One-to-Many Disjoint Path Covers in a Graph with Faulty Elements*

Jung-Heum Park

School of Computer Science and Information Engineering
The Catholic University of Korea, Puchon, Kyonggi-do 420-743, Korea
j.h.park@catholic.ac.kr

Abstract. In a graph G, k disjoint paths joining a single source and k distinct sinks that cover all the vertices in the graph are called a *one-to-many k-disjoint path cover* of G. We consider a k-disjoint path cover in a graph with faulty vertices and/or edges obtained by merging two graphs H_0 and H_1, $|V(H_0)| = |V(H_1)| = n$, with n pairwise nonadjacent edges joining vertices in H_0 and vertices in H_1. We present a sufficient condition for such a graph to have a k-disjoint path cover and give the construction scheme. Applying our main result to interconnection graphs, we observe that when there are f or less faulty elements, all of recursive circulant $G(2^m, 4)$, twisted cube TQ_m, and crossed cube CQ_m of degree m have k-disjoint path covers for any $f \geq 0$ and $k \geq 2$ such that $f + k \leq m - 1$.

1 Introduction

Usually a measure of reliability (or fault-tolerance) of an interconnection network is given by the maximum number of nodes which can fail simultaneously without prohibiting the ability of each fault-free node to communicate with all other fault-free nodes. Connectivity of an interconnection graph corresponds to the reliability of the interconnection network which is subject to node failures.

It is well-known that connectivity of a graph G was characterized in terms of disjoint paths joining a pair of vertices in G. According to Menger's theorem, a graph G is k-connected if and only if for every pair of vertices s and t, G has k disjoint paths (of type one-to-one) joining them. It is straightforward to verify that a graph G is k-connected if and only if G has k disjoint paths (of type one-to-many) joining every source s and k distinct sinks $t_1, t_2, \ldots, t_k$ such that $t_i \neq s$ for all $1 \leq i \leq k$.

Sometimes "restricted" one-to-many disjoint paths joining a source s and k distinct sinks $t_1, t_2, \ldots, t_k$ are required. When we analyze Rabin number[1] of a graph G, we are to find k disjoint paths $P_1, P_2, \ldots, P_k$ such that $\max\{l_i\}$ is as small as possible, where P_i is an s-t_i path of length l_i, $1 \leq i \leq k$. A star-like tree[2] is a subdivision of a star, a tree with only one vertex which is not an endvertex. Given a source s and lengths $l_1, l_2, \ldots, l_k$ of k paths, the star-like tree

* This work was supported by grant No. R05-2003-000-11506-0 from the Basic Research Program of the Korea Science & Engineering Foundation.

problem is essentially to find one-to-many disjoint paths joining s and some k sinks whose lengths are $l_1, l_2, \ldots, l_k$, respectively.

Given a source s and a set of k sinks $T = \{t_1, t_2, \ldots, t_k\}$ in a graph G, we are concerned with one-to-many disjoint paths $P_1, P_2, \ldots, P_k$ in G joining s and T that *cover* all the vertices in the graph, that is, $\bigcup_{1 \le i \le k} V(P_i) = V(G)$ and $V(P_i) \cap V(P_j) = \{s\}$ for all $i \ne j$. Here $V(P_i)$ and $V(G)$ denote the vertex sets of P_i and G, respectively. We call such k disjoint paths a *one-to-many k-disjoint path cover* (in short, *one-to-many k-DPC*) of G.

Embedding of linear arrays and rings into a faulty interconnection graph is one of the central issues in parallel processing. The problem is modelled as finding as long fault-free paths and cycles as possible in the graph with some faulty vertices and/or edges. A graph G is called *f-fault hamiltonian* (resp. *f-fault hamiltonian-connected*) if there exists a hamiltonian cycle (resp. if each pair of vertices are joined by a hamiltonian path) in $G \backslash F$ for any set F of faulty elements such that $|F| \le f$. For a graph G to be f-fault hamiltonian (resp. f-fault hamiltonian-connected), it is necessary that $f \le \delta(G) - 2$ (resp. $f \le \delta(G) - 3$), where $\delta(G)$ is the minimum degree of G.

To a graph G with a set of faulty elements F, the definition of a one-to-many disjoint path cover can be extended. Given a source s and a set of k sinks $T = \{t_1, t_2, \ldots, t_k\}$ in $G \backslash F$, a one-to-many k-disjoint path cover joining s and T is k disjoint paths P_i joining s and t_i, $1 \le i \le k$, such that $\bigcup_{1 \le i \le k} V(P_i) = V(G) \backslash F$, $V(P_i) \cap V(P_j) = \{s\}$ for all $i \ne j$, and every edge on each path P_i is fault-free. Such a one-to-many k-DPC is denoted by k-DPC$[s, \{t_1, t_2, \ldots, t_k\} | G, F]$. A graph G is called *f-fault one-to-many k-disjoint path coverable* if for any set F of faulty elements such that $|F| \le f$, G has k-DPC$[s, \{t_1, t_2, \ldots, t_k\} | G, F]$ for every source s and k distinct sinks $t_1, t_2, \ldots, t_k$ in $G \backslash F$.

Proposition 1. *Let G be a graph. The following statements are equivalent.*
(a) G is f-fault hamiltonian-connected.
(b) G is f-fault one-to-many 1*-disjoint path coverable.*
(c) G is f-fault one-to-many 2*-disjoint path coverable.*

We are given two graphs H_0 and H_1 with n vertices. We denote by V_i and E_i the vertex set and edge set of H_i, $i = 0, 1$, respectively. We let $V_0 = \{v_1, v_2, \ldots, v_n\}$ and $V_1 = \{w_1, w_2, \ldots, w_n\}$. With respect to a permutation $M = (i_1, i_2, \ldots, i_n)$ of $\{1, 2, \ldots, n\}$, we can "merge" the two graphs into a graph $H_0 \oplus_M H_1$ with $2n$ vertices in such a way that the vertex set $V = V_0 \cup V_1$ and the edge set $E = E_0 \cup E_1 \cup E_2$, where $E_2 = \{(v_j, w_{i_j}) | 1 \le j \le n\}$. We denote by $H_0 \oplus H_1$ an arbitrary graph obtained by merging H_0 and H_1 w.r.t. some permutation M. Here, H_0 and H_1 are called *components* of $H_0 \oplus H_1$.

In this paper, we will show that we can always construct f-fault one-to-many $k+1$-DPC in $H_0 \oplus H_1$ by using f-fault one-to-many k-DPC of H_i and fault-hamiltonicity of H_i, $i = 0, 1$. Precisely speaking, we will prove the following.

Theorem 1. *For $f \ge 0$ and $k \ge 2$, let H_i, $i = 0, 1$, be a graph with n vertices satisfying the following three conditions:*
(a) H_i is f-fault one-to-many k-disjoint path coverable.

(b) H_i is $f+k-2$-fault hamiltonian-connected(2-disjoint path coverable).
(c) H_i is $f+k-1$-fault hamiltonian.
Then, $H_0 \oplus H_1$ is f-fault one-to-many $k+1$-disjoint path coverable.

By applying the above theorem to interconnection graphs, we will show that all of recursive circulant $G(2^m, 4)$, twisted cube TQ_m, and crossed cube CQ_m of degree m are f-fault one-to-many k-disjoint path coverable for every pair of $f \geq 0$ and $k \geq 2$ such that $f + k \leq m - 1$.

2 Construction of One-to-Many DPC in $H_0 \oplus H_1$

In this section, we will prove Theorem 1 by constructing a one-to-many $k+1$-DPC in $H_0 \oplus H_1$ when the number of faulty elements is f or less. Since each H_i satisfies the condition (c) of Theorem 1, it is necessary that $f + k - 1 \leq \delta_i - 2$, where $\delta_i = \delta(H_i)$. That is,

$$f + k \leq \delta - 1,$$

where $\delta = \min\{\delta_0, \delta_1\}$. We assume w.l.o.g. that the source s is in H_0, and there are $k+1$ distinct sinks $T = \{t_1, t_2, \ldots, t_{k+1}\}$, k_i sinks in H_i, $i = 0, 1$. Thus,

$$k_0 + k_1 = k + 1.$$

We assume that $\{t_1, t_2, \ldots, t_{k_1}\}$ is the set of sinks in H_1. We denote by F the set of faulty elements in $H_0 \oplus H_1$. We let f_0, f_1, and f_2 be the numbers of faulty elements in H_0, in H_1, and between H_0 and H_1, respectively. Let f_i^v and f_i^e be the numbers of faulty vertices and edges in H_i, respectively. Then,

$$|F| = f_0 + f_1 + f_2 \leq f, \text{ and } f_i^v + f_i^e = f_i, i = 0, 1.$$

Before going on, we need some definitions and notation. For a vertex v in $H_0 \oplus H_1$, we denote by $\bar{v}$ the vertex adjacent to v which is in a different component from v. A vertex v in $H_0 \oplus H_1$ is called *free* if $v \neq s$, $v \notin T$, and $v \notin F$. A *free bridge* of a vertex w is the path $(w, \bar{w})$ of length one if $\bar{w} \notin T$, $\bar{w} \notin F$, and $(w, \bar{w}) \notin F$; otherwise, it is a path $(w, v, \bar{v})$ of length two such that $v \neq \bar{w}$, v and $\bar{v}$ are free vertices, and $(w, v), (v, \bar{v}) \notin F$.

Lemma 1. *For any source and sink w in $H_0 \oplus H_1$, there exists a free bridge of w.*

Proof. There are at least $\delta + 1$ candidates for a free bridge of w, and at most $f + k + 1$ elements "block" the candidates. For the source w, at most f faulty elements and $k+1$ sinks form blocking elements. For a sink w, blocking elements are faulty elements, k sinks (excluding w itself), and the source. Since $f + k \leq \delta - 1$, the lemma is proved. □

We define L to be a set of edges joining a sink w in H_1 and $\bar{w}$ in H_0 different from s such that $(w, \bar{w})$ is not the free bridge of w. That is,

$$L = \{(w, \bar{w}) | w \text{ is a sink in } H_1, \bar{w} \neq s, \text{ and } \bar{w} \in T \text{ or } \bar{w} \in F \text{ or } (w, \bar{w}) \in F\}.$$

An edge $(w, \bar{w})$ in L is called *matched* and its endvertices are called *matched* vertices. We denote by f_2^L be the number of matched faulty edges, faulty edges in L. We let $l = |L|$. Then,

$$l \le k_0 + f_0^v + f_2^L.$$

If $\bar{s} \in T$ and $(s, \bar{s}) \in F$, we let $\alpha = 1$; otherwise, let $\alpha = 0$. Obviously,

$$l + \alpha \le k_1, \text{ and } f_2^L + \alpha \le f_2.$$

First we will construct f-fault one-to-many $k+1$-DPC for the case of $k_1 \ge 1$, and then we will consider the case of $k_1 = 0$ later in Section 2.3. Construction I gives an f-fault one-to-many $k+1$-DPC unless the following conditions of C1, C2, and C3 are satisfied:

C1: $k_1 = 1$, $\bar{s} \in T$, $f_0 = f$,

C2: $k_1 + f_1 \ge k + f$,

C3: $l + \alpha \ge 2$, $l = k_0 + f_0^v + f_2^L$, $f = f_0^v + f_1 + f_2^L + \alpha$.

Construction II considers the case that satisfies C3. Remaining cases are considered later in Section 2.1 and 2.2. We denote by $H[v, w|H_i, F]$ a hamiltonian path in $H_i \backslash F$ joining a pair of fault-free vertices v and w. We let $F_i = F \cap V_i$.

Construction I

UNDER the condition of $k_1 \ge 1$.
UNLESS C1, C2, and C3

1. Let $t_1 = \begin{cases} \text{an arbitrary matched sink,} & \text{if } l \ge 1; \\ \bar{s}, & \text{if } l = 0 \text{ and } \bar{s} \in T; \\ \text{an arbitrary sink,} & \text{if } l = 0 \text{ and } \bar{s} \notin T. \end{cases}$
2. Find a free bridge of s and free bridges $B_j = (t_j, \ldots, t'_j)$ of t_j for all j, $2 \le j \le k_1$. They should be pairwise disjoint.
3. When $(s, \bar{s})$ is the free bridge of s:
   - Find k-DPC$[s, \{t'_2, \ldots, t'_{k_1}, t_{k_1+1}, \ldots, t_{k+1}\}|H_0, F_0]$.
   - Find $H[\bar{s}, t_1|H_1, F_1 \cup F']$, where $F' = \bigcup_{2 \le j \le k_1} \{V(B_j) \cap V_1\}$.
   - Merge the k-DPC and the hamiltonian path obtained in the previous step with the edge $(s, \bar{s})$ and the free bridges B_j, $2 \le j \le k_1$.
4. When $(s, \bar{s})$ is not the free bridge of s: Let (s, x, y) be the free bridge of s.
   4.1 When $(s, \bar{s}) \notin F$, $\bar{s} \in T$, and $k_1 \ge 2$: Let z be t_1 if $\bar{s} \ne t_1$; otherwise, let z be t_2. If $\bar{s} \ne t_1$, we assume w.l.o.g. that $\bar{s} = t_2$.
   - Find k-DPC$[s, \{t'_3, \ldots, t'_{k_1}, t_{k_1+1}, \ldots, t_{k+1}\} \cup \{x\}|H_0, F_0]$.
   - Find $H[y, z|H_1, F_1 \cup F']$, where $F' = \{\bar{s}\} \cup \bigcup_{3 \le j \le k_1} \{V(B_j) \cap V_1\}$.
   - Merge the k-DPC and the hamiltonian path obtained with the edges $(s, \bar{s})$, (x, y) and the free bridges B_j, $3 \le j \le k_1$.

   4.2 When $(s, \bar{s}) \notin F$, $\bar{s} \in T$, and $k_1 = 1$, or $(s, \bar{s}) \in F$, or $\bar{s} \in F$: In this case, it holds true that $f_0 < f$. (Recall C1.)
   - Find k-DPC$[s, \{t'_2, \ldots, t'_{k_1}, t_{k_1+1}, \ldots, t_{k+1}\}|H_0, F_0 \cup \{x\}]$.
   - Find $H[y, t_1|H_1, F_1 \cup F']$, where $F' = \bigcup_{2 \le j \le k_1} \{V(B_j) \cap V_1\}$.
   - Merge the k-DPC and the hamiltonian path obtained with the edge (x, y) and the free bridges B_j, $2 \le j \le k_1$.

When we find a k-DPC or a hamiltonian path, sometimes we regard some fault-free vertices and/or edges as faulty elements. They are called *virtual* faults. For example, in step 4.2 of Construction I, the vertex x is a virtual fault when we find a k-DPC, and F' is a set of virtual faults when we find a hamiltonian path.

The existence of pairwise disjoint free bridges in step 2 of Construction I can be shown by extending Lemma 1. The proof is omitted in this paper.

Lemma 2. *We can always find k-DPC's of H_0 in step 3, 4.1, and 4.2 of Construction I.*

Proof. In each step, the number of sinks is k and the number of faulty elements including virtual faults is at most f. The k-DPC's can be found since H_0 is f-fault one-to-many k-disjoint path coverable. □

Lemma 3. *We can always find hamiltonian paths of H_1 in step 3, 4.1, and 4.2 of Construction I unless C2 and C3.*

Proof. Let $f' = |F'|$. A sink t_j in H_1 contributes to f' by 2 if t_j is matched or if $t_j = \bar{s}$ and $(s, \bar{s}) \in F$. For $l \geq 1$, $f' + f_1 = (k_1 - l) + 2(l - 1) + \alpha + f_1 = k_1 + l + \alpha + f_1 - 2$. When $l + \alpha \geq 2$, $f' + f_1 \leq k_1 + (k_0 + f_0^v + f_2^L) + \alpha + f_1 - 2 \leq (k + 1) + f - 2 \leq f + k - 1$. The equality holds only when $l = k_0 + f_0^v + f_2^L$ and $f = f_0^v + f_2^L + \alpha + f_1$. Unless C3, $f' + f_1 \leq f + k - 2$, and thus there exists a hamiltonian path between every pair of fault-free vertices in H_1. When $l + \alpha < 2$, that is, $l = 1$ and $\alpha = 0$, $f' + f_1 = k_1 + 1 + f_1 - 2$. Unless C2, $f' + f_1 \leq f + k - 2$. For $l = 0$, $f' + f_1 \leq (k_1 - 1) + f_1$. Also in this case, unless C2, $f' + f_1 \leq f + k - 2$. This completes the proof. □

Let us consider the case that satisfies C3. Obviously, $k_1 \geq 2$. Observe that for every free vertex $w (\neq \bar{s})$ in H_1, $(w, \bar{w})$ is a free bridge of w.

Construction II

UNDER the condition C3: $l + \alpha \geq 2$, $l = k_0 + f_0^v + f_2^L$, $f = f_0^v + f_1 + f_2^L + \alpha$.

1. Let t_1 be a sink as defined in step 1 of Construction I.
2. Find a free bridge of s and free bridges $B_j = (t_j, \ldots, t'_j)$ of t_j for all j, $2 \leq j \leq k_1$. They should be pairwise disjoint.
3. When $(s, \bar{s})$ is the free bridge of s ($\alpha = 0$, $l \geq 2$): We assume that t_2 is a matched sink.
   - Find $H[\bar{s}, t_1 | H_1, F_1 \cup F']$, where $F' = \{(t_1, t_2)\} \cup \bigcup_{3 \leq j \leq k_1} \{V(B_j) \cap V_1\}$. Let the hamiltonian path $(\bar{s}, \ldots, t_2, z, \ldots, t_1)$. Note that z is a free vertex.
   - Find k-DPC$[s, \{t'_3, \ldots, t'_{k_1}, t_{k_1+1}, \ldots, t_{k+1}\} \cup \{\bar{z}\} | H_0, F_0]$.
   - Merge the k-DPC and the hamiltonian path with the edges $(s, \bar{s})$, $(\bar{z}, z)$ and the free bridges B_j, $3 \leq j \leq k_1$.
4. When $(s, \bar{s})$ is not the free bridge of s: Let (s, x, y) be the free bridge of s.

4.1 When $(s,\bar{s}) \notin F$ and $\bar{s} \in T$ ($\alpha = 0$, $l \geq 2$, $k_1 \geq 3$): We assume that t_2 is a matched sink and t_3 is $\bar{s}$.
- Find $H[y, t_1|H_1, F_1 \cup F']$, where $F' = \{(t_1,t_2)\} \cup \bigcup_{3\leq j\leq k_1}\{V(B_j) \cap V_1\}$. Let the hamiltonian path $(y, \dots, t_2, z, \dots, t_1)$.
- Find k-DPC$[s, \{t'_4, \dots, t'_{k_1}, t_{k_1+1}, \dots, t_{k+1}\} \cup \{x, \bar{z}\}|H_0, F_0]$.
- Merge the k-DPC and the hamiltonian path with the edges $(s, \bar{s})$, (x, y), $(\bar{z}, z)$ and the free bridges B_j, $4 \leq j \leq k_1$.

4.2 When $(s,\bar{s}) \in F$ or $\bar{s} \in F$: Let t_2 be a matched sink if $\bar{s} \notin T$; otherwise, let t_2 be $\bar{s}$. Note that $f_0 < f$.
- Find $H[y, t_1|H_1, F_1 \cup F']$, where $F' = \{(t_1,t_2)\} \cup \bigcup_{3\leq j\leq k_1}\{V(B_j) \cap V_1\}$. Let the hamiltonian path $(y, \dots, t_2, z, \dots, t_1)$.
- Find k-DPC$[s, \{t'_3, \dots, t'_{k_1}, t_{k_1+1}, \dots, t_{k+1}\} \cup \{\bar{z}\}|H_0, F_0 \cup \{x\}]$.
- Merge the k-DPC and the hamiltonian path with the edge (x, y), $(\bar{z}, z)$ and the free bridges B_j, $3 \leq j \leq k_1$.

It is easy to observe the existence of k-DPC's of H_0 in step 3, 4.1, and 4.2 of Construction II.

Lemma 4. *We can always find hamiltonian paths of H_1 in step 3, 4,1, and 4.2 of Construction II.*

Proof. Let $f' = |F'|$. For $\alpha = 0$ ($l \geq 2$), $f' + f_1 = (k_1 - l) + 2(l-2) + 1 + f_1 = k_1 + l + f_1 - 3 = k_1 + (k_0 + f_0^v + f_2^L) + f_1 - 3 \leq (k+1) + f - 3 = f + k - 2$. For $\alpha = 1$, $f' + f_1 = (k_1 - l - 1) + 2(l-1) + 1 + f_1 = k_1 + l + f_1 - 2 = k_1 + (k_0 + f_0^v + f_2^L) + f_1 - 2 = (k+1) + (f-1) - 2 = f + k - 2$. This completes the proof. □

Hereafter in this section, we will construct f-fault one-to-many $k+1$-DPC's for two exceptional cases of C1 and C2 and for the case of $k_1 = 0$.

2.1 When C2: $k_1 + f_1 \geq k + f$

The condition C2 is equivalent to that $k_0 + (f_0 + f_2) \leq 1$. When $(s, \bar{s})$ is not the free bridge of s, we denote by (s, x, y) the free bridge of s. Let v be the endvertex of the free bridge of s in H_1. Of course, $v = \bar{s}$ if $(s, \bar{s})$ is the free bridge of s; otherwise, $v = y$. We let

$$F' = \begin{cases} \emptyset, & \text{if } (s,\bar{s}) \text{ is the free bridge of } s; \\ \{x\}, & \text{if at least one of } (s,\bar{s}) \text{ and } \bar{s} \text{ are faulty.} \end{cases}$$

Case 1: $f_0 + f_2 = 0$ and $k_0 = 0$.

We first consider the case that $\bar{s}$ is a free vertex or a faulty vertex. We find $H[v, t_1|H_1, F_1]$ and let the path be $(v, \dots, t_{k+1}, z_{k+1}, \dots, t_k, z_k, \dots, t_2, z_2, \dots, t_1)$. We find k-DPC$[s, \{\bar{z_2}, \dots, \bar{z_{k+1}}\}|H_0, F']$, and then merge the k-DPC and the hamiltonian path with the free bridge of s and the edges $(\bar{z_j}, z_j)$ for all $2 \leq j \leq k+1$. The existence of a hamiltonian path in H_1 and a k-DPC in H_0 is straightforward.

For the remaining case that $\bar{s}$ is a sink, we let $t_1 = \bar{s}$. Find a hamiltonian cycle $C = (t_2, \ldots, z, t_3, \ldots, u)$ in $H_1 \backslash (F_1 \cup F'')$, where $F'' = \{t_1\} \cup \{t_4, \ldots, t_{k+1}\}$. The C exists since H_1 is $f+k-1$-fault hamiltonian and $f_1 + |F''| = f_1 + (k-1) \leq f+k-1$. We let $Q_1 = (t_2, \ldots, z)$ and $Q_2 = (t_3, \ldots, u)$ be two disjoint paths such that $C = (Q_1, Q_2)$. Find k-DPC$[s, \{\bar{z}, \bar{u}, \bar{t_4}, \ldots, \bar{t_{k+1}}\}|H_0, \emptyset]$, and then merge the k-DPC and C with the edges $(s, \bar{s})$, $(\bar{z}, z)$, $(\bar{u}, u)$, and $(\bar{t_j}, t_j)$ for all $4 \leq j \leq k+1$.

Case 2: $f_0 + f_2 = 0$ and $k_0 = 1$.

Let t_{k+1} be the sink in H_0. We consider the case that $\bar{s}$ is either a free vertex or a faulty vertex first. We find k-DPC$[v, \{t_1, t_2, \ldots, t_k\}|H_1, F_1]$. We let the v-t_j path in the k-DPC be $(v, z_j, \ldots, t_j)$ for $1 \leq j \leq k$, and assume w.l.o.g. that $\bar{z_j} \neq t_{k+1}$ for every $2 \leq j \leq k$. We find k-DPC$[s, \{\bar{z_2}, \ldots, \bar{z_k}, t_{k+1}\}|H_0, F']$, and then merge the two k-DPC's with the free bridge of s and $(\bar{z_j}, z_j)$ for all $2 \leq j \leq k$. A $k+1$-DPC of $H_0 \oplus H_1$ can be obtained by removing all the edges (v, z_j), $2 \leq j \leq k$. It is easy to observe the existence of two k-DPC's.

When $\bar{s} \in T$, we let $t_1 = \bar{s}$ and assume w.l.o.g. that $\bar{t_j} \neq t_{k+1}$ for all $3 \leq j \leq k$. Find a hamiltonian cycle $C = (t_2, z, \ldots, u)$ in $H_1 \backslash (F_1 \cup F'')$, where $F'' = \{t_1\} \cup \{t_3, \ldots, t_k\}$. We assume w.l.o.g. that $\bar{z} \neq t_{k+1}$. Find k-DPC$[s, \{\bar{z}, \bar{t_3}, \ldots, \bar{t_k}, t_{k+1}\}|H_0, \emptyset]$, and then merge the k-DPC and C with the edges $(s, \bar{s})$, $(\bar{z}, z)$, and $(\bar{t_j}, t_j)$ for all $3 \leq j \leq k$.

Case 3: $f_0 + f_2 = 1$ and $k_0 = 0$.

We assume w.l.o.g. that $(t_1, \bar{t_1})$ is the free bridge of t_1. Let us first consider the case that either $(s, \bar{s})$ is a free bridge of s or at least one of $(s, \bar{s})$ and $\bar{s}$ are faulty elements. We find k-DPC$[v, \{t_2, \ldots, t_{k+1}\}|H_1, F_1 \cup \{t_1\}]$. The k-DPC exists since $f_1 + 1 = f_1 + (f_0 + f_2) \leq f$. We let the v-t_j path in the k-DPC be $(v, z_j, \ldots, t_j)$, $2 \leq j \leq k+1$. We assume w.l.o.g. that $(z_j, \bar{z_j})$ is the free bridge of z_j for all $2 \leq j \leq k$. Find k-DPC$[s, \{\bar{t_1}, \bar{z_2}, \ldots, \bar{z_k}\}|H_0, F_0 \cup F']$. The existence of the k-DPC is due to the fact that $|F'| = 1$ if and only if $f_1 + f_2 \geq 1$ (equivalently, $f_0 \leq f-1$). Finally, we merge the two k-DPC's with the free bridge of s and the edges $(\bar{t_1}, t_1)$, and $(\bar{z_j}, z_j)$ for all $2 \leq j \leq k$.

Now, $(s, \bar{s})$ is a fault-free edge and $\bar{s}$ is a sink. We let $t_1 = \bar{s}$. Find k-DPC$[y, \{t_2, \ldots, t_{k+1}\}|H_1, F_1 \cup \{t_1\}]$, and let the y-t_j path in the k-DPC be $(y, z_j, \ldots, t_j)$, $2 \leq j \leq k+1$. We assume w.l.o.g. that $(z_j, \bar{z_j})$ is the free bridge of z_j for every $2 \leq j \leq k$. We find k-DPC$[s, \{\bar{z_2}, \ldots, \bar{z_k}\} \cup \{x\}|H_0, F_0]$. Finally, we merge the two k-DPC's in a very similar way to the previous case.

2.2 When C1: $k_1 = 1$, $\bar{s} \in T$, $f_0 = f$

We let $\bar{s} = t_1$. Find k-DPC$[s, \{t_2, \ldots, t_{k+1}\}|H_0, F_0]$. Since $f + k \leq \delta - 1$, there exists j, $2 \leq j \leq k+1$, such that the s-t_j path P_j in the k-DPC passes through at least two vertices z and u adjacent to s satisfying $(s, z), (s, u) \notin F$. Let $P_j = (Q_1, Q_2)$, where $Q_1 = (s, z, \ldots, x)$, $Q_2 = (u, \ldots, t_j)$. That is, $P_j = (s, z, \ldots, x, u, \ldots, t_j)$. We redefine P_j to be $P_j = (s, Q_2) = (s, u, \ldots, t_j)$, and construct s-t_1 path in a way that $P_1 = (Q_1, \bar{x}, H[\bar{x}, t_1|H_1, \emptyset])$. All the s-t_j paths form a $k+1$-DPC in $H_0 \oplus H_1$.

2.3 When $k_1 = 0$

If $(s, \bar{s})$ is not the free bridge of s, we let (s, x, y) be the free bridge of s. First, let us consider the case that either $f_1 + f_2 = 0$ or $f_1 + f_2 = 1$ and $(s, \bar{s})$ is not the free bridge of s. Obviously, when $f_1 + f_2 = 1$ and $(s, \bar{s})$ is not the free bridge of s, $\bar{s} \in F$ or $(s, \bar{s}) \in F$. Thus, in this case, for every vertex v in H_0 different from s, $(v, \bar{v})$ is fault-free. We let

$$F' = \begin{cases} \emptyset, & \text{if } f_1 + f_2 = 0; \\ \{x\}, & \text{if } f_1 + f_2 = 1 \text{ and } (s, \bar{s}) \text{ is not the free bridge of } s. \end{cases}$$

We first find k-DPC$[s, \{t_1, t_2, \ldots, t_k\}|H_0, F_0 \cup F']$. Here, sink t_{k+1} is regarded as a free vertex. The k-DPC exists since $f_0 + |F'| \le f$. Some s-t_j path P_j in the k-DPC passes through t_{k+1}, and let $P_j = (s, Q_1, t_{k+1}, z, Q_2, t_j)$. Define s-t_{k+1} path to be (s, Q_1, t_{k+1}). We denote by B_s the free bridge of s, and let v be the endvertex of B_s in H_1. Redefine P_j to be $P_j = (B_s, H[v, \bar{z}|H_1, F_1], z, Q_2, t_j)$. All of the s-t_j paths form a $k+1$-DPC in $H_0 \oplus H_1$.

Now, we consider the case that either $f_1 + f_2 \ge 2$ or $f_1 + f_2 = 1$ and $(s, \bar{s})$ is the free bridge of s. If there is a sink t_j such that $(t_j, \bar{t_j})$ is the free bridge of t_j, then we let t_1 be such a sink. If there is no such a sink, we let t_1 be an arbitrary sink and let $(t_1, z, \bar{z})$ be the free bridge of t_1. We let $F' = F'_s \cup F'_{t_1}$, where

$$F'_s = \begin{cases} \emptyset, & \text{if } (s, \bar{s}) \text{ is the free bridge of } s; \\ \{x\}, & \text{if } (s, \bar{s}) \text{ is not the free bridge of } s, \text{ and} \end{cases}$$

$$F'_{t_1} = \begin{cases} \{t_1\}, & \text{if } (t_1, \bar{t_1}) \text{ is the free bridge of } t_1; \\ \{t_1, z\}, & \text{if } (t_1, \bar{t_1}) \text{ is not the free bridge of } t_1. \end{cases}$$

We find k-DPC$[s, \{t_2, \ldots, t_{k+1}\}|H_0, F_0 \cup F']$, and then find $H[v, w|H_1, F_1]$, where w is the endvertex of the free bridge of t_1 in H_1. The existence of such a k-DPC is due to the following lemma. The k-DPC and an s-t_1 path $P_1 = (s, B_s, H[v, w|H_1, F_1], B_1, t_1)$ form a $k+1$-DPC, where B_1 is the free bridge of t_1.

Lemma 5. *If either $f_1 + f_2 \ge 2$ or $f_1 + f_2 = 1$ and $(s, \bar{s})$ is a free bridge of s, then $f_0 + f' \le f$, where $f' = |F'|$.*

Proof. Observe that if $(s, \bar{s})$ is not the free bridge of s, then at least one of $\bar{s}$ and $(s, \bar{s})$ are faulty elements. Similarly, if $(t_j, \bar{t_j})$ is not the free bridge of t_j, then $\bar{t_j} \in F$ or $(t_j, \bar{t_j}) \in F$. When $(t_1, \bar{t_1})$ is not the free bridge of t_1, $f_1 + f_2 \ge k + 1$, and thus $f_0 \le f - k - 1$ and $f_0 + f' \le f_0 + 3 \le (f - k - 1) + 3 \le f$. Recall that $k \ge 2$. Now, let us consider the case that $(t_1, \bar{t_1})$ is the free bridge of t_1. If $(s, \bar{s})$ is the free bridge of s, $f_0 + f' = f_0 + 1 \le f$ since $f_1 + f_2 \ge 1$. If $(s, \bar{s})$ is not the free bridge of s, $f_1 + f_2 \ge 2$ by assumption, and thus $f_0 + f' = f_0 + 2 \le f$. □

3 Application

A graph G is called *fully one-to-many disjoint path coverable* if for any $f \ge 0$ and $k \ge 2$ such that $f + k \le \delta(G) - 1$, G is f-fault one-to-many k-disjoint path coverable. A graph G is called *almost fully one-to-many disjoint path coverable* if for any $f \geq 0$ and $k \geq 3$ such that $f + k \leq \delta(G) - 1$, G is f-fault one-to-many k-disjoint path coverable. Note that almost fully one-to-many disjoint path coverable graph G which is $\delta(G) - 3$-fault hamiltonian-connected is fully one-to-many disjoint path coverable. The following is immediate from Theorem 1.

Corollary 1. *Let H_i be a graph with n vertices satisfying the following two conditions, where $\delta_i = \delta(H_i)$, $i = 0, 1$.*
(a) H_i is fully one-to-many disjoint path coverable.
(b) H_i is $\delta_i - 2$-fault hamiltonian.
Then, $H_0 \oplus H_1$ is almost fully one-to-many disjoint path coverable.

3.1 Recursive circulants $G(2^m, 4)$

$G(2^m, 4)$ is an m-regular graph with 2^m vertices. According to the recursive structure of recursive circulants[6], we can observe that $G(2^m, 4)$ is isomorphic to $G(2^{m-2}, 4) \times K_2 \oplus_M G(2^{m-2}, 4) \times K_2$ for some permutation M. Obviously, $G(2^{m-2}, 4) \times K_2$ is isomorphic to $G(2^{m-2}, 4) \oplus_{M'} G(2^{m-2}, 4)$ for some M'. First we consider fault-hamiltonicity of $G(2^m, 4)$ and $G(2^m, 4) \times K_2$, and then we show that $G(2^m, 4)$, $m \geq 3$, is fully one-to-many disjoint path coverable. Lemma 7 implies that $G(2^{m-1}, 4) \times K_2$ of degree m, $m \geq 4$, is $m - 3$-fault hamiltonian-connected and $m - 2$-fault hamiltonian. The proof of Lemma 7 is omitted in this paper due to space limit.

Lemma 6. *[7] $G(2^m, 4)$, $m \geq 3$, is $m - 3$-fault hamiltonian-connected and $m - 2$-fault hamiltonian.*

Lemma 7. *If a graph G is $\delta(G) - 3$-fault hamiltonian-connected and $\delta(G) - 2$-fault hamiltonian for $\delta(G) \geq 3$, then $G \times K_2$ is $\delta(G \times K_2) - 3$-fault hamiltonian-connected and $\delta(G \times K_2) - 2$-fault hamiltonian.*

Theorem 2. *$G(2^m, 4)$, $m \geq 3$, is fully one-to-many disjoint path coverable.*

Proof. The proof is by induction on m. For $m = 3$, the theorem holds true by Lemma 6. For $m = 4$, by Lemma 6, it is sufficient to show that $G(2^4, 4)$ is 0-fault one-to-many 3-disjoint path coverable. The proof is mainly by case analysis, and omitted here. For $m \geq 5$, by Corollary 1 and Lemma 7, $G(2^{m-2}, 4) \times K_2$ is fully one-to-many disjoint path coverable, and thus, by Corollary 1 and Lemma 6, $G(2^m, 4)$ is fully one-to-many disjoint path coverable. □

3.2 Twisted cube TQ_m, crossed cube CQ_m

Originally, twisted cube TQ_m is defined for odd m. We let $TQ_m = TQ_{m-1} \times K_2$ for even m. Then, TQ_m is isomorphic to $TQ_{m-1} \oplus_M TQ_{m-1}$ for some M. Also, crossed cube CQ_m is isomorphic to $CQ_{m-1} \oplus_{M'} CQ_{m-1}$ for some M'. Both TQ_m and CQ_m are m-regular graphs with 2^m vertices. The fault-hamiltonicity of them were studied in the literature.

Lemma 8. *(a)* TQ_m*,* $m \geq 3$*, is* $m-3$*-fault hamiltonian-connected and* $m-2$*-fault hamiltonian[4]. (b)* CQ_m*,* $m \geq 3$*, is* $m-3$*-fault hamiltonian-connected and* $m-2$*-fault hamiltonian[3].*

By Corollary 1 and Lemma 7 and 8, we have the following.

Theorem 3. TQ_m *and* CQ_m*,* $m \geq 3$*, are fully one-to-many disjoint path coverable.*

3.3 One-to-one disjoint path covers

Similar to the definition of a one-to-many disjoint path cover, we can define a one-to-one disjoint path cover in a graph with faulty elements. In a straightforward manner, when $f + k \leq \delta(G) - 1$, we can construct f-fault one-to-one k-DPC of a graph G joining s and t using f-fault one-to-many k-DPC joining s and $\{t_1, t_2, \ldots, t_k\}$, where $t_1 = t$ and for every $2 \leq j \leq k$, t_j is a fault-free vertex adjacent to t such that (t_j, t) is also fault-free.

Proposition 2. *Let* G *be a graph, and let* $f \geq 0$*. (a)* G *is* f*-fault one-to-one* 1*-disjoint path coverable if and only if* G *is* f*-fault hamiltonian-connected. (b)* G *is* f*-fault one-to-one* 2*-disjoint path coverable if and only if* G *is* f*-fault hamiltonian. (c) Let* $k \geq 1$*, and* $f+k \leq \delta(G)-1$*. If* G *is* f*-fault one-to-many* k*-disjoint path coverable, then* G *is* f*-fault one-to-one* k*-disjoint path coverable.*

Theorem 4. *For* $m \geq 3$*,* $G(2^m, 4)$*,* TQ_m*, and* CQ_m *are* f*-fault one-to-one* k*-disjoint path coverable for any* $f \geq 0$ *and* $k \geq 2$ *such that* $f + k \leq m - 1$*.*

It was shown that fault-free $G(2^m, 4)$, $m \geq 3$, is (0-fault) one-to-one k-disjoint path coverable for any $1 \leq k \leq m$[5]. The construction of f-fault one-to-one k-DPC's of $G(2^m, 4)$, TQ_m, and CQ_m satisfying $f + k = m$ remains open.

References

1. D.-R. Duh and G.-H. Chen, "On the Rabin number problem," *Networks* **30(3)**, pp. 219-230, 1997.
2. F. Harary and M. Lewinter, "The starlike trees which span a hypercube," *Computers & Mathematics with Applications* **15(4)**, pp. 299-302, 1988.
3. W.T. Huang, M.Y. Lin, J.M. Tan, and L.H. Hsu, "Fault-tolerant ring embedding in faulty crossed cubes," in *Proc. SCI'2000*, pp. 97-102, 2000.
4. W.T. Huang, J.M. Tan, C.N. Huang, L.H. Hsu, "Fault-tolerant hamiltonicity of twisted cubes," *J. Parallel Distrib. Comput.* **62**, pp. 591-604, 2002.
5. J.-H. Park, "One-to-one disjoint path covers in recursive circulants," *Journal of KISS* **30(12)**, pp. 691-698, 2003 (in Korean).
6. J.-H. Park and K.Y. Chwa, "Recursive circulants and their embeddings among hypercubes," *Theoretical Computer Science* **244**, pp. 35-62, 2000.
7. C.-H. Tsai, J.M. Tan, Y.-C. Chuang, and L.-H. Hsu, "Fault-free cycles and links in faulty recursive circulant graphs," in *Proc. of Workshop on Algorithms and Theory of Computation ICS2000*, pp. 74-77, 2000.