired k-disjoint path coverable for $k \geq 2$ if and only if G is $(f+k)$-connected and either $n = f+2k$ or $n \geq f+2k+1$ and $(v_i, v_{i+f+k+1}) \in E(G)$ for every i, $1 \leq i \leq n-f-2k$ or $k \leq i \leq n-f-k-1$.*

Proof Sufficiency. Let F_v be an arbitrary vertex fault set of G with $|F_v| \leq f$. Then, $G \setminus F_v$ is a proper interval graph and is $(f+k-|F_v|)$-connected. Let $(w_1, w_2, \ldots, w_{n'})$ be the subsequence of the consecutive ordering of G which contains all fault-free vertices, where $n' = n - |F_v|$. Then, the subsequence forms a consecutive ordering of $G \setminus F_v$. To conclude $G \setminus F_v$ is unpaired k-disjoint path coverable, we will show that $G \setminus F_v$ with its consecutive ordering satisfies the condition of Theorem 5.4. Assume $|F_v| = f \geq 1$; otherwise we are done since if $|F_v| < f$, then $G \setminus F_v$ is $(k+1)$-connected and if $f = 0$, then the two conditions of Theorems 5.4 and 5.5 are the same. Then, we have $n' = n - f$ and $G \setminus F_v$ is k-connected. If $n' = 2k$, then we are done. For $n' \geq 2k+1$, it suffices to show that $(w_i, w_{i+k+1}) \in E(G \setminus F_v)$ for each i, $1 \leq i \leq n'-2k$ or $k \leq i \leq n'-k-1$. Let $v_p = w_i$ and $v_q = w_{i+k+1}$. Then, we have $q \leq p+f+k+1$ since there are at most f faulty vertices between v_p and v_q exclusively. If $q \leq p+f+k$, then (v_p, v_q) is an edge of $G \setminus F_v$ as well as G since G is $(f+k)$-connected. For the remaining case of $q = p+f+k+1$, we have $p = i$. The condition of this theorem says that $(v_p, v_q) \in E(G)$ for every p, $1 \leq p \leq n-f-2k$ or $k \leq p \leq n-f-k-1$, which is equivalent to that $(w_i, w_{i+k+1}) \in E(G \setminus F_v)$ for every i, $1 \leq i \leq n'-2k$ or $k \leq i \leq n'-k-1$. Therefore, $G \setminus F_v$ is unpaired k-disjoint path coverable.

Necessity. The condition that G is $(f+k)$-connected is necessary from Lemma 2.1. Suppose $n \geq f+2k+1$ and $(v_i, v_{i+f+k+1}) \notin E(G)$ for some i, $1 \leq i \leq n-f-2k$ or $k \leq i \leq n-f-k-1$. We will show for some vertex fault set F_v with $|F_v| = f$, $G \setminus F_v$ is not unpaired k-disjoint path coverable. Let $F_v := \{v_j : i+1 \leq j \leq i+f\}$ and $(w_1, w_2, \ldots, w_{n'})$ be the subsequence of the consecutive ordering of G that contains all vertices not in F_v. Then, $w_j = v_j$ for $1 \leq j \leq i$ and $w_j = v_{j+f}$ for $i+1 \leq j \leq n-f = n'$. In this case, we have $(w_i, w_{i+k+1}) \notin E(G \setminus F_v)$ for some

i, $1 \leq i \leq n' - 2k$ or $k \leq i \leq n' - k - 1$. By Theorem 5.4, $G \setminus F_v$ is not unpaired k-disjoint path coverable. This completes the proof. ■

COROLLARY 5.6 *For an integer $B \geq 2$, a proper interval graph G with $n \geq 2B$ vertices is f-vertex-fault unpaired k-disjoint path coverable for any f and $k \geq 2$ with $f + k \leq B$ if and only if G is B-connected and either $n = 4$ or $n \geq 5$ and G is $(B + 1)$-connected.*

A proper interval graph G with $n = 4$ vertices is not general-demand 2-disjoint path coverable if $\kappa(G) = 2$; the graph is not even one-to-many 2-disjoint path coverable by Theorem 3.1. Excluding this exceptional case leads to the following.

THEOREM 5.7 *For an integer $B \geq 2$, a proper interval graph G with $n \geq 2B$ vertices is f-vertex-fault general-demand k-disjoint path coverable for any f and $k \geq 2$ with $f + k \leq B$ if and only if G is $(B + 1)$-connected.*

Proof The reduction algorithm of Section 2 cannot be applied directly since it produces a virtual edge fault in Step 2(c). Instead, to obtain a reduction of f-vertex-fault general-demand k-DPC problem into f'-vertex-fault unpaired k'-DPC problem such that $f' + k' \leq f + k$ and $k' \geq 2$, we modify Step 2 of the original reduction algorithm as follows: Pick up any terminal, say source s_i, of demand $d(s_i) \geq 2$. Let $p := d(s_i)$ and let $N(s_i)$ be the set of neighbors of s_i. (i) If there exists a free vertex $w \in N(s_i)$, then apply Step 2(a) of the original algorithm; (ii) else if there exists a sink $t_j \in N(s_i)$ of demand one, then apply Step 2(b) of the algorithm; (iii) otherwise there exists at least p sinks $t_1, t_2, ..., t_p \in N(s_i)$ of demand two or greater since $|N(s_i)| \geq B + 1 \geq f + k + 1$ and there are at most f faults and $k - p$ sources in $N(s_i)$. Decrement $d(t_j)$ by one for every $1 \leq j \leq p$. Let $S' := S \setminus s_i$ and $F' := F \cup \{s_i\}$. Find $\mathcal{P} := (k - p)$-DPC$[S', T|G, F']$ and return $\mathcal{P} \cup \{(s_i, t_j) : 1 \leq j \leq p\}$. ■

For an edge fault set F_e of a proper interval graph G, $G \setminus F_e$ is not necessarily a proper interval graph. So, to deal with edge fault, it needs to take a different approach from Theorem 5.5. The *f-edge-fault* unpaired k-disjoint path coverability of a proper interval graph is open.

6. Concluding Remarks

In this paper, we presented a framework that enables the generalization of three DPC problems: one-to-one, one-to-many, and unpaired many-to-many. The general-demand DPC problem reduces to the unpaired many-to-many DPC problem, and the single-source DPC problem reduces to the one-to-many DPC problem. As a result, an f-fault unpaired k-disjoint path coverable graph for any f and $k \geq 1$ with $f + k \leq B$ is also f-fault general-demand k-disjoint path coverable for any f and $k \geq 1$ with $f + k \leq B$. Furthermore, an f-fault one-to-many k-disjoint path coverable graph is f-fault single-source k-disjoint path coverable. We obtained some results on f-fault general-demand/single-source k-disjoint path coverability of restricted HL-graphs and f-vertex-fault general-demand k-disjoint path coverability of proper interval graphs.

Acknowledgement

We are grateful to the anonymous reviewers for their comments and suggestions. This work was supported by the Korea Research Foundation Grant funded by the

Korean Government (MOEHRD, Basic Research Promotion Fund) (KRF-2008-521-D00363).

References

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory*, 2nd printing, Springer, 2008.
[2] C.-H. Chang, C.-K. Lin, H.-M. Huang, and L.-H. Hsu, "The super laceability of the hypercubes," *Inform. Proc. Lett.* **92(1)**, pp. 15-21, 2004.
[3] X.-B. Chen, "Many-to-many disjoint paths in faulty hypercubes," *Information Sciences* **179(18)**, pp. 3110-3115, 2009.
[4] X.-B. Chen, "Unpaired many-to-many vertex-disjoint path covers of a class of bipartite graphs," *Inform. Proc. Lett.* **110(6)**, pp. 203-205, 2010.
[5] C. Chen, C.-C. Chang and G.J. Chang, "Proper interval graphs and the guard problem," *Discrete Mathematics* **170(1-3)**, pp. 223-230, 1997.
[6] T. Dvořák and P. Gregor, "Partitions of faulty hypercubes into paths with prescribed endvertices," *SIAM J. Discrete Mathematics* **22(4)**, pp. 1448-1461, 2008.
[7] P. Gregor and T. Dvořák, "Path partitions of hypercubes," *Inform. Proc. Lett.* **108(6)**, pp. 402-406, 2008.
[8] P.-L. Lai and H.-C. Hsu, "The two-equal-disjoint path cover problem of matching composition network," *Inform. Proc. Lett.* **107(1)**, pp. 18-23, 2008.
[9] D. Liu and J. Li, "Many-to-many n-disjoint path covers in n-dimensional hypercubes," *Inform. Proc. Lett.* **110(14-15)**, pp. 580-584, 2010.
[10] J.-H. Park, "One-to-one disjoint path covers in recursive circulants," *Journal of KISS* **30(12)**, pp. 691-698, 2003 (in Korean).
[11] J.-H. Park, "One-to-many disjoint path covers in a graph with faulty elements," in *Proc. of the International Computing and Combinatorics Conference COCOON 2004*, pp. 392-401, Aug. 2004.
[12] J.-H. Park, "Unpaired many-to-many disjoint path covers in hypercube-like interconnection networks," *Journal of KISS* **33(10)**, pp. 789-796, 2006 (in Korean).
[13] J.-H. Park, H.-C. Kim, and H.-S. Lim, "Fault-hamiltonicity of hypercube-like interconnection networks," in *Proc. IEEE International Parallel and Distributed Processing Symposium IPDPS 2005*, Denver, Apr. 2005.
[14] J.-H. Park, H.-C. Kim, and H.-S. Lim, "Many-to-many disjoint path covers in hypercube-like interconnection networks with faulty elements," *IEEE Transactions on Parallel and Distributed Systems* **17(3)**, pp. 227-240, Mar. 2006.
[15] J.-H. Park, H.-C. Kim, and H.-S. Lim, "Many-to-many disjoint path covers in the presence of faulty elements," *IEEE Transactions on Computers* **58(4)**, pp. 528-540, Apr. 2009.
[16] F.S. Roberts, "Indifference graphs," in F. Harary (ed.), *Proof Techniques in Graph Theory*, Academic Press, NY, pp. 139-146, 1969.
[17] C.-H. Tsai, J.J.M. Tan, and L.-H. Hsu, "The super-connected property of recursive circulant graphs," *Inform. Proc. Lett.* **91(6)**, pp. 293-298, 2004.
[18] A.S. Vaidya, P.S.N. Rao, and S.R. Shankar, "A class of hypercube-like networks," *Proc. 5th IEEE Symposium on Parallel and Distributed Processing(SPDP)*, pp. 800-803, 1993.