Many-to-Many Disjoint Path Covers in the Presence of Faulty Elements

Jung-Heum Park, *Member, IEEE,* Hee-Chul Kim, and Hyeong-Seok Lim

***Abstract*— A *many-to-many k-disjoint path cover (k-DPC)* of a graph G is a set of k disjoint paths joining k sources and k sinks in which each vertex of G is covered by a path. It is called a *paired* many-to-many disjoint path cover when each source should be joined to a specific sink, and it is called an *unpaired* many-to-many disjoint path cover when each source can be joined to an arbitrary sink. In this paper, we discuss about paired and unpaired many-to-many disjoint path covers including their relationships, application to strong hamiltonicity, and necessary conditions. And then, we give a construction scheme for paired many-to-many disjoint path covers in the graph $H_0 \oplus H_1$ obtained from connecting two graphs H_0 and H_1 with $|V(H_0)| = |V(H_1)|$ by $|V(H_0)|$ pairwise nonadjacent edges joining vertices in H_0 and vertices in H_1, where $H_0 = G_0 \oplus G_1$ and $H_1 = G_2 \oplus G_3$ for some graphs G_j's. Using the construction, we show that every m-dimensional restricted HL-graph and recursive circulant $G(2^m, 4)$ with f or less faulty elements have a paired k-DPC for any f and $k \geq 2$ with $f + 2k \leq m$.**

***Index Terms*— Fault tolerance, disjoint path covers, interconnection networks, restricted HL-graphs, recursive circulants, strong hamiltonicity, fault-hamiltonicity.**

I. Introduction

VARIOUS interconnection networks were proposed and their graph-theoretic properties have been investigated with their applications in parallel computing. Among the properties, finding parallel paths among nodes in interconnection networks is one of the important problems concerned with efficient data transmission. Usually interconnection networks are represented as graphs and parallel paths are studied in terms of disjoint paths in graphs. In this paper, we will use standard terminology in graphs (see [2]).

Let $G = (V, E)$ be an undirected simple graph. A set of paths in G is called *disjoint* if they do not share any vertices. In disjoint path problems, one or more source vertices and one or more sink vertices are given to find disjoint paths between them. Depending on the number of sources or sinks, there are one-to-one[22], [3], [33], one-to-many[6], [23], and many-to-many disjoint path problems[24], [25], [28]. Among them, the many-to-many disjoint path problem is the most generalized one, and will be mainly discussed in this paper.

For a set $S = \{s_1, s_2, \ldots, s_k\}$ of k sources and a set $T = \{t_1, t_2, \ldots, t_k\}$ of k sinks in $V(G)$, the many-to-many k-disjoint path problem is to determine whether there exist k disjoint paths each joining a source and a sink. There are *paired* and *unpaired* types of many-to-many k-disjoint path problems. In the paired type, each source should be joined to a specific sink, that is, s_j should be joined to t_j. In the unpaired type, each source can be joined to an arbitrary sink. The sources and sinks are called *terminals* in general.

Disjoint path cover of a graph G is a set of disjoint paths covering all the vertices of G. The problem of finding disjoint path covers is closely related to a well-known hamiltonian path problem and concerned with the application where the full utilization of vertices is important. The hamiltonian path problem can be viewed as a specific case of the disjoint path cover problem.

The disjoint path cover problem can be extended to a graph with some faulty elements (vertices and/or edges). Fault tolerance is one of the important measures in networks. Especially, fault-hamiltonicity of various interconnection networks was widely investigated in the literature[14], [16], [17], [27], [32], [34]. A graph G is called *f-fault hamiltonian* (resp. *f-fault hamiltonian-connected*) if there exists a hamiltonian cycle (resp. if each pair of vertices are joined by a hamiltonian path) in $G \backslash F$ for any set F of faulty elements with $|F| \leq f$.

Considering all the above versions of disjoint path cover problems, we give definitions for a graph G with a set F of faulty elements.

Definition 1: Given a set of k sources $S = \{s_1, s_2, \ldots, s_k\}$ and a set of k sinks $T = \{t_1, t_2, \ldots, t_k\}$ in $G \backslash F$ such that $S \cap T = \emptyset$, a *paired many-to-many k-disjoint path cover* joining S and T is a set of k fault-free disjoint paths P_j joining s_j and t_j, $1 \leq j \leq k$, that cover all the fault-free vertices of G.

Definition 2: Given a set of k sources $S = \{s_1, s_2, \ldots, s_k\}$ and a set of k sinks $T = \{t_1, t_2, \ldots, t_k\}$ in $G \backslash F$ such that $S \cap T = \emptyset$, an *unpaired many-to-many k-disjoint path cover* joining S and T is a set of k fault-free disjoint paths P_j joining s_j and t_{i_j}, $1 \leq j \leq k$, with an arbitrary permutation $(i_1, i_2, \ldots, i_k)$ of $\{1, 2, \ldots, k\}$ that cover all the fault-free vertices of G.

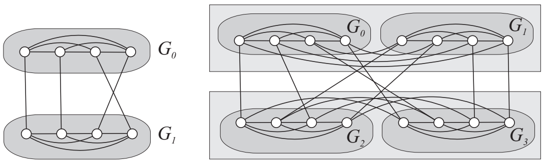
We can think of a situation in an interconnection network where k_p source-sink pairs are the paired type so that specific sources should be joined to specific sinks, and k_u sources and k_u sinks are the unpaired type so that they can be freely matched. We call the $(k_p + k_u)$-disjoint path cover of the mixed type as *hybrid many-to-many (k_p, k_u)-disjoint path cover*. Hereafter in this paper, paired (resp. unpaired) k-DPC refers to a paired (resp. unpaired) many-to-many k-disjoint path cover. Similarly, hybrid (k_p, k_u)-DPC refers to a hybrid many-to-many (k_p, k_u)-disjoint path cover.

Given S and T in a graph G, the problem of determining whether there exists a paired k-DPC between S and T in G was shown to be NP-complete for any fixed $k \geq 1$ [28]. The problem of determining the existence of an unpaired k-DPC (resp. a hybrid (k_p, k_u)-DPC) is also NP-complete for any fixed k (resp. any pair of fixed k_p and k_u). They can be reduced from the HAMILTONIAN PATH BETWEEN TWO VERTICES problem

J.-H. Park is with the School of Computer Science and Information Engineering, The Catholic University of Korea, Korea. E-mail: j.h.park@catholic.ac.kr

H.-C. Kim is with the School of Computer Science and Information Communications Engineering, Hankuk University of Foreign Studies, Korea. E-mail: hckim@hufs.ac.kr

H.-S. Lim is with the School of Electronics and Computer Engineering, Chonnam National University, Korea. E-mail: hslim@chonnam.ac.kr



(a) $G_0 \oplus G_1$ (b) $H_0 \oplus H_1$

Fig. 1. Examples of $G_0 \oplus G_1$ and $H_0 \oplus H_1$.

[12] as the paired k-DPC done in [28].

Only a few works can be found for many-to-many k-disjoint path cover problem with $k \geq 2$. Paired k-DPC problems in restricted HL-graphs and recursive circulants were investigated in [28], and unpaired k-DPC problems in restricted HL-graphs and recursive circulants were studied in [25] and [29], respectively. The restricted HL-graphs are known to contain twisted cubes[15], crossed cubes[11], multiply twisted cubes[10], Möbius cubes[8], Mcubes[31], and generalized twisted cubes[4]. It was shown that both m-dimensional restricted HL-graphs and recursive circulant $G(2^m, 4)$ with f or less faulty elements have a paired k-DPC for any f and $k \geq 1$ with $f + 2k \leq m - 1$, and have an unpaired k-DPC for any f and $k \geq 1$ with $f + k \leq m - 2$. Every m-dimensional crossed cube, $m \geq 5$, was shown to have a paired 2-DPC consisting of two paths of equal length by Lai *et al.* in [18].

In this paper, we consider a graph with faulty elements which has a k-DPC for arbitrary k sources and k sinks rather than fixed sources and sinks, which is called a many-to-many k-disjoint path coverable graph. It is defined as follows.

Definition 3: A graph G is called *f-fault paired (resp. unpaired) many-to-many k-disjoint path coverable* if $f + 2k \leq |V(G)|$ and for any set F of faulty elements with $|F| \leq f$, G has a paired (resp. unpaired) k-DPC for any set S of k sources and any set T of k sinks in $G \backslash F$ such that $S \cap T = \emptyset$.

Many-to-many disjoint path coverability can also be defined for the hybrid type. A graph G is called *f-fault hybrid many-to-many (k_p, k_u)-disjoint path coverable* if $f + 2(k_p + k_u) \leq |V(G)|$ and for any set F of faulty elements with $|F| \leq f$, G has a hybrid (k_p, k_u)-DPC for any set of k_p source-sink pairs and k_u sources and k_u sinks of unpaired type. Of course, all the terminals are distinct.

Many interconnection networks such as restricted HL-graphs and recursive circulant $G(2^m, 4)$ can be constructed by connecting two lower dimensional networks. We represent the construction as follows. Given two graphs G_0 and G_1 with n vertices each, we denote by V_j and E_j the vertex set and edge set of G_j, $j = 0, 1$, respectively. Let $V_0 = \{v_1, v_2, \ldots, v_n\}$ and $V_1 = \{w_1, w_2, \ldots, w_n\}$. With respect to a permutation $M = (i_1, i_2, \ldots, i_n)$ of $\{1, 2, \ldots, n\}$, we can "merge" the two graphs into a graph $G_0 \oplus_M G_1$ with $2n$ vertices in such a way that the vertex set $V = V_0 \cup V_1$ and the edge set $E = E_0 \cup E_1 \cup E_2$, where $E_2 = \{(v_j, w_{i_j}) | 1 \leq j \leq n\}$. We denote by $G_0 \oplus G_1$ a graph obtained by merging G_0 and G_1 w.r.t. an arbitrary permutation M. Here, G_0 and G_1 are called *components* of $G_0 \oplus G_1$. Fig. 1(a) shows an example of $G_0 \oplus G_1$.

In this paper, we discuss about some interesting properties of f-fault paired, unpaired, and hybrid many-to-many disjoint path covers from a graph-theoretic point of view, which include their relationships, application to strong hamiltonicity, and some necessary conditions. And then, for $H_0 = G_0 \oplus G_1$ and $H_1 = G_2 \oplus G_3$, we investigate how paired and unpaired many-to-many disjoint path coverability of G_i's and H_j's are translated into paired many-to-many disjoint path coverability of $H_0 \oplus H_1$.

By applying our result to restricted HL-graphs and recursive circulants, we show that every m-dimensional restricted HL-graph and recursive circulant $G(2^m, 4)$ are f-fault paired many-to-many k-disjoint path coverable for any f and $k \geq 2$ with $f + 2k \leq m$. The bound of "$f + 2k \leq m - 1$" in [28] is improved by one. It also implies that there exists a hybrid (k_p, k_u)-DPC in m-dimensional restricted HL-graphs and recursive circulant $G(2^m, 4)$ for any k_p and k_u with $k = k_p + k_u$ such that $f + 2k \leq m$.

The organization of this paper is as follows. In the next section, we will address properties of f-fault paired, unpaired, and hybrid many-to-many disjoint path coverable graphs and their relationships. In Section III, the paired many-to-many disjoint path coverability of $H_0 \oplus H_1$, where $H_0 = G_0 \oplus G_1$ and $H_1 = G_2 \oplus G_3$, will be developed and applied to restricted HL-graphs and recursive circulants. Finally, in Section IV, the concluding remarks of this paper will be given.

II. Many-to-Many Disjoint Path Covers

In this section, we discuss about the relationships among many-to-many disjoint path covers of the three types, their applicability to strong hamiltonicity, and their fundamental properties. Some aspects of paired many-to-many disjoint path covers were addressed in [28].

A path in a graph is represented as a sequence of vertices. A v-w path refers to a path from vertex v to w, and a v-path refers to a path whose starting vertex is v.

A. Unpaired many-to-many DPC

Given k sources $s_1, s_2, \ldots, s_k$ and sinks $t_1, t_2, \ldots, t_k$ in a graph G with fault set F, construction of an unpaired k-DPC is easier, in general, than construction of a paired k-DPC. A paired k-DPC is, by definition, an unpaired k-DPC.

Proposition 1: An f-fault paired many-to-many k-disjoint path coverable graph is f-fault unpaired many-to-many k-disjoint path coverable.

It is known from [28] that for any $k \geq 2$, an f-fault paired many-to-many k-disjoint path coverable graph is always f-fault paired many-to-many $(k-1)$-disjoint path coverable. It is a quite natural question of whether or not the corresponding property holds true in unpaired many-to-many disjoint path covers. Unfortunately, the answer is negative even when $f = 0$ as shown in the following lemma.

Lemma 1: A complete bipartite graph $K_{m,m}$, $m \geq 2$, is (0-fault) unpaired many-to-many m-disjoint path coverable. However, it is not (0-fault) unpaired many-to-many k-disjoint path coverable for any $1 \leq k < m$.

Proof: Let X, Y be the bipartition sets of $K_{m,m}$. Since there are m sources and m sinks in $K_{m,m}$, every vertex is a terminal. Let S_X and T_X be the set of sources and the set of sinks in X, respectively. S_Y and T_Y are defined similarly. Then, $|S_X| = |T_Y|$ and $|T_X| = |S_Y|$. The subgraph induced by $S_X \cup T_Y$ is a complete bipartite graph and it has a perfect matching M_1. Similarly, the subgraph induced by $T_X \cup S_Y$ also has a perfect matching M_2. $M_1 \cup M_2$ constitutes an unpaired m-DPC.

Now, we show $K_{m,m}$ is not unpaired k-disjoint path coverable for any k, $1 \leq k < m$. Suppose all the sinks are in Y. Also suppose the sources $s_1, s_2, \ldots, s_{k-1}$ are in X, and s_k is in Y. Assume there exists an unpaired k-DPC. Every s_i-path, $1 \le i < k$, in the k-DPC has even number of vertices, and thus has the same number of vertices in X and Y. But the s_k-path should have one more vertex in Y than in X, which contradicts $|X| = |Y|$. ■

B. Strong hamiltonicity

Many-to-many disjoint path coverability is closely related to hamiltonicity. When $k = 1$, an unpaired k-DPC is equivalent to a paired k-DPC by definition. Also, the 1-DPC is a hamiltonian path joining a source and a sink. It was known that for a graph G to be f-fault hamiltonian-connected, it is necessary that $f \le \delta(G) - 3$, where $\delta(G)$ is the minimum degree of G.

Proposition 2: The following statements are equivalent.
(a) A graph G is f-fault paired many-to-many 1-disjoint path coverable.
(b) A graph G is f-fault unpaired many-to-many 1-disjoint path coverable.
(c) A graph G is f-fault hamiltonian-connected.

The problem of finding a hamiltonian path/cycle in a graph passing through all the edges in a given set of prescribed edges of the graph was considered in the literature. For hypercubes, hamiltonian paths with prescribed edges and hamiltonian cycles with prescribed edges were studied in [9] and [5], respectively. In [28], the construction of a hamiltonian path/cycle passing through prescribed edges using a paired many-to-many k-DPC was suggested. It was shown that if G is f-fault paired many-to-many $k(\ge 2)$-disjoint path coverable, then for any fault set F with $|F| \le f$ and for any vertices s, t and any sequence of pairwise nonadjacent $k-1$ edges $((x_1, y_1), (x_2, y_2), \ldots, (x_{k-1}, y_{k-1}))$ in $G\backslash F$ such that $s \ne x_i, y_i$ and $t \ne x_i, y_i$ for all $1 \le i \le k-1$, there exists an s-t hamiltonian path in $G\backslash F$ that passes through the edges in the order given. The s-t hamiltonian path passes through the edge (x_i, y_i) from x_i to y_i, that is, we can assign the direction of how to pass through the edge.

Given a single prescribed edge (x, y), we can find a hamiltonian path passing through the edge using an unpaired 2-DPC by Theorem 1 in the following. The difference in this construction of hamiltonian path compared with that using a paired 2-DPC is that we cannot assign the direction of how to pass through the edge. That is, we cannot force the s-t hamiltonian path to pass through the edge from x to y, or vice versa.

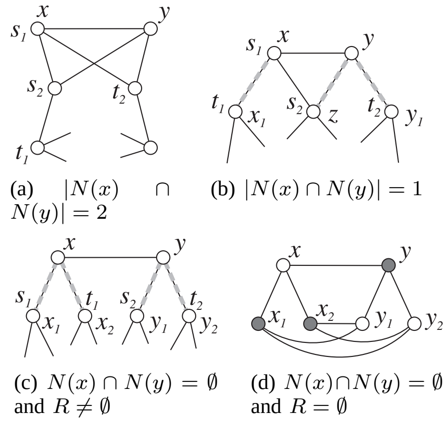
Theorem 1: If G is f-fault unpaired many-to-many 2-disjoint path coverable, then for any fault set F with $|F| \le f$ and any vertices s, t and edge (x, y) such that $\{s, t\} \cap \{x, y\} = \emptyset$, there exists an s-t hamiltonian path in $G\backslash F$ that passes through the edge (x, y).

Proof: There exists an unpaired 2-DPC joining $\{s, t\}$ and $\{x, y\}$ in $G\backslash F$. Let P_s and P_t be the s-path and the t-path in the unpaired 2-DPC, respectively. Then, (P_s, P_t^R) is the desired hamiltonian path. Here, P_t^R is the reverse of the path P_t. ■

Corollary 1: If G is f-fault unpaired many-to-many 2-disjoint path coverable, then for any fault set F with $|F| \le f$ and any two nonadjacent edges (v, w) and (x, y), there exists a hamiltonian cycle in $G\backslash F$ that passes through both edges (v, w) and (x, y).

C. Necessity

A necessary condition for a graph G to be f-fault paired many-to-many k-disjoint path coverable was studied in [28] as follows.



(a) $|N(x) \cap N(y)| = 2$ (b) $|N(x) \cap N(y)| = 1$
(c) $N(x) \cap N(y) = \emptyset$ and $R \ne \emptyset$ (d) $N(x) \cap N(y) = \emptyset$ and $R = \emptyset$

Fig. 2. Illustration of the proof of Theorem 2.

Lemma 2: [28] If G is f-fault paired many-to-many k-disjoint path coverable, then $\kappa(G) \ge f + 2k - 1$, where $\kappa(G)$ is the connectivity of G.

There exists a graph in which equality of the condition given in Lemma 2 holds true for some f and k. A complete graph K_4 with four vertices is such a graph for $f = 0$ and $k = 2$. However, Theorem 2 shown below is suggestive of room for improvement in the necessary condition of Lemma 2.

Theorem 2: With the unique exception of K_4, no 3-regular graph is 0-fault paired many-to-many 2-disjoint path coverable.

Proof: Let G be an arbitrary 3-regular graph, which is not isomorphic to K_4. Then, the number of vertices in G is even and at least 6. Let x and y be two vertices adjacent to each other such that $|N(x) \cap N(y)|$ is the maximum possible. Here, $N(v)$ is the neighborhood set of v, which is defined as $\{w | (v, w) \in E(G)\}$. Let $R = V(G)\backslash(N(x) \cup N(y))$. If $|N(x) \cap N(y)| = 2$, we are done due to Lemma 2. Observe in this case that $N(x) \cap N(y)$ is a vertex cut of size 2 and thus $\kappa(G) \le 2$. See Fig. 2. If $|N(x) \cap N(y)| = 1$ ($|R| \ge 1$), we let $z \in N(x) \cap N(y)$, $N(x) = \{z, y, x_1\}$ and $N(y) = \{z, x, y_1\}$. It is straightforward to check that there does not exist a paired 2-DPC for $\{s_1, t_1\} = \{x, x_1\}$ and $\{s_2, t_2\} = \{z, y_1\}$.

Finally, we assume $N(x) \cap N(y) = \emptyset$, and let $N(x) = \{y, x_1, x_2\}$ and $N(y) = \{x, y_1, y_2\}$. If $R \ne \emptyset$, there is no paired 2-DPC for $\{s_1, t_1\} = \{x_1, x_2\}$ and $\{s_2, t_2\} = \{y_1, y_2\}$. If $R = \emptyset$, we have $|V(G)| = 6$ and $(x_i, y_j) \in E(G)$ for all $1 \le i, j \le 2$. Note that $(x_1, x_2), (y_1, y_2) \notin E(G)$ by the choice of x and y. The graph is isomorphic to a complete bipartite $K_{3,3}$. It was shown not to be unpaired 2-disjoint path coverable in Lemma 1, and thus it is not paired 2-disjoint path coverable. This completes the proof. ■

In terms of connectivity and the minimum degree, necessary conditions for a graph G to be f-fault unpaired many-to-many k-disjoint path coverable can be derived as follows.

Theorem 3: Let G be an f-fault unpaired many-to-many $k(\ge 2)$-disjoint path coverable graph, then $\kappa(G) \ge f + k$. Furthermore, if $\kappa(G) = f + k$, then for any vertex cut C of size $f + k$, the number of vertices contained in any $p-1$ connected components of $G\backslash C$ is strictly less than k, where p is the number of connected components of $G\backslash C$.

Proof: Suppose $\kappa(G) \le f + k - 1$. G is not isomorphic to a complete graph; suppose otherwise, $|V(G)| \ge f + 2k$ by definition and $\kappa(G) \ge f + 2k - 1$, which is a contradiction. Thus, there exists a vertex cut C of size $f + k - 1$ or less. Let X be the vertex set of the smallest connected component in $G\backslash C$ and let Y be the vertex set of all the other connected components. If $|C| \leq f$, letting the fault set $F = C$, we have a disconnected graph $G\backslash F$. It is obvious that for some sets S of sources and T of sinks, there exists no unpaired k-DPC joining S and T. Assume $|C| > f$. Let F be an arbitrary subset of f vertices in C. If $|Y| \geq k$, we let $T \subseteq Y$ and let S be a set such that $S \supset C\backslash F$ and $S \cap X \neq \emptyset$. Then, there exists no fault-free path joining a source in X and a sink in Y without passing through any source in C. Finally, we assume $|Y| < k$. We let all the vertices in Y and any $k - |Y|$ vertices in $C\backslash F$ be sinks, and let S be any set of k fault-free vertices such that $S \cap T = \emptyset$. In this case, the number of vertices in $C\backslash F$ which are not contained in T is less than $|Y|$ since $|C\backslash F| \leq k - 1$. Thus, not every sink in Y can be joined to a source by a fault-free path. Therefore, it was proved that $\kappa(G) \geq f + k$.

Suppose $\kappa(G) = f + k$ and there is a vertex cut C of size $f + k$ such that the set Y of vertices contained in some $p - 1$ connected components of $G\backslash C$ has at least k vertices. Let $X = V(G)\backslash(C \cup Y)$. X is the vertex set of a connected component of $G\backslash C$. We let $F \subset C$ be a fault set with $|F| = f$. For $S = C\backslash F$ and any $T \subset Y$ with $|T| = k$, there exists no unpaired k-DPC joining S and T. No path joining a source and a sink can pass through a vertex in X as an intermediate vertex. Thus, we have the theorem. ■

Notice that equality in the necessary condition of $\kappa(G) \geq f + k$ given in Theorem 3 holds true in a complete bipartite graph $K_{m,m}$, $m \geq 2$, for $f = 0$ and $k = m$, by Lemma 1.

Lemma 3: Let G be an f-fault unpaired many-to-many $k(\geq 2)$-disjoint path coverable graph with $f + 2k + 1$ or more vertices. Then, $\delta(G) \geq f + k + 1$.

Proof: Suppose $\delta(G) \leq f + k$. By Theorem 3, we have $\delta(G) = \kappa(G) = f + k$. There exists a vertex v in G whose degree is equal to $\delta(G)$. The neighborhood set $N(v)$ of v forms a vertex cut of size $f + k$. In $G\backslash N(v)$, there is a connected component consisting of the vertex v only. Therefore, the number of vertices in all connected components but v is at least k, which contradicts Theorem 3. This completes the proof. ■

D. Hybrid many-to-many DPC

When $k_p = 0$, a hybrid (k_p, k_u)-DPC is equivalent to an unpaired k_u-DPC, and when $k_u = 0$ or 1, it is equivalent to a paired $(k_p + k_u)$-DPC. By definition, a paired $(k_p + k_u)$-DPC is a hybrid (k_p, k_u)-DPC, and a hybrid (k_p, k_u)-DPC is an unpaired $(k_p + k_u)$-DPC.

Proposition 3: (a) An f-fault paired many-to-many k-disjoint path coverable graph is f-fault hybrid many-to-many (k_p, k_u)-disjoint path coverable for any k_p and k_u with $k = k_p + k_u$.
(b) An f-fault hybrid many-to-many (k_p, k_u)-disjoint path coverable graph is f-fault unpaired many-to-many $(k_p + k_u)$-disjoint path coverable.

Recall that a necessary condition for a graph G to be paired many-to-many k-disjoint path coverable is $\kappa(G) \geq f + 2k - 1$, and that a necessary condition for G to be unpaired many-to-many k-disjoint path coverable is $\kappa(G) \geq f + k$. For each of the two conditions, there exist a graph satisfying equality for some f and k. One might expect that for a graph G to be f-fault hybrid many-to-many (k_p, k_u)-disjoint path coverable, it is necessary that $\kappa(G) \geq f + 2k_p + k_u + c$ for some $c = 0$ or -1, and one might guess that there exists a graph in which equality holds true for some f, k_p, and k_u. However, as shown in the following theorem, the situation is worse than expected.

Theorem 4: Let $k_p \geq 1$. If a graph G is f-fault hybrid many-to-many (k_p, k_u)-disjoint path coverable, then $\kappa(G) \geq f + 2k - 1$, where $k = k_p + k_u$.

Proof: Suppose G is f-fault hybrid many-to-many (k_p, k_u)-disjoint path coverable and $\kappa(G) \leq f + 2k - 2$. Then, G is not a complete graph; suppose otherwise, we have $\kappa(G) = |V(G)| - 1 \leq f + 2k - 2$, which contradicts $|V(G)| \geq f + 2k$. There exists a vertex cut C of size $f + 2k - 2$ or less. If a source s_1 and its pair sink t_1 are placed in different connected components of $G\backslash C$, and if every vertex in C is either a faulty vertex or a terminal (other than s_1 and t_1), then every s_1-t_1 path should pass through a faulty vertex or a terminal as an intermediate vertex. This is a contradiction. ■

III. Construction of Paired Disjoint Path Covers

It was shown in [28] that every m-dimensional restricted HL-graph and recursive circulant $G(2^m, 4)$, $m \geq 3$, are f-fault paired many-to-many k-disjoint path coverable for any f and $k \geq 1$ with $f + 2k \leq m - 1$. Both graphs are of degree m and have 2^m vertices. Their connectivities are equal to degree m. The necessary condition given in Lemma 2 says "$f + 2k \leq m + 1$." In this section, we construct an f-fault paired k-DPC for any f and $k \geq 2$ with $f + 2k \leq m$ in m-dimensional restricted HL-graphs and recursive circulant $G(2^m, 4)$, $m \geq 3$.

The bound on $f + 2k$ is improved by one as compared with [28]. Thus, the gap between the bound achieved and the bound $m + 1$ of necessity is just one. An f-fault paired many-to-many k-disjoint path coverable graph is always f-fault hybrid many-to-many (k_p, k_u)-disjoint path coverable provided $k_p + k_u = k$. Therefore, the gap between the bound of $f + 2k$ for hybrid many-to-many disjoint path covers and the bound of necessity given in Theorem 4 is one also. On the other hand, it has been shown in [25] and [29] that both m-dimensional restricted HL-graphs and $G(2^m, 4)$, $m \geq 3$, are f-fault unpaired many-to-many k-disjoint path coverable for any f and $k \geq 1$ with $f + k \leq m - 2$. Interesting enough, the gap for unpaired many-to-many disjoint path covers is also one. Refer to Lemma 3.

Let $H_0 = G_0 \oplus G_1$ and $H_1 = G_2 \oplus G_3$. Here, G_0 and G_1 are called *subcomponents* of $H_0 \oplus H_1$. The main problem studied in this section is how paired many-to-many disjoint path coverability and unpaired many-to-many disjoint path coverability of G_i's and H_j's are translated into paired many-to-many disjoint path coverability of $H_0 \oplus H_1$. To achieve simpler construction, we make an assumption that each G_i has 2^{m-2} vertices and is of degree $m - 2$. Thus, H_j has 2^{m-1} vertices and is of degree $m - 1$. The main theorem will be stated as follows.

Theorem 5: Let $m \geq 5$. Let G_i, $i = 0, 1, 2, 3$, be a graph of degree $m - 2$ having 2^{m-2} vertices. Suppose each G_i is (a) f-fault paired many-to-many k-disjoint path coverable for any f and $k \geq 2$ with $f + 2k \leq \delta(G_i)$ and (b) f-fault unpaired many-to-many k-disjoint path coverable for any f and $k \geq 1$ with $f + k \leq \delta(G_i) - 2$. Let $H_0 = G_0 \oplus G_1$ and $H_1 = G_2 \oplus G_3$. Furthermore, we suppose each H_j is (c) f-fault paired many-to-many k-disjoint path coverable for any f and $k \geq 2$ with $f + 2k \leq \delta(H_i)$ and (d) f-fault unpaired many-to-many k-disjoint path coverable for any f and $k \geq 1$ with $f + k \leq \delta(H_i) - 2$. Then, $H_0 \oplus H_1$ is

f-fault paired many-to-many k-disjoint path coverable for any f and $k \geq 2$ with $f + 2k \leq \delta(H_0 \oplus H_1) = m$.

Proof of Theorem 5 will be addressed in the next subsection. And then, the theorem is applied to the construction of an f-fault paired k-DPC of restricted HL-graphs in Subsection III-B and of recursive circulant $G(2^m, 4)$ in Subsection III-C.

A. Proof of Theorem 5

Given a fault set F, a set of k sources $S = \{s_1, s_2, \ldots, s_k\}$, and a set of k sinks $T = \{t_1, t_2, \ldots, t_k\}$ in a graph G, a paired many-to-many k-disjoint path cover joining S and T in $G \backslash F$ is denoted by k-DPC$[\{(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)\}|G, F]$. We are to construct a k-DPC$[\{(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)\}|H_0 \oplus H_1, F]$ for any given F with $|F| \leq f$, S and T with $|S| = |T| = k \geq 2$ such that $f + 2k \leq m$.

F_0 and F_1 denote the sets of faulty elements in H_0 and H_1, respectively, and F_2 denotes the set of faulty edges joining vertices in H_0 and vertices in H_1, so that $F = F_0 \cup F_1 \cup F_2$. Let $f_0 = |F_0|$, $f_1 = |F_1|$, and $f_2 = |F_2|$. We also denote by k_i the number of source-sink pairs in H_i, $i = 0, 1$, and by k_2 the number of source-sink pairs between H_0 and H_1. We assume w.l.o.g. that

$$k_0 \geq k_1, \text{ and if } k_0 = k_1, f_0 \geq f_1.$$

We let $I_0 = \{1, 2, \ldots, k_0\}$, $I_2 = \{k_0 + 1, k_0 + 2, \ldots, k_0 + k_2\}$, and $I_1 = \{k_0 + k_2 + 1, k_0 + k_2 + 2, \ldots, k_0 + k_2 + k_1\}$. We assume that $\{s_j, t_j | j \in I_0\} \cup \{s_j | j \in I_2\} \subseteq V(H_0)$ and $\{s_j, t_j | j \in I_1\} \cup \{t_j | j \in I_2\} \subseteq V(H_1)$.

We have $|F| \leq f$, $k = k_0 + k_1 + k_2 \geq 2$, and $f + 2k \leq m$. Observe that a paired many-to-many k-disjoint path cover in $H_0 \oplus H_1$ with a *virtual* fault set $F \cup F'$, where F' is a set of arbitrary $m - 2k - |F|$ fault-free edges, is also a paired many-to-many k-disjoint path cover in $H_0 \oplus H_1$ with the fault set F. Thus, we can assume

$$f + 2k = m \text{ and } |F| = f.$$

By the condition (d), each H_i is $(m-4)$-fault hamiltonian-connected, or equivalently, $(f + 2k - 4)$-fault hamiltonian-connected. Since $m \geq 5$ and $k \geq 2$, we have that

H_i is 1-fault hamiltonian-connected and
f-fault hamiltonian-connected.

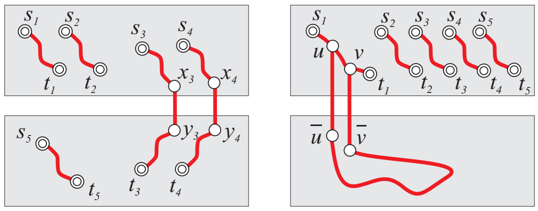
Hereafter in this section, an f-fault k-DPC refers to an f-fault paired many-to-many k-disjoint path cover joining the set of sources and the set of sinks. There are four cases, Cases I through IV.

Case I: $k_1 \geq 1$ or $f_0 \leq f - 1$.

In this case, H_0 is f_0-fault paired many-to-many $(k_0 + k_2)$-disjoint path coverable. By the assumption of $k_0 \geq k_1$, if $k_1 + k_2 \geq 1$, H_1 is f_1-fault paired many-to-many $(k_1 + k_2)$-disjoint path coverable. For a vertex v in $H_0 \oplus H_1$, we denote by $\bar{v}$ the vertex adjacent to v which is in a component different from the component in which v is contained.

Definition 4: A vertex v is called *free* if v is fault-free and not a terminal, that is, $v \notin F$ and $v \notin S \cup T$. An edge (v, w) is called *free* if v and w are free and $(v, w) \notin F$.

We denote by $H[v, w|G, F]$ a hamiltonian path in $G \backslash F$ joining a pair of fault-free vertices v and w in a graph G with a fault set F, that is, 1-DPC$[\{(v, w)\}|G, F]$.



(a) $k_1 + k_2 \geq 1$ (b) $k_1 + k_2 = 0$

Fig. 3. Illustration of Procedure PairedDPC-A.

Procedure PairedDPC-A($H_0 \oplus H_1, S, T, F$)

/* Under the condition of $k_1 \geq 1$ or $f_0 \leq f - 1$. See Fig. 3. */

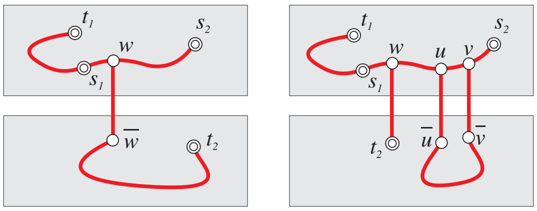
1) Pick up k_2 free edges joining vertices in H_0 and vertices in H_1. Let the free edges be (x_j, y_j), $j \in I_2$, with $x_j \in V(H_0)$.
2) Find $(k_0 + k_2)$-DPC$[\{(s_j, t_j)|j \in I_0\} \cup \{(s_j, x_j)|j \in I_2\}|H_0, F_0]$.
3) Case $k_1 + k_2 \geq 1$:
   a) Find $(k_1 + k_2)$-DPC$[\{(s_j, t_j)|j \in I_1\} \cup \{(y_j, t_j)|j \in I_2\}|H_1, F_1]$.
   b) Merge the two DPC's with the k_2 free edges.
4) Case $k_1 + k_2 = 0$:
   a) Let (u, v) be an edge on some path in the $(k_0 + k_2)$-DPC such that all the $\bar{u}$, $(u, \bar{u})$, $\bar{v}$, and $(v, \bar{v})$ are fault-free.
   b) Find $H[\bar{u}, \bar{v}|H_1, F_1]$.
   c) Merge the $(k_0 + k_2)$-DPC and the hamiltonian path with edges $(u, \bar{u})$ and $(v, \bar{v})$. Discard the edge (u, v).

Lemma 4: When $k_1 \geq 1$ or $f_0 \leq f - 1$, Procedure PairedDPC-A constructs an f-fault k-DPC.

Proof: We claim that the k_2 free edges in Step 1 exist. There are 2^{m-1} candidate free edges and $f + 2k$ blocking elements (f faults and $2k$ terminals). The number of nonblocked candidates is at least $2^{m-1} - (f + 2k) = 2^{m-1} - m > m > k_2$ for any $m \geq 5$. Thus, the claim is proved. The $(k_0 + k_2)$-DPC in H_0 exists when $k_0 + k_2 \geq 2$; if $k_1 \geq 1$, we have $f_0 + 2(k_0 + k_2) \leq f + 2(k - 1) \leq m - 1$, and if $f_0 \leq f - 1$, we have $f_0 + 2(k_0 + k_2) \leq (f - 1) + 2k \leq m - 1$. When $k_0 + k_2 = 1$, the $(k_0 + k_2)$-DPC is a hamiltonian path between two vertices in H_0. The hamiltonian path exists since H_0 is f-fault hamiltonian-connected and $f_0 \leq f$. Similarly, we can show the existence of $(k_1 + k_2)$-DPC in Step 3(a) and the hamiltonian path in Step 4(b). We claim the edge (u, v) in Step 4(a) exists. There are at least $|V(H_0)| - f_0 - k$ candidate edges, and at most $f_1 + f_2$ elements can block the candidates. Since each element blocks at most two candidates, the number of nonblocked candidates is at least $|V(H_0)| - f_0 - k - 2(f_1 + f_2) \geq 2^{m-1} - k - 2f > 2^{m-1} - 2m \geq 6$ for any $m \geq 5$. Note that $f + 2k = m$. ■

Case II: $k_1 = 0$, $f_0 = f$, $k_0 \geq 1$, $k_2 \geq 1$, and for some $a \in I_2$, $\bar{s_a}$ is not a terminal.

All the sources and all the faulty elements, if any, are contained in H_0. Notice that H_0 may not be f_0-fault many-to-many $(k_0 + k_2)$-disjoint path coverable since $f_0 + 2(k_0 + k_2) = f + 2k \not\leq m - 1$. Nevertheless, if $k \geq 3$, there always exists an $(f_0 + 1)$-fault $(k_0 + k_2 - 1)$-DPC in H_0 with s_a being a *virtual* fault. The $(k_0 + k_2 - 1)$-DPC (instead of $(k_0 + k_2)$-DPC) can be utilized to construct an f-fault k-DPC in $H_0 \oplus H_1$. In fact, $(s_a, \bar{s_a})$ plays a role of the free edge for s_a-t_a path.



(a) $\bar{w} \neq t_2$ (b) $\bar{w} = t_2$

Fig. 4. Illustration of Procedure PairedDPC-B.

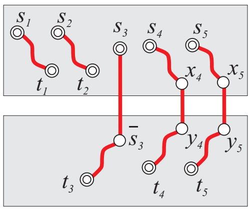


Fig. 5. Illustration of Procedure PairedDPC-C.

When $k = 2$, this approach will not be applied since the existence of an (f_0+1)-fault (k_0+k_2-1)-DPC, or equivalently an (f_0+1)-fault hamiltonian path in H_0 is not guaranteed. We consider the subcase $k = 2$ first, as shown in the following Procedure PairedDPC-B. The procedure is applicable for the case $k_1 = 0$, $f_0 = f$, and $k_0 = k_2 = 1$, regardless of whether the $\bar{s_2}$, $2 \in I_2$, is a terminal or not. It utilizes fault-hamiltonicity of components H_0 and H_1. Its correctness is straightforward since each H_i is f-fault hamiltonian-connected and 1-fault hamiltonian-connected.

Procedure PairedDPC-B($H_0 \oplus H_1, S, T, F$)
/* Under the condition of $k_1 = 0$, $f_0 = f$, and $k_0 = k_2 = 1$. See Fig. 4. */

1) Regarding s_1 as a *virtual* free vertex, find a hamiltonian path $P_h = H[s_2, t_1|H_0, F_0]$. Let $P_h = (s_2, P_w, w, s_1, P_1', t_1)$.
2) Case $\bar{w} \neq t_2$:
    a) Find a hamiltonian path $P_h' = H[\bar{w}, t_2|H_1, \emptyset]$.
    b) Let $P_1 = (s_1, P_1', t_1)$ and $P_2 = (s_2, P_w, w, P_h')$.
3) Case $\bar{w} = t_2$:
    a) Pick up an arbitrary edge (u, v) on P_h with $u, v \neq w$.
    b) Find a hamiltonian path $P_h' = H[\bar{u}, \bar{v}|H_1, \{t_2\}]$.
    c) Let $P_1 = (s_1, P_1', t_1)$ and $P_2 = (s_2, P_w, w, t_2)$, and then replace the edge (u, v) with (u, P_h', v).

Procedure PairedDPC-C($H_0 \oplus H_1, S, T, F$)
/* Under the condition of $k_1 = 0$, $f_0 = f$, $k_0 \geq 1$, $k_2 \geq 1$, $k \geq 3$, and there exists a source s_a, $a \in I_2$, with $\bar{s_a}$ being not a terminal. See Fig. 5. */

1) Pick up $k_2 - 1$ free edges joining vertices in H_0 and vertices in H_1. Let the free edges be (x_j, y_j), $j \in I_2 \backslash a$, with $x_j \in V(H_0)$.
2) Regarding s_a as a *virtual* fault, find $(k_0 + k_2 - 1)$-DPC$[\{(s_j, t_j)|j \in I_0\} \cup \{(s_j, x_j)|j \in I_2 \backslash a\}|H_0, F_0 \cup \{s_a\}]$.
3) Find k_2-DPC$[\{(\bar{s_a}, t_a)\} \cup \{(y_j, t_j)|j \in I_2 \backslash a\}|H_1, \emptyset]$.
4) Merge the two DPC's with $(s_a, \bar{s_a})$ and the $k_2 - 1$ free edges.

Lemma 5: When $k_1 = 0$, $f_0 = f$, $k_0 \geq 1$, $k_2 \geq 1$, $k \geq 3$, and there exists a source s_a, $a \in I_2$, with $\bar{s_a}$ being not a terminal,

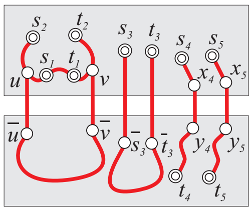


Fig. 6. Illustration of Procedure PairedDPC-D.

Procedure PairedDPC-C constructs an f-fault k-DPC.

Proof: The existence of $k_2 - 1$ free edges can be proved in the same way as in the proof of Lemma 4. The (k_0+k_2-1)-DPC exists since $f_0 + 1 + 2(k_0 + k_2 - 1) = f + 1 + 2(k - 1) = m - 1$. The existence of k_2-DPC is obvious. ■

Case III: $k_1 = 0$, $f_0 = f$, $k_0 \geq 1$, either $k_2 = 0$ or $k_2 \geq 1$ and for every $j \in I_2$, $\bar{s_j}$ is a terminal.

This is one of the hardest cases. An f_0-fault paired (k_0+k_2)-disjoint path coverability of H_0 is not guaranteed. The construction of an f-fault k-DPC relies on the construction of $(k-1)$-DPC in H_1 or when $f \geq 1$, k-DPC in H_1. Notice that if v is a free vertex or a terminal in $\{s_j, t_j|j \in I_0\}$, then $\bar{v}$ is always a free vertex. We consider the subcase $k_0 \geq 2$ first. In this case, the fault-hamiltonicity of H_0 and the paired $(k-1)$-disjoint path coverability of H_1 are employed.

Procedure PairedDPC-D($H_0 \oplus H_1, S, T, F$)
/* Under the condition of $k_1 = 0$, $f_0 = f$, $k_0 \geq 2$, and either $k_2 = 0$ or $k_2 \geq 1$ and $\bar{s_j}$ is a sink for every $j \in I_2$. See Fig. 6. */

1) Pick up k_2 free edges (x_j, y_j), $j \in I_2$, with $x_j \in V(H_0)$ and $y_j \in V(H_1)$ such that (s_j, x_j) is an edge and fault-free.
2) Regarding s_1 and t_1 as *virtual* free vertices, find a hamiltonian path $H[s_2, t_2|H_0, F_0 \cup F' \cup F'']$, where $F' = \{s_j, x_j|j \in I_2\}$ and $F'' = \{s_j, t_j|j \in I_0 \backslash \{1, 2\}\}$. Here, F' and F'' are *virtual* fault sets. Let the hamiltonian path be $(s_2, P_u, u, s_1, P_1', t_1, v, P_v, t_2)$.
3) Find $(k_0 + k_2 - 1)$-DPC$[\{(y_j, t_j)|j \in I_2\} \cup \{(\bar{s_j}, \bar{t_j})|j \in I_0 \backslash \{1, 2\}\} \cup \{(\bar{u}, \bar{v})\}|H_1, \emptyset]$.
4) Merge the hamiltonian path and the DPC with $\{(s_j, x_j, y_j)|j \in I_2\}$, $\{(s_j, \bar{s_j}), (t_j, \bar{t_j})|j \in I_0 \backslash \{1, 2\}\}$, and $\{(u, \bar{u}), (v, \bar{v})\}$. Discard edges (s_1, u) and (t_1, v).

Lemma 6: When $k_1 = 0$, $f_0 = f$, $k_0 \geq 2$, and either $k_2 = 0$ or $k_2 \geq 1$ and $\bar{s_j}$ is a terminal for every $j \in I_2$, Procedure PairedDPC-D constructs an f-fault k-DPC.

Proof: For each $j \in I_2$, we can pick up a free edge (x_j, y_j) one by one since there are $\delta(H_0) = m - 1$ candidates and at most $f + 2(k-1) = m - 2$ blocking elements (f faulty elements, $2k_0$ terminals, $k_2 - 1$ sources, and $k_2 - 1$ free edges picked up). The hamiltonian path in H_0 exists since $f_0 + 2(k_0 - 2) + 2k_2 = f + 2k - 4 = m - 4$. Obviously, the $(k_0 + k_2 - 1)$-DPC exists in H_1. ■

We come to the case that $k_1 = 0$, $f_0 = f$, $k_0 = 1$, and either $k_2 = 0$ or $k_2 \geq 1$ and $\bar{s_j}$ is a terminal for every $j \in I_2$. By the assumption of $k \geq 2$, we have $k_2 \geq 1$. Furthermore, the case $k_2 = 1$ was already considered in Procedure PairedDPC-B, and thus we assume $k_2 \geq 2$. Therefore, we have $k \geq 3$ and $m \geq 6$. Remember $t_1 \in V(H_0)$ and $t_j \in V(H_1)$ for all $j \geq 2$. There are two procedures depending on whether $f \geq 1$ or not. For the case $f \geq 1$, we utilize the fault-hamiltonicity of H_0 and the 0-fault paired k-disjoint path coverability of H_1.

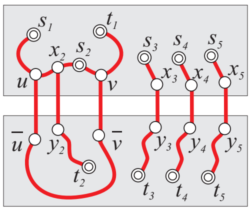
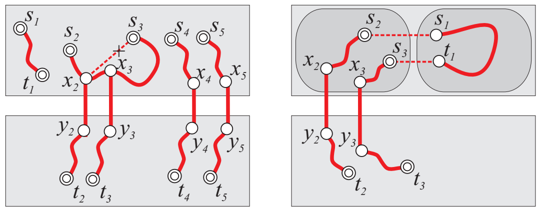


Fig. 7. Illustration of Procedure PairedDPC-E.



(a) $k_2'' \geq 1$ or $k_2'' = 0$ and $\hat{s_a}$ is a free vertex for some $a \in I_2'$

(b) $k_2'' = 0$ and $\hat{s_i}$ is a terminal for every $i \in I_2'$

Fig. 8. Illustration of Procedure PairedDPC-F.

Procedure PairedDPC-E($H_0 \oplus H_1, S, T, F$)
/* Under the condition of $k_1 = 0$, $f_0 = f \geq 1$, $k_0 = 1$, $k_2 \geq 2$, and $\bar{s_j}$ is a sink for every $j \in I_2$. See Fig. 7. */

1) Pick up $k_2 - 1$ free edges (x_j, y_j), $j \in I_2 \backslash 2$, with $x_j \in V(H_0)$ and $y_j \in V(H_1)$ such that (s_j, x_j) is an edge and fault-free.
2) Regarding s_2 as a *virtual* free vertex, find a hamiltonian path $P_h = H[s_1, t_1 | H_0, F_0 \cup F']$, where $F' = \{s_j, x_j | j \in I_2 \backslash 2\}$.
3) There exists a free vertex x_2 such that (s_2, x_2) is an edge of P_h. Removing s_2 and x_2 from P_h results in two subpaths (s_1, P_u, u) and (v, P_v, t_1). Let $y_2 = \bar{x_2}$.
4) Find $(k_0 + k_2)$-DPC$[\{(y_j, t_j) | j \in I_2\} \cup \{(\bar{u}, \bar{v})\} | H_1, \emptyset]$.
5) Merge the hamiltonian path and the DPC with $\{(s_j, x_j, y_j) | j \in I_2\}$ and $\{(u, \bar{u}), (v, \bar{v})\}$.

Lemma 7: When $k_1 = 0$, $f_0 = f \geq 1$, $k_0 = 1$, $k_2 \geq 2$, and $\bar{s_j}$ is a sink for every $j \in I_2$, Procedure PairedDPC-E constructs an f-fault k-DPC.

Proof: The existence of $k_2 - 1$ free edges can be proved in a very similar way as in the proof of Lemma 6. The hamiltonian path P_h exists since $f_0 + 2(k_2 - 1) = f + 2k - 4 = m - 4$. The $(k_0 + k_2)$-DPC exists in H_1 since $2(k_0 + k_2) = m - f \leq m - 1$. ∎

Finally, we have $f = 0$. We will show that for 'some' k_2 free edges joining vertices in H_0 and vertices in H_1, there exist two DPC's: a $(k_0 + k_2)$-DPC from sources to the union of sink t_1 and endvertices of the free edges in H_0, and k_2-DPC between sinks and endvertices of the free edges in H_1. The construction of a $(k_0 + k_2)$-DPC in H_0 is a little complicated. It consists of two subcases, as shown in Steps 1 and 2 of the following procedure.

For a vertex v in G_0 (resp. G_1), $\hat{v}$ denotes the vertex in G_1 (resp. G_0) which is adjacent to v. Let $I_2' = \{j \in I_2 | s_j \in V(G_0)\}$ and $I_2'' = I_2 \backslash I_2'$, and let $k_2' = |I_2'|$ and $k_2'' = |I_2''|$, so that $k_2' + k_2'' = k_2$. It is assumed w.l.o.g. that $k_2' \geq k_2''$.

Procedure PairedDPC-F($H_0 \oplus H_1, S, T, F$)
/* Under the condition of $k_1 = 0$, $f = 0$, $k_0 = 1$, $k_2 \geq 2$, and $\bar{s_j}$ is a sink for every $j \in I_2$. See Fig. 8. */

1) Case $k_2'' \geq 1$ or $k_2'' = 0$ and $\hat{s_a}$ is a free vertex for some $a \in I_2'$:
   a) Let x_a be a free vertex in H_0 such that $(s_a, x_a) \in E$ and $(s_b, x_a) \notin E$ for some $a, b \in I_2$.
   b) Pick up $k_2 - 2$ free edges (x_j, y_j), $j \in I_2 \backslash \{a, b\}$, with $x_j \in V(H_0)$ and $y_j \in V(H_1)$ such that $x_j \neq x_a$.
   c) Find $(k_0 + k_2 - 1)$-DPC$[\{(s_1, t_1), (s_b, x_a)\} \cup \{(s_j, x_j) | j \in I_2 \backslash \{a, b\}\} | H_0, F']$, where $F' = \{s_a\}$. Let the s_b-path in the DPC be (s_b, P', x_b, x_a).
   d) Let s_a-x_a path be (s_a, x_a) and let s_b-x_b path be (s_b, P', x_b). Let $y_a = \bar{x_a}$ and $y_b = \bar{x_b}$.

   a) Pick up two free edges (x_2, y_2) and (x_3, y_3) with $x_2, x_3 \in V(G_0)$ and $y_2, y_3 \in V(H_1)$.
   b) Find 2-DPC$[\{(s_2, x_2), (s_3, x_3)\} | G_0, \emptyset]$.
   c) Find $H[s_1, t_1 | G_1, \emptyset]$.
2) Case $k_2'' = 0$ and $\hat{s_i}$ is a terminal for every $i \in I_2'$:
   /* $k_2 = 2$, $s_2, s_3 \in V(G_0)$, and $s_1, t_1 \in V(G_1)$ */
3) Find k_2-DPC$[\{(y_j, t_j) | j \in I_2\} | H_1, \emptyset]$.
4) Merge the two DPC's with edges (x_j, y_j), $j \in I_2$.

Lemma 8: When $k_1 = 0$, $f = 0$, $k_0 = 1$, $k_2 \geq 2$, and $\bar{s_j}$ is a sink for every $j \in I_2$, Procedure PairedDPC-F constructs an f-fault k-DPC.

Proof: We first claim the existence of x_a in Step 1(a). When $k_2'' \geq 1$, let $a \in I_2'$ and $b \in I_2''$. Then, s_a and s_b are sources contained in G_0 and G_1, respectively. There are $m - 2$ candidates for x_a in G_0 and at most $2k_0 + (k_2 - 1)$ blocking terminals. Since $2k_0 + (k_2 - 1) = k = m - k \leq m - 3$, there exists such a vertex x_a. When $k_2'' = 0$ and $\hat{s_a}$ is a free vertex for some $a \in I_2'$, let s_b be an arbitrary source in G_0 with $b \in I_2 \backslash a$. By the structure of $G_0 \oplus G_1$, $(s_b, \hat{s_a}) \notin E$. Letting $x_a = \hat{s_a}$, the claim is proved. The existence of the $k_2 - 2$ free edges in Step 1(b) is straightforward. The $(k_0 + k_2 - 1)$-DPC in Step 1(c) exists since $1 + 2(k_0 + k_2 - 1) = 2k - 1 = m - 1$. By the choice of x_a, x_b is a free vertex different from x_a. Thus, a $(k_0 + k_2)$-DPC in H_0 is constructed successfully in Step 1. If $k_2'' = 0$ and $\hat{s_i}$ is a terminal for every $i \in I_2'$, we can see that $k_2 = 2$ and $\{\hat{s_2}, \hat{s_3}\} = \{s_1, t_1\}$. Since G_0 is 0-fault paired many-to-many $(k-1)$-disjoint path coverable and G_1 is hamiltonian-connected, a $(k_0 + k_2)$-DPC can be constructed in Step 2. Existence of the k_2-DPC in Step 3 is due to $k_2 < k$, precisely speaking, due to $2k_2 = 2(k-1) \leq m-1$. This completes the proof. ∎

Case IV: $k_2 = k$ and $f_0 = f$.

To construct an f-fault k-DPC in this case, we mainly utilize the *unpaired* many-to-many disjoint path coverability of H_0 and the *paired* many-to-many disjoint path coverability and the hamiltonicity of *subcomponents* G_2 and G_3. By virtue of unpaired many-to-many disjoint path coverability, we are able to keep out of some troublesome subcases although this is one of the hardest cases.

However, there is an exceptional case in which we cannot apply the unpaired many-to-many disjoint path coverability of H_0, the case of $k = 2$. We consider the exceptional case first in the following Procedure PairedDPC-G. Its correctness is straightforward since each H_i is f-fault hamiltonian-connected and 0-fault paired many-to-many 2-disjoint path coverable.

Procedure PairedDPC-G($H_0 \oplus H_1, S, T, F$)
/* Under the condition of $k_2 = k = 2$, $f_0 = f$. See Fig. 9. */

1) Find $H[s_1, s_2 | H_0, F_0]$. Let the hamiltonian path be $(s_1, P_u, u, v, P_v, s_2)$ for some edge (u, v) with $\{\bar{u}, \bar{v}\} \cap$

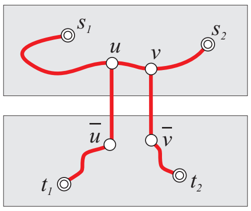
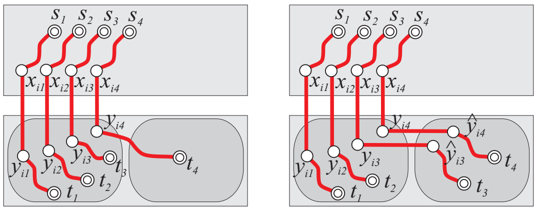


Fig. 9. Illustration of Procedure PairedDPC-G.



(a) $f \geq 1$ (b) $f = 0$ and $2 \leq k_2' \leq k_2''$

Fig. 10. Illustration of Procedure PairedDPC-H.

$\{t_1, t_2\} = \emptyset$.

2) Find 2-DPC$[\{(\bar{u}, t_1), (\bar{v}, t_2)\}|H_1, \emptyset]$.
3) Merge the hamiltonian path and 2-DPC with edges $(u, \bar{u})$ and $(v, \bar{v})$.

We assume $k \geq 3$ and thus $m \geq 6$. For a vertex v in G_2 (resp. G_3), $\hat{v}$ denotes the vertex in G_3 (resp. G_2) which is adjacent to v. We let $I_2' = \{j \in I_2 | t_j \in V(G_2)\}$ and $I_2'' = I_2 \backslash I_2'$, and let $k_2' = |I_2'|$ and $k_2'' = |I_2''|$. We assume w.l.o.g. either $2 \leq k_2' \leq k_2''$ or $k_2' \geq k_2 - 1$.

Procedure PairedDPC-H($H_0 \oplus H_1, S, T, F$)
/* Under the condition of $k_2 = k \geq 3$, $f_0 = f$, and $f \geq 1$ or $2 \leq k_2' \leq k_2''$. See Fig. 10. */

1) Pick up k_2 free edges (x_j, y_j), $j \in I_2$, with $x_j \in V(H_0)$ and $y_j \in V(G_2)$ such that $\hat{y_j}$ is not a terminal.
2) Find an f_0-fault *unpaired* many-to-many k_2-disjoint path cover joining $\{s_j | j \in I_2\}$ and $\{x_j | j \in I_2\}$ in H_0. Let s_j-path in the unpaired k_2-DPC join s_j and x_{i_j}, $j \in I_2$.
3) Case $f \geq 1$: Find k_2-DPC$[\{(y_{i_j}, t_j) | j \in I_2\}|H_1, \emptyset]$.
4) Case $f = 0$ and $2 \leq k_2' \leq k_2''$: Find k_2-DPC in H_1 as follows.
   a) Find k_2'-DPC$[\{(y_{i_j}, t_j) | j \in I_2'\}|G_2, F']$, where $F' = \{y_{i_j} | j \in I_2''\}$.
   b) Find k_2''-DPC$[\{(\hat{y_{i_j}}, t_j) | j \in I_2''\}|G_3, \emptyset]$.
   c) Merge the k_2'-DPC and k_2''-DPC with edges $(y_{i_j}, \hat{y_{i_j}})$, $j \in I_2''$.
5) Merge the *unpaired* k_2-DPC in H_0 and k_2-DPC in H_1 with edges (x_{i_j}, y_{i_j}), $j \in I_2$.

Lemma 9: When $k_2 = k \geq 3$, $f_0 = f$, and $f \geq 1$ or $2 \leq k_2' \leq k_2''$, Procedure PairedDPC-H constructs an f-fault k-DPC.

Proof: The k_2 free edges in Step 1 exist since there are 2^{m-2} candidates and at most $f + 2k$ elements (f faults and $2k$ terminals) block the candidates. Of course, $2^{m-2} - (f + 2k) = 2^{m-2} - m \geq m \geq k_2$ for any $m \geq 6$. The existence of unpaired k_2-DPC is due to that $f_0 + k_2 = f + k = m - k \leq m - 3$. The k_2-DPC in Step 3 exists since $2k_2 \leq (f - 1) + 2k_2 = f + 2k - 1 = m - 1$. The existence of k_2'-DPC in Step 4(a) is due to $|F'| + 2k_2' = k_2'' + 2k_2' = 2k - k_2'' \leq 2k - 2 \leq m - 2$. The k_2''-DPC in Step 4(b) also exists since $2k_2'' = 2k - 2k_2' \leq m - 2$. ■

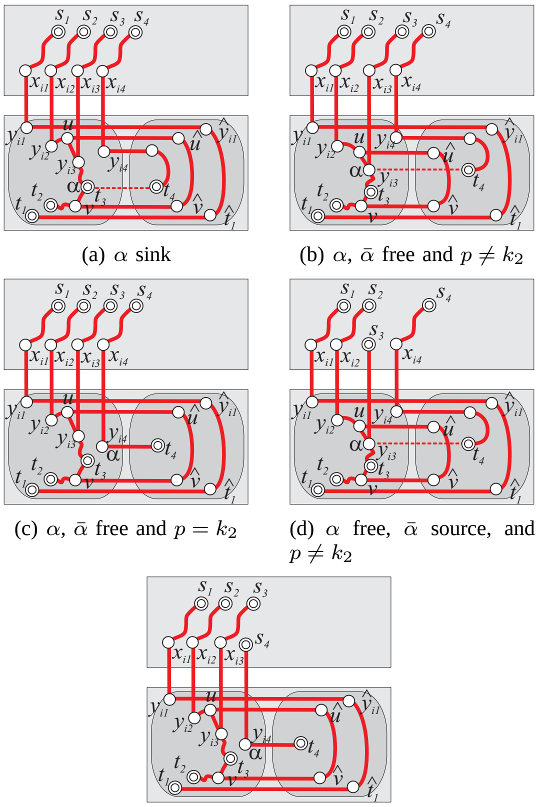
Now, we have $k_2 = k \geq 3$, $f = 0$, and $k_2' \geq k_2 - 1$. The subcase $k_2' = k_2 - 1$ is considered first in the following. The vertex α in G_2, which is adjacent to the sink in G_3, plays an extraordinary role in the construction. The unpaired many-to-many disjoint path coverability of H_0, the hamiltonicity of G_2, and the paired many-to-many disjoint path coverability of G_3 are utilized.

Procedure PairedDPC-I($H_0 \oplus H_1, S, T, F$)
/* Under the condition of $k_2 = k \geq 3$, $f = 0$, and $k_2' = k_2 - 1$. See Fig. 11. */

1) Let t_{k_2} be the sink in G_3, and let $\alpha = \hat{t_{k_2}}$.
2) a) Case α is a sink:
   Pick up k_2 free edges (x_j, y_j), $j \in I_2$, with $x_j \in V(H_0)$ and $y_j \in V(G_2)$.
   b) Case both α and $\bar{\alpha}$ are free vertices:
   Inclusive of $(\bar{\alpha}, \alpha)$, pick up k_2 free edges (x_j, y_j), $j \in I_2$, with $x_j \in V(H_0)$ and $y_j \in V(G_2)$.
   c) Case α is a free vertex and $\bar{\alpha}$ is a source, say s_p:
   Pick up $k_2 - 1$ free edges (x_j, y_j), $j \in I_2 \backslash p$, with $x_j \in V(H_0)$ and $y_j \in V(G_2)$.
3) a) Case α is a sink or both α and $\bar{\alpha}$ are free vertices:
   Find an *unpaired* k_2-DPC joining $\{s_j | j \in I_2\}$ and $\{x_j | j \in I_2\}$ in H_0. Let s_j-path in the unpaired DPC join s_j and x_{i_j}, $j \in I_2$. We let $t_p = \alpha$ if α is a sink, and let $y_{i_p} = \alpha$ if both α and $\bar{\alpha}$ are free vertices.
   b) Case α is a free vertex and $\bar{\alpha}$ is a source s_p:
   Regarding s_p as a *virtual* fault, find an *unpaired* $(k_2 - 1)$-DPC joining $\{s_j | j \in I_2 \backslash p\}$ and $\{x_j | j \in I_2 \backslash p\}$ in H_0. Let s_j-path in the unpaired DPC join s_j and x_{i_j}, $j \in I_2 \backslash p$. Let s_p-path be (s_p), and let $x_{i_p} = s_p$ and $y_{i_p} = \alpha$.
4) a) Case $p \neq k_2$:
   Let $q \in I_2$ with $q \neq p, k_2$. Find $H[y_{i_q}, t_q | G_2, F']$, where $F' = \{y_{i_j}, t_j | j \in I_2 \backslash \{p, q, k_2\}\} \cup \{y_{i_{k_2}}\}$. Let the hamiltonian path be $(y_{i_q}, P_u, u, y_{i_p}, P', t_p, v, P_v, t_q)$. Find $(k_2 - 1)$-DPC$[\{(\hat{u}, \hat{v}), (\hat{y_{i_{k_2}}}, t_{k_2})\} \cup \{(\hat{y_{i_j}}, \hat{t_j}) | j \in I_2 \backslash \{p, q, k_2\}\}|G_3, \emptyset]$. Merge the hamiltonian path and $(k_2 - 1)$-DPC with edges $(u, \hat{u})$, $(v, \hat{v})$, $(y_{i_{k_2}}, \hat{y_{i_{k_2}}})$, and $(y_{i_j}, \hat{y_{i_j}})$, $(t_j, \hat{t_j})$, $j \in I_2 \backslash \{p, q, k_2\}$.
   b) Case $p = k_2$:
   Let $q, r \in I_2$ with $q, r \neq k_2$. Find $H[y_{i_q}, t_q | G_2, F']$, where $F' = \{y_{i_j}, t_j | j \in I_2 \backslash \{p, q, r\}\} \cup \{y_{i_p}\}$. Let the hamiltonian path be $(y_{i_q}, P_u, u, y_{i_r}, P', t_r, v, P_v, t_q)$. Find $(k_2 - 2)$-DPC$[\{(\hat{u}, \hat{v})\} \cup \{(\hat{y_{i_j}}, \hat{t_j}) | j \in I_2 \backslash \{p, q, r\}\}|G_3, F'']$, where $F'' = \{t_{k_2}\}$. Merge the hamiltonian path and $(k_2 - 2)$-DPC with edges $(u, \hat{u})$, $(v, \hat{v})$, (y_{i_p}, t_{k_2}), and $(y_{i_j}, \hat{y_{i_j}})$, $(t_j, \hat{t_j})$, $j \in I_2 \backslash \{p, q, r\}$.
5) Merge the k_2 disjoint paths joining s_j and x_{i_j} in H_0 and k_2 disjoint paths joining y_{i_j} and t_j in H_1 with edges (x_{i_j}, y_{i_j}), $j \in I_2$.

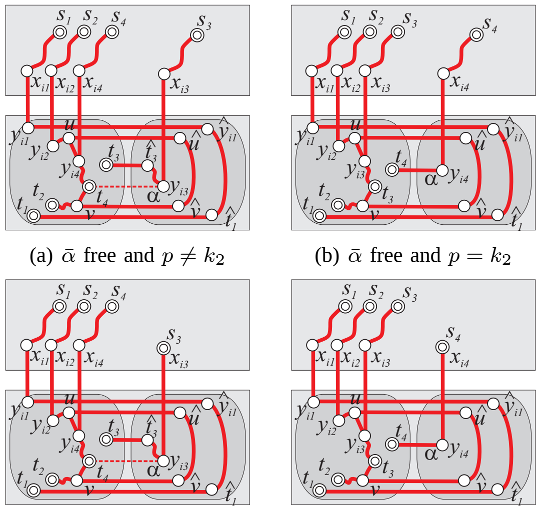
Lemma 10: When $k_2 = k \geq 3$, $f = 0$, and $k_2' = k_2 - 1$, Procedure PairedDPC-I constructs a k-DPC.

Proof: The existence of free edges in Step 2 can be shown in a similar way to the proof of Lemma 9. Both the unpaired k_2-DPC in Step 3(a) and the 1-fault unpaired $(k_2 - 1)$-DPC in Step 3(b) exist since $k_2 = k = m - k \leq m - 3$. When $p \neq k_2$



(a) α sink (b) α, $\bar{\alpha}$ free and $p \neq k_2$ (c) α, $\bar{\alpha}$ free and $p = k_2$ (d) α free, $\bar{\alpha}$ source, and $p \neq k_2$ (e) α free, $\bar{\alpha}$ source, and $p = k_2$

Fig. 11. Illustration of Procedure PairedDPC-I.



(a) $\bar{\alpha}$ free and $p \neq k_2$ (b) $\bar{\alpha}$ free and $p = k_2$ (c) $\bar{\alpha}$ source and $p \neq k_2$ (d) $\bar{\alpha}$ source and $p = k_2$

Fig. 12. Illustration of Procedure PairedDPC-J.

(Step 4(a)) the hamiltonian path between y_{i_q} and t_q in G_2 exists since $|F'| \leq 2(k_2-3)+1 = 2k-5 = m-5$. By the construction, $t_{k_2} \notin \{\hat{u}, \hat{v}, \hat{y_{i_{k_2}}}\} \cup \{\hat{y_{i_j}}, \hat{t_j} | j \in I_2 \backslash \{p, q, k_2\}\}$. The (k_2-1)-DPC in G_3 exists since $2(k_2-1) = 2k-2 = m-2$. Similarly, when $p = k_2$ (Step 4(b)), we can see $t_{k_2} \notin \{\hat{u}, \hat{v}\} \cup \{\hat{y_{i_j}}, \hat{t_j} | j \in I_2 \backslash \{p, q, r\}\}$ and the existence of the hamiltonian path in G_2 and 1-fault (k_2-2)-DPC in G_3. ∎

When $k_2 = k \geq 3$, $f = 0$, and $k_2' = k_2$, the following Procedure PairedDPC-J constructs a k_2-DPC. The procedure is very similar to Procedure PairedDPC-I. Its correctness can be shown in a similar way to the proof of Lemma 10, and it is omitted in this paper.

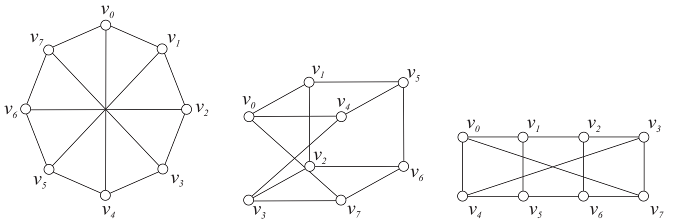
Procedure PairedDPC-J($H_0 \oplus H_1, S, T, F$)
/* Under the condition of $k_2 = k \geq 3$, $f = 0$, and $k_2' = k_2$. See Fig. 12. */

1) Let $\alpha = \hat{t_{k_2}}$. Here, α is a free vertex in G_3.
2) a) Case $\bar{\alpha}$ is a free vertex:
   Let $(x_1, y_1) = (\bar{\alpha}, \alpha)$. Pick up $k_2 - 1$ free edges (x_j, y_j), $j \in I_2 \backslash 1$, with $x_j \in V(H_0)$ and $y_j \in V(G_2)$.
   b) Case $\bar{\alpha}$ is a source, say s_p:
   Pick up $k_2 - 1$ free edges (x_j, y_j), $j \in I_2 \backslash p$, with $x_j \in V(H_0)$ and $y_j \in V(G_2)$.
3) a) Case $\bar{\alpha}$ is a free vertex:
   Find an *unpaired* k_2-DPC joining $\{s_j | j \in I_2\}$ and $\{x_j | j \in I_2\}$ in H_0. Let s_j-path in the unpaired k_2-DPC join s_j and x_{i_j}, $j \in I_2$. We let $y_{i_p} = \alpha$.
   b) Case $\bar{\alpha}$ is a source s_p:
   Regarding s_p as a *virtual* fault, find an *unpaired* $(k_2 - 1)$-DPC joining $\{s_j | j \in I_2 \backslash p\}$ and $\{x_j | j \in I_2 \backslash p\}$ in H_0. Let s_j-path in the unpaired DPC join s_j and x_{i_j}, $j \in I_2 \backslash p$. Let s_p-path be (s_p), and let $x_{i_p} = s_p$ and $y_{i_p} = \alpha$.
4) a) Case $p \neq k_2$:
   Let $q \in I_2$ with $q \neq p, k_2$. Find $H[y_{i_q}, t_q | G_2, F']$, where $F' = \{y_{i_j}, t_j | j \in I_2 \backslash \{p, q, k_2\}\} \cup \{t_p\}$. Let the hamiltonian path be $(y_{i_q}, P_u, u, y_{i_{k_2}}, P', t_{k_2}, v, P_v, t_q)$. Find (k_2-1)-DPC$[\{(\hat{u}, \hat{v}), (y_{i_p}, \hat{t_p})\} \cup \{(\hat{y_{i_j}}, \hat{t_j}) | j \in I_2 \backslash \{p, q, k_2\}\} | G_3, \emptyset]$. Merge the hamiltonian path and (k_2-1)-DPC with edges $(u, \hat{u})$, $(v, \hat{v})$, $(t_p, \hat{t_p})$, and $(y_{i_j}, \hat{y_{i_j}})$, $(t_j, \hat{t_j})$, $j \in I_2 \backslash \{p, q, k_2\}$.
   b) Case $p = k_2$:
   Let $q, r \in I_2$ with $q, r \neq k_2$. Find $H[y_{i_q}, t_q | G_2, F']$, where $F' = \{y_{i_j}, t_j | j \in I_2 \backslash \{p, q, r\}\} \cup \{t_{k_2}\}$. Let the hamiltonian path be $(y_{i_q}, P_u, u, y_{i_r}, P', t_r, v, P_v, t_q)$. Find (k_2-2)-DPC$[\{(\hat{u}, \hat{v})\} \cup \{(\hat{y_{i_j}}, \hat{t_j}) | j \in I_2 \backslash \{p, q, r\}\} | G_3, F'']$, where $F'' = \{y_{i_p}\}$. Merge the hamiltonian path and (k_2-2)-DPC with edges $(u, \hat{u})$, $(v, \hat{v})$, (y_{i_p}, t_{k_2}), and $(y_{i_j}, \hat{y_{i_j}})$, $(t_j, \hat{t_j})$, $j \in I_2 \backslash \{p, q, r\}$.
5) Merge the k_2 disjoint paths joining s_j and x_{i_j} in H_0 and k_2 disjoint paths joining y_{i_j} and t_j in H_1 with edges (x_{i_j}, y_{i_j}), $j \in I_2$.

B. Restricted HL-graphs

Vaidya *et al.*[35] introduced a class of hypercube-like interconnection networks, called *HL-graphs*, which can be defined by applying the $\oplus$ operation repeatedly as follows: $HL_0 = \{K_1\}$; for $m \geq 1$, $HL_m = \{G_0 \oplus G_1 | G_0, G_1 \in HL_{m-1}\}$. Then, $HL_1 = \{K_2\}$; $HL_2 = \{C_4\}$; $HL_3 = \{Q_3, G(8,4)\}$. Here, C_4 is a cycle graph with 4 vertices, Q_3 is a 3-dimensional hypercube, and $G(8,4)$ is a recursive circulant which is defined as follows: the vertex set is $\{v_i | 0 \leq i \leq 7\}$ and the edge set is $\{(v_i, v_j) | i+1$ or $i+4 \equiv j \pmod 8\}$. $G(8,4)$ is isomorphic to twisted cube TQ_3 and Möbius ladder with four spokes as shown in Figure 13.

In [27], a subclass of nonbipartite HL-graphs, called *restricted HL-graphs*, was introduced by the authors, which is defined



(a) $G(8,4)$ (b) TQ_3 (c) Möbius ladder

Fig. 13. Isomorphic graphs.

recursively as follows: $RHL_m = HL_m$ for $0 \le m \le 2$; $RHL_3 = HL_3 \backslash Q_3 = \{G(8,4)\}$; $RHL_m = \{G_0 \oplus G_1 | G_0, G_1 \in RHL_{m-1}\}$ for $m \ge 4$. A graph which belongs to RHL_m is called an *m-dimensional restricted HL-graph.* Many of the nonbipartite hypercube-like interconnection networks such as crossed cube, Möbius cube, twisted cube, multiply twisted cube, Mcube, generalized twisted cube, etc. proposed in the literature are restricted HL-graphs with the exception of recursive circulant $G(2^m, 4)$ and "near" bipartite interconnection networks such as twisted m-cube. In fact, every $G(2^m, 4)$ with odd m is an m-dimensional restricted HL-graph. Some works on HL-graphs and restricted HL-graphs appeared in the literature; for example, hamiltonicity of HL-graphs[20], fault-hamiltonicity of restricted HL-graphs[27], and fault-panconnectivity and fault-pancyclicity of restricted HL-graphs[30].

In this subsection, we are to construct an f-fault paired many-to-many k-DPC in an m-dimensional restricted HL-graph for any f and $k \ge 2$ with $f + 2k \le m$ by employing Theorem 5. For our purpose, we need the unpaired many-to-many disjoint path coverability of restricted HL-graphs with faulty elements. It was considered in [25] as follows.

Lemma 11: [25] Every m-dimensional restricted HL-graph, $m \ge 3$, is f-fault unpaired many-to-many k-disjoint path coverable for any f and $k \ge 1$ with $f + k \le m - 2$.

The existence of a paired many-to-many 2-DPC in 4-dimensional restricted HL-graphs is checked by a computer program for each $G(8,4) \oplus G(8,4)$ in RHL_4, sources s_1 and s_2, and sinks t_1 and t_2. Thus, we have the lemma.

Lemma 12: Every 4-dimensional restricted HL-graph is 0-fault paired many-to-many 2-disjoint path coverable.

Now, we are ready to state the paired many-to-many disjoint path coverability of restricted HL-graphs.

Theorem 6: Every m-dimensional restricted HL-graph, $m \ge 3$, is f-fault paired many-to-many k-disjoint path coverable for any f and $k \ge 2$ with $f + 2k \le m$.

Proof: The proof is by induction on m. For $m = 3$, the theorem is vacantly true since $f + 2k \ge 4 > m$. For $m = 4$, the theorem holds true by Lemma 12. Let $m \ge 5$. Theorem 5 and Lemma 11 lead to the theorem. ∎

Corollary 2: Every m-dimensional restricted HL-graph, $m \ge 3$, is f-fault hybrid many-to-many (k_p, k_u)-disjoint path coverable for any f, k_p, and k_u with $k_p + k_u \ge 2$ and $f + 2(k_p + k_u) \le m$.

C. Recursive circulant $G(2^m, 4)$

Recursive circulant is an interconnection network proposed in [26]. Recursive circulant $G(N, d)$, $d \ge 2$, is defined as follows: the vertex set $V = \{v_0, v_1, v_2, \cdots, v_{N-1}\}$, and the edge set $E = \{(v_i, v_j) \mid$ there exists k, $0 \le k \le \lceil \log_d N \rceil - 1$, such that $i + d^k \equiv j \pmod N\}$. $G(N, d)$ is a circulant graph with N vertices and jumps of powers of d, $d^0, d^1, \cdots, d^{\lceil \log_d N \rceil - 1}$.

In this work, our attention is restricted to $G(N, d)$ with $N = 2^m$ and $d = 4$. $G(2^m, 4)$, whose degree is m, compares favorably to the hypercube Q_m. While retaining the attractive properties of hypercube Q_m such as node-symmetry, recursive structure, the connectivity, etc., it achieves noticeable improvements in diameter[26] and possesses a complete binary tree with $2^m - 1$ vertices as a subgraph[19]. $G(N, d)$ with degree three or higher is hamiltonian-connected[7]. $G(N, d)$ with $N = cd^m$ and $1 \le c < d$ is hamiltonian decomposable[1], [13], [21], that is, the set of edges can be partitioned into edge-disjoint hamiltonian cycles (and a 1-factor when the degree is odd). In [13], the edge forwarding index and the bisection width of recursive circulants were also analyzed.

In this subsection, we will construct an f-fault paired many-to-many k-DPC in recursive circulant $G(2^m, 4)$ for any f and $k \ge 2$ with $f + 2k \le m$. The unpaired many-to-many disjoint path coverability of $G(2^m, 4)$ explored in [29] are shown below. It will be utilized to establish our result. We denote by $G \times G'$ the product of graphs G and G'.

Lemma 13: [29] $G(2^m, 4)$ with $m \ge 3$, $G(2^{m-1}, 4) \times K_2$ with $m \ge 4$, and $G(2^{m-2}, 4) \times C_4$ with $m \ge 5$ are all f-fault unpaired many-to-many k-disjoint path coverable for any f and $k \ge 1$ with $f + k \le m - 2$.

Now, we consider the paired many-to-many disjoint path coverability of recursive circulant $G(2^m, 4)$. Due to its recursive structure, $G(2^m, 4)$ is isomorphic to some graph $[G(2^{m-2}, 4) \times K_2] \oplus [G(2^{m-2}, 4) \times K_2]$. To employ Theorem 5, we need to develop the paired many-to-many disjoint path coverability of $G(2^{m-2}, 4) \times K_2$ as well as $G(2^{m-2}, 4)$. For this kind of technical reasons, we will show a stronger result than the aforementioned, as stated in Theorem 7. The proof proceeds by induction on m. Basis will be shown in Lemma 14.

Lemma 14: (a) $G(2^4, 4)$ is 0-fault paired many-to-many 2-disjoint path coverable[24].
(b) $G(2^4, 4) \times K_2$ is 1-fault paired many-to-many 2-disjoint path coverable.

Proof: The proof of (b) is completed according to the proof of Theorem 5 given in Subsection III-A. Notice that Procedures PairedDPC-C, PairedDPC-E, PairedDPC-F, PairedDPC-H, PairedDPC-I, and PairedDPC-J are never employed since all of the procedures assume $k \ge 3$ and $m \ge 6$. ∎

Theorem 7: $G(2^m, 4)$ with $m \ge 3$, $G(2^{m-1}, 4) \times K_2$ with $m \ge 4$, and $G(2^{m-2}, 4) \times C_4$ with $m \ge 5$ are all f-fault paired many-to-many k-disjoint path coverable for any f and $k \ge 2$ with $f + 2k \le m$.

Proof: Let $G' = G(2^{m-2}, 4)$ and $G'' = G(2^{m-3}, 4) \times K_2$. Observe that $G(2^{m-1}, 4) \times K_2$ is isomorphic to some graph $[G'' \times K_2] \oplus [G'' \times K_2]$, and that each of $G(2^m, 4)$ and $G(2^{m-2}, 4) \times C_4$ is isomorphic to some graph $[G' \times K_2] \oplus [G' \times K_2]$. Furthermore, $G'' \times K_2$ is isomorphic to $G(2^{m-3}, 4) \times C_4$. The proof is by induction on m. For $G(2^3, 4)$, the theorem is vacantly true. Base cases hold for $G(2^4, 4)$ by Lemma 14(a), for $G(2^3, 4) \times K_2$ by Lemma 12, and for $G(2^4, 4) \times K_2$ by Lemma 14(b). For $G(2^m, 4)$ with $m \ge 5$, $G(2^{m-1}, 4) \times K_2$ with $m \ge 6$, and $G(2^{m-2}, 4) \times C_4$ with $m \ge 5$, by Theorem 5 and Lemma 13, paired many-to-many disjoint path covers are constructed. ∎

IV. Concluding Remarks

In this paper, we considered many-to-many disjoint path covers of the three types: paired, unpaired, and hybrid type, and investigated some interesting properties including their relationships, application to strong hamiltonicity, and necessary conditions. Also, we gave a construction scheme for paired many-to-many disjoint path covers in $H_0 \oplus H_1$, where $H_0 = G_0 \oplus G_1$ and $H_1 = G_2 \oplus G_3$. Mainly utilizing the construction, we proved that every m-dimensional restricted HL-graph and recursive circulant $G(2^m, 4)$ are f-fault paired many-to-many k-disjoint path coverable for any f and $k \geq 2$ with $f + 2k \leq m$. The bound m on $f + 2k$ is improved by one as compared with [28], and thus the gap between the bound achieved and the bound $m+1$ of necessity is now one.

To construct an f-fault paired many-to-many k-disjoint path cover for the hardest case of $k_2 = k$ and $f_0 = f$ (Case IV), the unpaired many-to-many disjoint path coverability of component H_0 of $H_0 \oplus H_1$ was employed. It has the effect of keeping out some troublesome cases, and thus the construction is greatly simplified. If hybrid many-to-many disjoint path coverability of H_0 were available, the construction for the case of $k_0 \geq 1$ and $k_2 \geq 2$ would be simplified, too. Of course, this approach is applicable for any graphs which can be defined recursively.

It is open to bridge the "gap" between the bounds achieved and the bounds of necessity for the three kinds of many-to-many disjoint path covers in restricted HL-graphs and recursive circulant $G(2^m, 4)$ addressed in Section III. Among them, we conjecture that for some constant m_0, every m-dimensional restricted HL-graph and $G(2^m, 4)$ with $m \geq m_0$ are f-fault hybrid many-to-many (k_p, k_u)-disjoint path coverable for any $k_p \geq 1$ and $k_u \geq 2$ with $f+2k \leq m+1$, where $k = k_p+k_u$. Under the same condition of $f + 2k \leq m + 1$, the hybrid many-to-many disjoint path cover is more possible compared with the paired counterpart.

Acknowledgment

This work was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD, Basic Research Promotion Fund) (KRF-2006-521-D00377).

References

[1] D.K. Biss, "Hamiltonian decomposition of recursive circulant graphs," *Discrete Mathematics* **214**, pp. 89-99, 2000.
[2] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications,* 5th printing, American Elsevier Publishing Co., Inc., 1976.
[3] C.-H. Chang, C.-K. Lin, H.-M. Huang, and L.-H. Hsu, "The super laceability of the hypercubes," *Inform. Proc. Lett.* **92(1)**, pp. 15-21, 2004.
[4] F.B. Chedid, "On the generalized twisted cube," *Inform. Proc. Lett.* **55**, pp. 49-52, 1995.
[5] X.-B. Chen, "Cycles passing through prescribed edges in a hypercube with some faulty edges," *Inform. Proc. Lett.* **104(6)**, pp. 211-215, 2007.
[6] C.C. Chen and J. Chen, "Nearly optimal one-to-many parallel routing in star networks," *IEEE Trans. on Parallel and Distributed Systems* **8(12)**, pp. 1196-1202, 1997.
[7] C.C. Chen and N.F. Quimpo, "On strongly hamiltonian abelian group graphs," *Lecture Notes in Mathematics* **884** (Springer, Berlin), Australian Conference on Combinatorial Mathematics, pp. 23-34, 1980.
[8] P. Cull and S. Larson, "The Möbius cubes," in *Proc. of the 6th IEEE Distributed Memory Computing Conf.*, pp. 699-702, 1991.
[9] T. Dvořák and P. Gregor, "Hamiltonian paths with prescribed edges in hypercubes," *Discrete Mathematics* **307**, pp. 1982-1998, 2007.
[10] K. Efe, "A variation on the hypercube with lower diameter," *IEEE Trans. on Computers* **40(11)**, pp. 1312-1316, 1991.
[11] K. Efe, "The crossed cube architecture for parallel computation," *IEEE Trans. on Parallel and Distributed Systems* **3(5)**, pp. 513-524, 1992.
[12] M.R. Garey and D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W.H. Freeman & Co., San Francisco, 1979.
[13] G. Gauyacq, C. Micheneau, and A. Raspaud, "Routing in recursive circulant graphs: edge forwarding index and hamiltonian decomposition," in *Proc. of the 24th International Workshop on Graph-Theoretic Concepts in Computer Science (WG98)*, pp. 227-241, 1998.
[14] Q.P. Gu and S. Peng, "Cluster fault-tolerant routing in star graphs," *Networks* **35(1)**, pp. 83-90, 2000.
[15] P.A.J. Hilbers, M.R.J. Koopman, J.L.A. van de Snepscheut, "The Twisted Cube," in J. Bakker, A. Nijman, P. Treleaven, eds., *PARLE: Parallel Architectures and Languages Europe, Vol. I: Parallel Architectures*, Springer, pp. 152-159, 1987.
[16] S.-Y. Hsieh, G.-H. Chen, and C.-W. Ho, "Fault-free hamiltonian cycles in faulty arrangement graphs," *IEEE Trans. Parallel and Distributed Systems* **10(3)**, pp. 223-237, Mar. 1999.
[17] H.-C. Hsu, T.-K. Li, J.J.M. Tan, and L.-H. Hsu, "Fault hamiltonicity and fault hamiltonian connectivity of the arrangement graphs," *IEEE Trans. Computers* **53(1)**, pp. 39-53, Jan. 2004.
[18] P.-L. Lai, H.-C. Hsu, "On the two-equal-disjoint path cover problem of crossed cubes," in *Proc. 9th Joint International Conference on Information Sciences JCIS 2006*, Taiwan, pp. 603-606, Oct. 2006.
[19] H.-S. Lim, J.-H. Park, and K.-Y. Chwa, "Embedding trees into recursive circulants," *Discrete Applied Mathematics* **69**, pp. 83-99, 1996.
[20] C.-D. Park and K.Y. Chwa, "Hamiltonian properties on the class of hypercube-like networks," *Inform. Proc. Lett.* **91**, pp. 11-17, 2004.
[21] J.-H. Park, "Hamiltonian decomposition of recursive circulants," in *Proc. International Symposium on Algorithms and Computation ISAAC'98*, Taejon, Korea, pp. 297-306, 1998.
[22] J.-H. Park, "One-to-one disjoint path covers in recursive circulants," *Journal of KISS* **30(12)**, pp. 691-698, 2003 (in Korean).
[23] J.-H. Park, "One-to-many disjoint path covers in a graph with faulty elements," in *Proc. of the International Computing and Combinatorics Conference COCOON 2004*, pp. 392-401, Aug. 2004.
[24] J.-H. Park, "Many-to-many disjoint path covers in double loop networks," *Journal of KISS* **32(8)**, pp. 426-431, 2005 (in Korean).
[25] J.-H. Park, "Unpaired many-to-many disjoint path covers in hypercube-like interconnection networks," *Journal of KISS* **33(10)**, pp. 789-796, 2006 (in Korean).
[26] J.-H. Park and K.Y. Chwa, "Recursive circulants and their embeddings among hypercubes," *Theoretical Computer Science* **244**, pp. 35-62, 2000.
[27] J.-H. Park, H.-C. Kim, and H.-S. Lim, "Fault-hamiltonicity of hypercube-like interconnection networks," in *Proc. IEEE International Parallel and Distributed Processing Symposium IPDPS 2005*, Denver, Apr. 2005.
[28] J.-H. Park, H.-C. Kim, and H.-S. Lim, "Many-to-many disjoint path covers in hypercube-like interconnection networks with faulty elements," *IEEE Trans. on Parallel and Distributed Systems* **17(3)**, pp. 227-240, Mar. 2006.
[29] J.-H. Park and J.-H. Lee, "Unpaired many-to-many disjoint path covers in recursive circulant $G(2^m, 4)$," manuscript, 2007 (submitted for publication).
[30] J.-H. Park, H.-S. Lim, and H.-C. Kim, "Panconnectivity and pancyclicity of hypercube-like interconnection networks with faulty elements," *Theoretical Computer Science* **377(1-3)**, pp. 170-180, 2007.
[31] N.K. Singhvi and K. Ghose, "The Mcube: a symmetrical cube based network with twisted links," in *Proc. of the 9th IEEE Int. Parallel Processing Symposium IPPS 1995*, pp. 11-16, 1995.
[32] C.-H. Tsai, J.J.M. Tan, Y.-C. Chuang, and L.-H. Hsu, "Fault-free cycles and links in faulty recursive circulant graphs," in *Proc. of the Workshop on Algorithms and Theory of Computation ICS 2000*, pp. 74-77, 2000.
[33] C.-H. Tsai, J.J.M. Tan, and L.-H. Hsu, "The super-connected property of recursive circulant graphs," *Inform. Proc. Lett.* **91(6)**, pp. 293-298, 2004.
[34] Y.-C. Tseng, S.-H. Chang, and J.-P. Sheu, "Fault-tolerant ring embedding in a star graph with both link and node failures," *IEEE Trans. Parallel and Distributed Systems* **8(12)**, pp. 1185-1195, 1997.
[35] A.S. Vaidya, P.S.N. Rao, S.R. Shankar, "A class of hypercube-like networks," in *Proc. of the 5th IEEE Symposium on Parallel and Distributed Processing SPDP 1993*, pp. 800-803, Dec. 1993.