Many-to-Many Disjoint Path Covers in a Graph with Faulty Elements*

Jung-Heum Park[1], Hee-Chul Kim[2], and Hyeong-Seok Lim[3]

[1] The Catholic University of Korea, Korea
j.h.park@catholic.ac.kr
[2] Hankuk University of Foreign Studies, Korea
hckim@hufs.ac.kr
[3] Chonnam National University, Korea
hslim@chonnam.ac.kr

Abstract. In a graph G, k vertex disjoint paths joining k distinct source-sink pairs that cover all the vertices in the graph are called a *many-to-many k-disjoint path cover(k-DPC)* of G. We consider an f-fault k-DPC problem that is concerned with finding many-to-many k-DPC in the presence of f or less faulty vertices and/or edges. We consider the graph obtained by merging two graphs H_0 and H_1, $|V(H_0)| = |V(H_1)| = n$, with n pairwise nonadjacent edges joining vertices in H_0 and vertices in H_1. We present sufficient conditions for such a graph to have an f-fault k-DPC and give the construction schemes. Applying our main result to interconnection graphs, we observe that when there are f or less faulty elements, all of recursive circulant $G(2^m, 4)$, twisted cube TQ_m, and crossed cube CQ_m of degree m have f-fault k-DPC for any $k \geq 1$ and $f \geq 0$ such that $f + 2k \leq m - 1$.

1 Introduction

One of the central issues in various interconnection networks is finding node-disjoint paths concerned with the routing among nodes and the embedding of linear arrays. Node-disjoint paths can be used as parallel paths for an efficient data routing among nodes. Also, each path in node-disjoint paths can be utilized in its own pipeline computation. An interconnection network is often modeled as a graph, in which vertices and edges correspond to nodes and links, respectively. In the rest of this paper, we will use standard terminology in graphs (see [1]).

Disjoint paths can be categorized as three types: one-to-one, one-to-many, and many-to-many. One-to-one type deals with the disjoint paths joining a single source s and a single sink t. One-to-many type considers the disjoint paths joining a single source s and k distinct sinks $t_1, t_2, \ldots, t_k$. Most of the works done on disjoint paths deal with the one-to-one or one-to-many. For a variety of networks one-to-one and one-to-many disjoint paths were constructed, e.g., hypercubes [3],

* This work was supported by grant No. R01-2003-000-11676-0 from the Basic Research Program of the Korea Science & Engineering Foundation.

star networks [2], etc. Many-to-many type deals with the disjoint paths joining k distinct sources $s_1, s_2, \ldots, s_k$ and k distinct sinks $t_1, t_2, \ldots, t_k$. In many-to-many type, several problems can be defined depending on whether specific sources should be joined to specific sinks or a source can be freely matched to a sink. The works on many-to-many type have a relative paucity because of its difficulty and some results can be found in [4, 7].

All of three types of disjoint paths in a graph G can be accommodated with the covering of vertices in G. A *disjoint path cover* in a graph G is to find disjoint paths containing all the vertices in G. A disjoint path cover problem originated from an interconnection network is concerned with the application where the full utilization of nodes is important. For an embedding of linear arrays in a network, the cover implies every node can be participated in a pipeline computation. One-to-one disjoint path covers in recursive circulants[8, 12] and one-to-many disjoint path covers in some hypercube-like interconnection networks[9] were studied.

Given a set of k sources $S = \{s_1, s_2, \ldots, s_k\}$ and a set of k sinks $T = \{t_1, t_2, \ldots, t_k\}$ in a graph G such that $S \cap T = \emptyset$, we are concerned with many-to-many disjoint paths $P_1, P_2, \ldots, P_k$ in G, P_i joining s_i and t_i, $1 \leq i \leq k$, that *cover* all the vertices in the graph, that is, $\bigcup_{1\leq i\leq k} V(P_i) = V(G)$ and $V(P_i) \cap V(P_j) = \emptyset$ for all $i \neq j$. Here $V(P_i)$ and $V(G)$ denote the vertex sets of P_i and G, respectively. We call such k disjoint paths a *many-to-many k-disjoint path cover* (in short, *many-to-many k-DPC*) of G.

On the other hand, embedding of linear arrays and rings into a faulty interconnection network is one of the important problems in parallel processing [5, 6, 11]. The problem is modeled as finding as long fault-free paths and cycles as possible in the graph with some faulty vertices and/or edges. A graph G is called *f-fault hamiltonian* (resp. *f-fault hamiltonian-connected*) if there exists a hamiltonian cycle (resp. if each pair of vertices are joined by a hamiltonian path) in $G \backslash F$ for any set F of faulty elements such that $|F| \leq f$. For a graph G to be f-fault hamiltonian (resp. f-fault hamiltonian-connected), it is necessary that $f \leq \delta(G) - 2$ (resp. $f \leq \delta(G) - 3$), where $\delta(G)$ is the minimum degree of G.

To a graph G with a set of faulty elements F, the definition of a many-to-many disjoint path cover can be extended. Given a set of k sources $S = \{s_1, s_2, \ldots, s_k\}$ and a set of k sinks $T = \{t_1, t_2, \ldots, t_k\}$ in $G \backslash F$ such that $S \cap T = \emptyset$, a many-to-many k-disjoint path cover joining S and T is k disjoint paths P_i joining s_i and t_i, $1 \leq i \leq k$, such that $\bigcup_{1\leq i\leq k} V(P_i) = V(G) \backslash F$, $V(P_i) \cap V(P_j) = \emptyset$ for all $i \neq j$, and every edge on each path P_i is fault-free. Such a many-to-many k-DPC is denoted by k-DPC$[\{(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)\}|G, F]$. A graph G is called *f-fault many-to-many k-disjoint path coverable* if for any set F of faulty elements such that $|F| \leq f$, G has k-DPC$[\{(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)\}|G, F]$ for every k distinct sources $s_1, s_2, \ldots, s_k$ and k distinct sinks $t_1, t_2, \ldots, t_k$ in $G \backslash F$.

Proposition 1. *For a graph G to be f-fault many-to-many k-disjoint path coverable, it is necessary that $f + 2k \leq \delta(G) + 1$.*

Proposition 2. *(a) A graph G is f-fault many-to-many 1-disjoint path coverable if and only if G is f-fault hamiltonian-connected.*

(b) If G is f-fault many-to-many $k(\geq 2)$-disjoint path coverable, then G is f-fault many-to-many $k-1$-disjoint path coverable.

Proposition 3. *If a graph G is f-fault many-to-many $k(\geq 2)$-disjoint path coverable, then for any pair of vertices s and t and any sequence of pairwise nonadjacent $k-1$ edges $((x_1, y_1), (x_2, y_2), \ldots, (x_{k-1}, y_{k-1}))$, there exists a hamiltonian path in $G \backslash F$ between s and t passing through the edges in the order given for any set F of faulty elements with $|F| \leq f$. That is, there exists a hamiltonian path of the form of $(s, \ldots, x_1, y_1, \ldots, x_{k-1}, y_{k-1}, \ldots, t)$.*

We are given two graphs G_0 and G_1 with n vertices. We denote by V_i and E_i the vertex set and edge set of G_i, $i = 0, 1$, respectively. We let $V_0 = \{v_1, v_2, \ldots, v_n\}$ and $V_1 = \{w_1, w_2, \ldots, w_n\}$. With respect to a permutation $M = (i_1, i_2, \ldots, i_n)$ of $\{1, 2, \ldots, n\}$, we can "merge" the two graphs into a graph $G_0 \oplus_M G_1$ with $2n$ vertices in such a way that the vertex set $V = V_0 \cup V_1$ and the edge set $E = E_0 \cup E_1 \cup E_2$, where $E_2 = \{(v_j, w_{i_j}) | 1 \leq j \leq n\}$. We denote by $G_0 \oplus G_1$ a graph obtained by merging G_0 and G_1 w.r.t. an arbitrary permutation M. Here, G_0 and G_1 are called *components* of $G_0 \oplus G_1$.

In this paper, we will show that by using f'-fault many-to-many k'-DPC of G_i for all f' and k' such that $f' + 2k' \leq f + 2k$, and fault-hamiltonicity of G_i, we can always construct an $f+1$-fault many-to-many k-DPC in $G_0 \oplus G_1$ and an f-fault many-to-many $k+1$-DPC in $H_0 \oplus H_1$, where $H_0 = G_0 \oplus G_1$ and $H_1 = G_2 \oplus G_3$. Precisely speaking, we will prove the following two theorems. Note that $\delta(G_0 \oplus G_1) = \delta + 1$ and $\delta(H_0 \oplus H_1) = \delta + 2$, where $\delta = \min_i \delta(G_i)$.

Theorem 1. *For $k \geq 2$ and $f \geq 0$, or for $k = 1$ and $f \geq 2$, let G_i be a graph with n vertices satisfying the following conditions, $i = 0, 1$:*
(a) G_i is $f + 2j$-fault many-to-many $k - j$-disjoint path coverable for every j, $0 \leq j < k$.
(b) G_i is $f + 2k - 1$-fault hamiltonian.
Then, $G_0 \oplus G_1$ is $f + 1$-fault many-to-many k-disjoint path coverable.

Note that the condition (a) of Theorem 1 is equivalent to that for any f' and k' such that $f' + 2k' \leq f + 2k$, G_i is f'-fault k'-disjoint path coverable. In this paper, we are concerned with a construction of f-fault many-to-many k-DPC of a graph G such that $f + 2k \leq \delta(G) - 1$.

Theorem 2. *For $k \geq 1$ and $f \geq 0$, let G_i be a graph with n vertices satisfying the following conditions, $i = 0, 1, 2, 3$:*
(a) G_i is $f + 2j$-fault many-to-many $k - j$-disjoint path coverable for every j, $0 \leq j < k$.
(b) G_i is $f + 2k - 1$-fault hamiltonian.
Then, $H_0 \oplus H_1$ is f-fault many-to-many $k + 1$-disjoint path coverable, where $H_0 = G_0 \oplus G_1$ and $H_1 = G_2 \oplus G_3$.

By applying the above two theorems to interconnection graphs, we will show that all of recursive circulant $G(2^m, 4)$, twisted cube TQ_m, and crossed cube CQ_m of degree m are f-fault many-to-many k-disjoint path coverable for every $k \geq 1$ and $f \geq 0$ such that $f + 2k \leq m - 1$.

Remark 1. Even when there are $p(< k)$ sources such that each source is identical with its corresponding sink, that is, when $s_i = t_i$ for all $1 \le i \le p$ and $S' \cap T' = \emptyset$, where $S' = \{s_{p+1}, \ldots, s_k\}$ and $T' = \{t_{p+1}, \ldots, t_k\}$, we can construct f-fault many-to-many k-DPC as follows: (a) we first define $P_i = (s_i)$, $1 \le i \le p$, a path with one vertex, and then (b) regarding them as virtual faulty vertices, find $f + p$-fault many-to-many $k - p$-DPC. Consequently, Proposition 3 can be extended so that adjacent edges are allowed.

2 Preliminaries

Let us consider fault-hamiltonicity of $G_0 \oplus G_1$. The following five lemmas are useful for our purpose. The proofs for them are omitted due to space limit.

Lemma 1. *For $f \ge 0$, if G_i is f-fault hamiltonian-connected and $f + 1$-fault hamiltonian, $i = 0, 1$, then $G_0 \oplus G_1$ is also f-fault hamiltonian-connected and $f + 1$-fault hamiltonian.*

Lemma 2. *For $f \ge 2$, if G_i is f-fault hamiltonian-connected and $f + 1$-fault hamiltonian, $i = 0, 1$, then $G_0 \oplus G_1$ is $f + 1$-fault hamiltonian-connected.*

Lemma 3. *For $f = 0, 1$, if G_i is f-fault hamiltonian-connected and $f + 1$-fault hamiltonian, $i = 0, 1$, then $G_0 \oplus G_1$ with $f + 1$ faulty elements has a hamiltonian path joining s and t unless s and t are contained in the same component and all the faulty elements are contained in the other component.*

Lemma 4. *For $f \ge 1$, if G_i is f-fault hamiltonian-connected and $f + 1$-fault hamiltonian, $i = 0, 1$, then $G_0 \oplus G_1$ is $f + 2$-fault hamiltonian.*

Lemma 5. *Let G be a δ-regular graph such that $\delta \ge 3$. If G is $\delta - 3$-fault hamiltonian-connected and $\delta - 2$-fault hamiltonian, then $G \times K_2$ is $\delta - 2$-fault hamiltonian-connected and $\delta - 1$-fault hamiltonian.*

For a vertex v in $G_0 \oplus G_1$, we denote by $\bar{v}$ the vertex adjacent to v which is in a component different from the component in which v is contained. We denote by U the set of terminals, the set of sources and sinks $S \cup T$, and denote by F the set of faulty elements.

Definition 1. *A vertex v in $G_0 \oplus G_1$ is called* free *if $v \notin F$ and $v \notin U$. An edge (v, w) is called* free *if v and w are free and $(v, w) \notin F$.*

Definition 2. *A* free bridge *of a fault-free vertex v is the path $(v, \bar{v})$ of length one if $\bar{v}$ is free and $(v, \bar{v}) \notin F$; otherwise, it is a path $(v, w, \bar{w})$ of length two such that $w \ne \bar{v}$, $(v, w) \notin F$, and $(w, \bar{w})$ is a free edge.*

Lemma 6. *Let $G_0 \oplus G_1$ have k source-sink pairs and at most f faulty elements such that $f + 2k \le \Delta - 1$, where Δ is the minimum degree of $G_0 \oplus G_1$.*
(a) For any terminal w in $G_0 \oplus G_1$, there exists a free bridge of w.
(b) For any set of terminals $W_l = \{w_1, w_2, \ldots, w_l\}$ in G_0 with $l \le 2k$, there exist l pairwise disjoint free bridges of w_i's, $1 \le i \le l$.
(c) For a single terminal w_1 in G_1 and a set of terminals $W_l \backslash w_1 = \{w_2, \ldots, w_l\}$ in G_0 with $l \le 2k$, there exist l pairwise disjoint free bridges of w_i's, $1 \le i \le l$.

Proof. There are at least Δ candidates for a free bridge of w, and at most $f+2k-1$ elements (f faulty elements and $2k-1$ terminals other than w) can "block" the candidates. Since each element block at most one candidate, there are at least $\Delta-(f+2k-1)\geq 2$ nonblocked candidates, and thus (a) is proved. We prove (b) by induction on l. Before going on, we need some definitions. We call vertices v and $\bar{v}$ and an edge joining them collectively a *column* of v. When $(v,\bar{v})$ (resp. $(v,w,\bar{w})$) is the free bridge of v, we say that the free bridge *occupies* a column of v (resp. two columns of v and w). We are to construct free bridges for W_l satisfying a condition that the number of occupied columns $c(l)$ is less than or equal to $f(l)+t(l)$, where $f(l)$ and $t(l)$ are the numbers of faulty elements and terminals contained in the $c(l)$ occupied columns, respectively. When $l=1$, there exists a free bridge which satisfies the condition. Assume that there exist pairwise disjoint free bridges for $W_{l-1}=W\backslash w_l$ satisfying the condition. If $(w_l,\bar{w_l})$ is the free bridge of w_l, we are done. Suppose otherwise. There are Δ candidates for a free bridge, and the number of blocking elements is at most $c(l-1)$ plus the number of terminals and faulty elements which are not contained in the $c(l-1)$ occupied columns. Thus, the number of blocking elements is at most $f+2k-1$, which implies the existence of pairwise disjoint free bridges for W_l. Obviously, $c(l)=c(l-1)+2$ and $f(l)+t(l)\geq f(l-1)+t(l-1)+2$, and thus it satisfies the condition.

Now, let us prove (c). If $(w_1,\bar{w_1})$ is the free bridge of w_1, it occupies one column. If $(w_1,x,\bar{x})$ is the free bridge of w_1 and $\bar{w_1}$ is not a terminal of which we are to find a free bridge, it occupies two columns. For these cases, in the same way as (b), we can construct pairwise disjoint free bridges satisfying the above condition. When $(w_1,x,\bar{x})$ is the free bridge of w_1 and $\bar{w_1}\in W_l$, letting $w_2=\bar{w_1}$ without loss of generality, we first find pairwise disjoint free bridges of w_1 and w_2. They occupy three columns, that is, $c(2)=3$. We proceed to construct free bridges with a relaxed condition that $c(l)\leq f(l)+t(l)+1$. This relaxation does not cause a problem since the number of blocking elements is at most $f+2k$, still less than the number of candidates for a free bridge, Δ. □

Remark 2. According to the proof of Lemma 6 (a) and (b), we have at least two choices when we find free bridges of terminals contained in one component.

Remark 3. If G_i satisfies the conditions of Theorem 1 or 2, then $f+2k\leq\delta-1$, where $\delta=\min_i\delta(G_i)$. Concerned with Theorem 1, free bridges of type Lemma 6 (b) and (c) exist in $G_0\oplus G_1$ since $(f+1)+2k\leq\delta(G_0\oplus G_1)-1$. Concerned with Theorem 2, free bridges of the two types also exist in $H_0\oplus H_1$ since $f+2(k+1)\leq\delta(H_0\oplus H_1)-1$.

3 Construction of Many-to-Many DPC

In this section, we will prove the main theorems. First of all, we will develop five basic procedures for constructing many-to-many disjoint path covers. They play a significant role in proving the theorems.

3.1 Five basic procedures

In a graph $C_0 \oplus C_1$ with two components C_0 and C_1, we are to define some notation. When we are concerned with Theorem 1, C_0 and C_1 correspond to G_0 and G_1, respectively. When we are concerned with Theorem 2, C_0 and C_1 correspond to H_0 and H_1, respectively. We denote by V_0 and V_1 the sets of vertices in C_0 and C_1, respectively. We let F_0 and F_1 be the sets of faulty elements in C_0 and C_1, respectively, and let F_2 be the set of faulty edges joining vertices in C_0 and vertices in C_1. Let $f_i = |F_i|$ for $i = 0, 1, 2$.

We denote by R the set of source-sink pairs in $C_0 \oplus C_1$. We also denote by k_i the number of source-sink pairs in C_i, $i = 0, 1$, and by k_2 the number of source-sink pairs between C_0 and C_1. Without loss of generality, we assume that $k_0 \geq k_1$. We let $I_0 = \{1, 2, \ldots, k_0\}$, $I_2 = \{k_0 + 1, k_0 + 2, \ldots, k_0 + k_2\}$, and $I_1 = \{k_0 + k_2 + 1, k_0 + k_2 + 2, \ldots, k_0 + k_2 + k_1\}$. We assume that $\{s_j, t_j | j \in I_0\} \cup \{s_j | j \in I_2\} \subseteq V_0$ and $\{s_j, t_j | j \in I_1\} \cup \{t_j | j \in I_2\} \subseteq V_1$. Among the k_2 sources s_j's, $j \in I_2$, we assume that the free bridges of k'_2 sources are of length one and the free bridges of $k''_2 (= k_2 - k'_2)$ sources are of length two.

First three procedures DPC-A, DPC-B, and DPC-C are applicable when $k_0 \geq 1$, and the last two procedures DPC-D and DPC-E are applicable when $k_2 = |R|$ (equivalently, $k_0 = k_1 = 0$). We denote by $H[v, w|G, F]$ a hamiltonian path in $G \backslash F$ joining a pair of fault-free vertices v and w in a graph G with a set F of faulty elements.

When we find a k-DPC or a hamiltonian path, sometimes we regard some fault-free vertices and/or edges as faulty elements. They are called *virtual* faults. For example, in step 2 of Procedure DPC-A, F' is the set of virtual vertex faults, and in step 2 of DPC-C, (s_2, s_1) in F' is a virtual edge fault.

Procedure DPC-A($C_0 \oplus C_1, R, F$)

UNDER the condition of $1 \leq k_0 < |R|$.

1. Find pairwise disjoint free bridges $B_{s_j} = (s_j, \ldots, s'_j)$ of s_j for all $j \in I_2$.
2. Find k_0-DPC$[\{(s_j, t_j) | j \in I_0\} | C_0, F_0 \cup F']$, where $F' = V_0 \cap \bigcup_{j \in I_2} V(B_{s_j})$.
3. Find $k_1 + k_2$-DPC$[\{(s'_j, t_j) | j \in I_2\} \cup \{(s_j, t_j) | j \in I_1\} | C_1, F_1]$.
4. Merge the two DPC's with the free bridges.

Procedure DPC-B($C_0 \oplus C_1, R, F$)

UNDER the condition of $k_0 = |R|$.

1. Let s_1 and t_1 be a pair such that $|X_1| \leq |X_j|$ for all $j \in I_0$, where $X_j = V_0 \cap \{V(B_{s_j}) \cup V(B_{t_j})\}$. Let $B_{s_1} = (s_1, \ldots, s'_1)$, $B_{t_1} = (t_1, \ldots, t'_1)$.
2. Find $k_0 - 1$-DPC$[\{(s_j, t_j) | j \in I_0 \backslash 1\} | C_0, F_0 \cup X_1]$.
3. Find $H[s'_1, t'_1 | C_1, F_1]$.
4. Merge the $k_0 - 1$-DPC and hamiltonian path with the free bridges.

Keep in mind that under the condition of procedure DPC-C below, for every s_j, $j \in I_2$, $\bar{s_j} = t_{j'}$ for some $j' \in I_2$, and thus for every other fault-free vertex v in G_0, $(v, \bar{v})$ is the free bridge of v.

Procedure DPC-C($C_0 \oplus C_1, R, F$)

UNDER the condition that $k_0 \geq 1$, $k_1 = 0$, $k_2' = 0$, and all the faulty elements are contained in C_0.

1. When $k_0 \geq 2$, find pairwise disjoint free bridges $B_{t_2} = (t_2, t_2')$, $B_{s_j} = (s_j, s_j')$ and $B_{t_j} = (t_j, t_j')$ for all $j \in I_0 \backslash \{1, 2\}$, and $B_{s_j} = (s_j, \ldots, s_j')$ for all $j \in I_2$. When $k_0 = 1$, find pairwise disjoint free bridges $B_{s_j} = (s_j, \ldots, s_j')$ for all $j \in I_2 \backslash 2$.
2. Find $H[s_2, t_1 | C_0, F_0 \cup F']$, where $F' = V_0 \cap [B_{t_2} \cup \bigcup_{j \in I_0 \backslash \{1,2\}} (V(B_{s_j}) \cup V(B_{t_j})) \cup \bigcup_{j \in I_2} V(B_{s_j})]$ if $k_0 \geq 2$; $F' = \{(s_2, s_1)\} \cup (V_0 \cap \bigcup_{j \in I_2 \backslash 2} V(B_{s_j}))$ otherwise. Let the hamiltonian path be $(s_2, Q_1, z, s_1, Q_2, t_1)$.
3. Let $u = t_2'$ if $k_0 \geq 2$; otherwise, $u = t_2$. Find $k_0 + k_2 - 1$-DPC$[\{\bar{z}, u)\} \cup \{(s_j', t_j') | j \in I_0 \backslash \{1, 2\}\} \cup \{(s_j', t_j) | j \in I_2 \backslash 2\} | C_1, \emptyset]$.
4. Merge the hamiltonian path and $k_0 + k_2 - 1$-DPC with the free bridges and the edge $(z, \bar{z})$. Discard the edge (z, s_1).

Procedures DPC-D and DPC-E are concerned with the case of $k_2 = |R|$. Without loss of generality, we assume that $f_0 \geq f_1$. This assumption does not conflict with the assumption of $k_0 \geq k_1$.

Procedure DPC-D($C_0 \oplus C_1, R, F$)

UNDER the condition that $k_2 = |R|$ ($k_0 = k_1 = 0$).

1. If $k_2'' \geq 1$, we assume that $(s_1, \bar{s_1})$ is not the free bridge of s_1. Find pairwise disjoint free bridges $B_{t_1} = (t_1, \ldots, t_1')$ and $B_{s_j} = (s_j, \ldots, s_j')$ for all $j \in I_2 \backslash 1$.
2. Find $H[s_1, t_1' | C_0, F_0 \cup F']$, where $F' = V_0 \cap \bigcup_{j \in I_2 \backslash 1} V(B_{s_j})$.
3. Find $k_2 - 1$-DPC$[\{(s_j', t_j) | j \in I_2 \backslash 1\} | C_1, F_1 \cup F'']$, where $F'' = V_1 \cap B_{t_1}$.
4. Merge the hamiltonian path and the $k_2 - 1$-DPC with the free bridges.

Observe that under the condition of procedure DPC-E below, for every source s_j in G_0, $\bar{s_j} = t_{j'}$ for some $j' \in I_2$, and thus for any free vertex v in G_0, $(v, \bar{v})$ is a free edge.

Procedure DPC-E($C_0 \oplus C_1, R, F$)

UNDER the condition that $k_2 = |R|$, $k_2' = 0$, and all the faulty elements are contained in C_0.

1. Find pairwise disjoint free bridges $B_{t_1} = (t_1, \ldots, t_1')$ and $B_{s_j} = (s_j, \ldots, s_j')$ for all $j \in I_2 \backslash \{1, 2\}$.
2. Find $H[s_2, t_1' | C_0, F_0 \cup F']$, where $F' = \{(s_1, s_2)\} \cup (V_0 \cap \bigcup_{j \in I_2 \backslash \{1,2\}} V(B_{s_j}))$. Let the hamiltonian path be $(s_2, \ldots, z, s_1, \ldots, t_1')$.
3. Find $k_2 - 1$-DPC$[\{(\bar{z}, t_2)\} \cup \{(s_j', t_j) | j \in I_2 \backslash \{1, 2\}\} | C_1, F'']$, where $F'' = V_1 \cap V(B_{t_1})$.
4. Merge the hamiltonian path and the $k_2 - 1$-DPC with the free bridges. Discard the edge (s_1, z).

3.2 Proof of Theorem 1

For $k = 1$ and $f \geq 2$, the theorem is exactly the same as Lemma 2. We assume that

$$k \geq 2,\ f_0 + f_1 + f_2 \leq f + 1, \text{ and } k_0 + k_1 + k_2 = k.$$

Lemmas 7, 8, and 9 are concerned with $k_0 \geq 1$, and Lemmas 10 and 11 are concerned with $k_2 = k$.

Lemma 7. *When $1 \leq k_0 < k$, Procedure DPC-A($G_0 \oplus G_1, R, F$) constructs an $f+1$-fault k-DPC unless $f_0 = f + 1$, $k_1 = 0$, and $k'_2 = 0$.*

Proof. The existence of pairwise disjoint free bridges in step 1 is due to Lemma 6 (b). Unless $f_0 = f + 1$, $k_1 = 0$, and $k'_2 = 0$, G_0 is $f_0 + k'_2 + 2k''_2$-fault k_0-disjoint path coverable since $2k_0 + f_0 + k'_2 + 2k''_2 \leq 2k + f$, and thus there exists a k_0-DPC in step 2. Similarly, G_1 is f_1-fault $k_1 + k_2$-disjoint path coverable since $2k_1 + 2k_2 + f_1 \leq 2k + f$. This completes the proof of the lemma. □

Lemma 8. *When $k_0 = k$, Procedure DPC-B($G_0 \oplus G_1, R, F$) constructs an $f+1$-fault k-DPC unless $f_0 = f + 1$ ($k_1 = 0$, and $k'_2 = 0$).*

Proof. To prove the existence of a $k-1$-DPC in step 2, we will show that $f_0 + |X_1| \leq f + 2$. When $|X_1| = 2$, the inequality holds true unless $f_0 = f + 1$. When $|X_1| = 3$, the number $f_1 + f_2$ of faulty elements in G_1 or between G_0 and G_1 is at least $k(\geq 2)$, and thus $f_0 + 3 \leq f_0 + f_1 + f_2 + 1 \leq f + 2$. When $|X_1| = 4$, analogously to the previous case, $f_0 + 4 \leq f_0 + f_1 + f_2 < f + 2$ since $f_1 + f_2 \geq 2k$. The existence of a hamiltonian path joining s'_1 and t'_1 is due to the fact that $f_1 \leq f + 2k - 2$. □

Lemma 9. *When $k_0 \geq 1$, $f_0 = f + 1$, $k_1 = 0$, and $k'_2 = 0$, Procedure DPC-C($G_0 \oplus G_1, R, F$) constructs an $f + 1$-fault k-DPC.*

Proof. Whether $k_0 \geq 2$ or not, it holds true that $f_0 + |F'| \leq f + 1 + 2(k-2) + 1 = f + 2k - 2$, which implies the existence of a hamiltonian path in step 2. By the construction, $(z, \bar{z})$ is the free bridge of z. Note that $z \neq s_2$ when $k_0 = 1$. The existence of a $k-1$-DPC in step 3 is straightforward. □

Lemma 10. *When $k_2 = k$, Procedure DPC-D($G_0 \oplus G_1, R, F$) constructs an $f + 1$-fault k-DPC unless $f_0 = f + 1$ and $k'_2 = 0$.*

Proof. The existence of pairwise disjoint free bridges is due to Lemma 6(c). To prove the existence of the hamiltonian path, we will show that $f_0 + |F'| \leq f + 2k - 2$. When $k''_2 \geq 1$, $f_0 + |F'| = f_0 + 2(k''_2 - 1) + k'_2 \leq f + 2k - 2$ unless $f_0 = f + 1$ and $k'_2 = 0$. When $k''_2 = 0$, $f_0 + |F'| = f_0 + k'_2 - 1 \leq f + 2k - 2$. The existence of $k_2 - 1$-DPC in step 3 is due to that $f_1 + |F''| \leq f + 2$. Note that the assumption that $f_0 \geq f_1$ implies that $f_1 < f + 1$. □

Lemma 11. *When $k_2 = k$, $f_0 = f + 1$, and $k'_2 = 0$, Procedure DPC-E($G_0 \oplus G_1, R, F$) constructs an $f + 1$-fault k-DPC.*

Proof. The existence of the hamiltonian path is due to the fact that $f_0 + |F'| = f_0 + 2(k_2 - 2) + 1 \leq f + 2k - 2$. Note that z is different from s_1 and s_2, and thus $(z, \bar{z})$ is a free edge. The existence of the $k_2 - 1$-DPC is straightforward. □

Consequently, the proof of Theorem 1 is completed. From Theorem 1 and Lemma 4, the following corollary is immediate.

Corollary 1. *For $k \geq 2$ and $f \geq 0$, or for $k = 1$ and $f \geq 2$, let G_i be a graph with n vertices satisfying the two conditions of Theorem 1, $i = 0, 1$. Then,*
(a) $G_0 \oplus G_1$ is $f + 2j + 1$-fault many-to-many $k - j$-disjoint path coverable for every j, $0 \leq j < k$, and
(b) $G_0 \oplus G_1$ is $f + 2k$-fault hamiltonian.

3.3 Proof of Theorem 2 for $k \geq 2$ and $f \geq 0$ or for $k = 1$ and $f \geq 2$

Corollary 1 implies that H_i, $i = 0, 1$, is $f + 2j + 1$-fault many-to-many $k - j$-disjoint path coverable for every j, $0 \leq j < k$, and that H_i is $f + 2k$-fault hamiltonian. In this subsection, by utilizing mainly these properties of H_i, we are to prove Theorem 2 for $k \geq 2$ and $f \geq 0$ or for $k = 1$ and $f \geq 2$. We assume that

$$f_0 + f_1 + f_2 \leq f \text{ and } k_0 + k_1 + k_2 = k + 1.$$

Similarly to the proof of Theorem 1, Lemmas 12, 13, and 14 are concerned with $k_0 \geq 1$, and Lemmas 15 and 17 are concerned with $k_2 = k + 1$.

Lemma 12. *When $1 \leq k_0 < k+1$, Procedure DPC-A($H_0 \oplus H_1, R, F$) constructs an f-fault $k + 1$-DPC unless $f_0 = f$, $k_1 = 0$, and $k_2' = 0$.*

Proof. Unless $f_0 = f$, $k_1 = 0$, and $k_2' = 0$, H_0 is $f_0 + k_2' + 2k_2''$-fault k_0-disjoint path coverable since $2k_0 + f_0 + k_2' + 2k_2'' \leq 2k + f + 1$, and thus there exists a k_0-DPC in step 2. Similarly, H_1 is f_1-fault $k_1 + k_2$-disjoint path coverable since $2k_1 + 2k_2 + f_1 \leq 2k + f + 1$. □

Lemma 13. *When $k_0 = k+1$, Procedure DPC-B($H_0 \oplus H_1, R, F$) constructs an f-fault $k + 1$-DPC unless $f_0 = f$ ($k_1 = 0$ and $k_2' = 0$).*

Proof. To prove the existence of a k-DPC in step 2, we will show that $f_0 + |X_1| \leq f + 1$. When $|X_1| = 2$, the inequality holds true unless $f_0 = f$. When $|X_1| = 3$, it holds true that $f_1 + f_2 \geq k + 1$, and thus $f_0 + 3 \leq f_0 + f_1 + f_2 + 1 \leq f + 1$. When $|X_1| = 4$, $f_0 + 4 \leq f_0 + f_1 + f_2 < f + 1$ since $f_1 + f_2 \geq 2(k + 1)$. Obviously, there exists a hamiltonian path in H_1 joining s_1' and t_1'. □

Lemma 14. *When $k_0 \geq 1$, $f_0 = f$, $k_1 = 0$, and $k_2' = 0$, Procedure DPC-C($H_0 \oplus H_1, R, F$) constructs an f-fault $k + 1$-DPC.*

Proof. There exists a hamiltonian path in H_0 joining s_2 and t_1 since $f_0 + |F'| \leq f + 2(k - 1) + 1 = f + 2k - 1$. The existence of a k-DPC is straightforward. □

Hereafter in this subsection, $k_2 = k + 1$ ($k_0 = k_1 = 0$). Due to Lemma 6(a) and Remark 2, we assume that F'' defined in step 3 of Procedures DPC-D and DPC-E is a subset of $V(G_2)$ or $V(G_3)$. That is, $F'' \cap V(G_2) \neq \emptyset$ if and only if $F'' \cap V(G_3) = \emptyset$.

Lemma 15. *When $k_2 = k+1$, Procedure DPC-D($H_0 \oplus H_1, R, F$) constructs an f-fault $k+1$-DPC unless $f_0 = f$ and $k'_2 = 0$.*

Proof. To prove the existence of a hamiltonian path in H_0, we will show that $f_0 + |F'| \leq f + 2k - 1$. When $k''_2 \geq 1$, $f_0 + |F'| = f_0 + 2(k''_2 - 1) + k'_2 \leq f + 2k - 1$ unless $f_0 = f$ and $k'_2 = 0$. When $k''_2 = 0$, $f_0 + |F'| = f_0 + k'_2 - 1 \leq f + 2k - 1$. Now, let us consider the existence of a $k_2 - 1$-DPC in step 3. When $f \geq 1$ or $|F''| = 1$, there exists a $k_2 - 1$-DPC in H_1 since $f_1 + |F''| \leq f + 1$. Note that from the assumption of $f_0 \geq f_1$, if $f \geq 1$, then $f_1 < f$. When $f = 0$ and $|F''| = 2$ ($k \geq 2$ by the assumption), the existence of a $k_2 - 1$-DPC is due to the following Lemma 16. □

The proof of Lemma 16 is omitted. Of course, Lemma 16 does not say that $G_0 \oplus G_1$ is 2-fault many-to-many k-disjoint path coverable.

Lemma 16. *For $k \geq 2$, let G_i be a graph with n vertices satisfying the following conditions, $i = 0, 1$: (a) G_i is $2j$-fault many-to-many $k-j$-disjoint path coverable for every j, $0 \leq j < k$, and (b) G_i is $2k-1$-fault hamiltonian. Then, $G_0 \oplus G_1$ with two faulty vertices in G_0 and no other faulty elements is many-to-many k-disjoint path coverable.*

Lemma 17. *When $k_2 = k + 1$, $f_0 = f$, and $k'_2 = 0$, Procedure DPC-E($H_0 \oplus H_1, R, F$) constructs an f-fault $k+1$-DPC.*

Proof. There exists a hamiltonian path in H_0 joining s_2 and t'_1 since $f_0 + |F'| = f_0 + 2(k_2 - 2) + 1 = f + 2k - 1$. When $f \geq 1$, there exists a $k_2 - 1$-DPC in H_1 since $|F''| = 2 \leq f + 1$. When $f = 0$ (and $|F''| = 2$), the existence of a $k_2 - 1$-DPC is due to Lemma 16. □

3.4 Proof of Theorem 2 for $k = 1$ and $f = 0, 1$

In $H_0 \oplus H_1$, H_0 and H_1 are called components and G_i, $0 \leq i \leq 3$, are called *subcomponents*. Contrary to the proof given in the previous subsection, we can not employ Corollary 1. Instead, Lemma 1 and 3 are utilized repeatedly in this subsection. We denote by $\hat{v}$ the vertex which is adjacent to v and contained in the same component with v and in a different subcomponent from v. Lemmas 18, 19, and 20 are concerned with $k_0 \geq 1$. It is assumed that $k_0 \geq k_1$. All the proofs of lemmas in this subsection are omitted.

Lemma 18. *When $k_0 = 1$, we can construct an f-fault 2-DPC unless $f_0 = f$, $k_1 = 0$, and $k'_2 = 0$.*

Lemma 19. *When $k_0 = 2$, we can construct an f-fault 2-DPC unless $f_0 = f$ ($k_1 = 0$, $k'_2 = 0$).*

Lemma 20. *When $k_0 \geq 1$, $f_0 = f$, $k_1 = 0$, and $k'_2 = 0$, we can construct an f-fault 2-DPC.*

Now, let us consider the case when $k_2 = 2$ ($k_0 = k_1 = 0$). We assume that $f_0 \geq f_1$. Then, $f_1 = 0$. We denote by $l_{i,j}$ the number of edges joining vertices in G_i and G_j, $i \neq j$. Observe that $l_{0,1} = n$, $l_{0,2} + l_{0,3} = n$, $l_{0,2} = l_{1,3}$, and $l_{0,3} = l_{1,2}$. Note that $n \geq f + 4$ since each G_i is $f + 1$-fault hamiltonian.

Lemma 21. *When $k_2 = 2$, we can construct an f-fault 2-DPC unless $f_0 = f$ and $k'_2 = 0$.*

Lemma 22. *When $k_2 = 2$, $f = 0$, (s_1, t_1) is an edge, and $k'_2 = 1$, we can construct an f-fault 2-DPC.*

Lemma 23. *When $k_2 = 2$, $f_0 = f$, and $k'_2 = 0$, we can construct an f-fault 2-DPC.*

At last, the proof of Theorem 2 is completed. From Theorem 2, we have the following corollary.

Corollary 2. *For $k \geq 1$ and $f \geq 0$, let G_i be a graph with n vertices satisfying the two conditions of Theorem 2, $i = 0, 1, 2, 3$. Then, $H_0 \oplus H_1$ is $f + 2j$-fault many-to-many $k + 1 - j$-disjoint path coverable for every j, $0 \leq j < k$, where $H_0 = G_0 \oplus G_1$ and $H_1 = G_2 \oplus G_3$.*

4 Hypercube-Like Interconnection Networks

A graph G is called *fully many-to-many disjoint path coverable* if for any $k \geq 1$ and $f \geq 0$ such that $f + 2k \leq \delta(G) - 1$, G is f-fault many-to-many k-disjoint path coverable.

4.1 Recursive circulants $G(2^m, 4)$

$G(2^m, 4)$ is an m-regular graph with 2^m vertices. According to the recursive structure of recursive circulants[10], we can observe that $G(2^m, 4)$ is isomorphic to $G(2^{m-2}, 4) \times K_2 \oplus_M G(2^{m-2}, 4) \times K_2$ for some permutation M. Obviously, $G(2^{m-2}, 4) \times K_2$ is isomorphic to $G(2^{m-2}, 4) \oplus_{M'} G(2^{m-2}, 4)$ for some M'. Fault-hamiltonicity of $G(2^m, 4)$ was studied in [11]. By utilizing Lemma 5, we can also obtain fault-hamiltonicity of $G(2^m, 4) \times K_2$.

Lemma 24. *(a) $G(2^m, 4)$, $m \geq 3$, is $m - 3$-fault hamiltonian-connected and $m-2$-fault hamiltonian[11]. (b) $G(2^m, 4) \times K_2$, $m \geq 3$, is $m-2$-fault hamiltonian-connected and $m - 1$-fault hamiltonian.*

Theorem 3. *$G(2^m, 4)$, $m \geq 3$, is fully many-to-many disjoint path coverable.*

Proof. The proof is by induction on m. For $m = 3, 4$, the theorem holds true by Lemma 24. For $m \geq 5$, from Corollary 2 and Lemma 24, the theorem follows immediately. □

4.2 Twisted cube TQ_m, crossed cube CQ_m

Originally, twisted cube TQ_m is defined for odd m. We let $TQ_m = TQ_{m-1} \times K_2$ for even m. Then, TQ_m is isomorphic to $TQ_{m-1} \oplus_M TQ_{m-1}$ for some M. Also, crossed cube CQ_m is isomorphic to $CQ_{m-1} \oplus_{M'} CQ_{m-1}$ for some M'. Both TQ_m and CQ_m are m-regular graphs with 2^m vertices. Fault-hamiltonicity of them were studied in the literature.

Lemma 25. *(a) TQ_m, $m \geq 3$, is $m-3$-fault hamiltonian-connected and $m-2$-fault hamiltonian[6]. (b) CQ_m, $m \geq 3$, is $m-3$-fault hamiltonian-connected and $m-2$-fault hamiltonian[5].*

From Lemma 5, Corollary 2, and Lemma 25, we have the following theorem.

Theorem 4. *TQ_m and CQ_m, $m \geq 3$, are fully many-to-many disjoint path coverable.*

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