Paired Many-to-Many Disjoint Path Covers in Restricted Hypercube-Like Graphs

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Abstract

Given two disjoint vertex-sets, $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ in a graph, a *paired many-to-many k-disjoint path cover* between S and T is a set of pairwise vertex-disjoint paths $\{P_1, \ldots, P_k\}$ that altogether cover every vertex of the graph, in which each path P_i runs from s_i to t_i. A family of hypercube-like interconnection networks, called *restricted hypercube-like graphs*, includes most non-bipartite hypercube-like networks found in the literature, such as twisted cubes, crossed cubes, Möbius cubes, recursive circulant $G(2^m, 4)$ of odd m, etc. In this paper, we show that every m-dimensional restricted hypercube-like graph, $m \geq 5$, with at most f vertex and/or edge faults being removed has a paired many-to-many k-disjoint path cover between arbitrary disjoint sets S and T of size k each, subject to $k \geq 2$ and $f + 2k \leq m + 1$. The bound $m + 1$ on $f + 2k$ is the best possible.

Keywords: Hypercube-like graph, disjoint path, path cover, path partition, fault tolerance, interconnection network.

1. Introduction

One of the central issues in the study of interconnection networks is to detect vertex-disjoint paths, which are naturally related to routing among nodes and fault tolerance of the network [12, 22]. Moreover, the disjoint path is one of the fundamental notions in graph theory, from which many properties of a graph can be deduced [2, 22]. An interconnection network is frequently represented as a graph, in which the vertices and edges correspond to nodes and links, respectively. Since node and/or link failure is inevitable in a large network, fault tolerance is essential to the network performance.

Let G be a simple undirected graph whose vertex and edge sets, respectively, are denoted by $V(G)$ and $E(G)$. A *path cover* of G is a set of paths in G, such that every vertex of G is contained in at least one path. A *vertex-disjoint path cover*, or simply a *disjoint path cover*, of G is a special kind of path cover in

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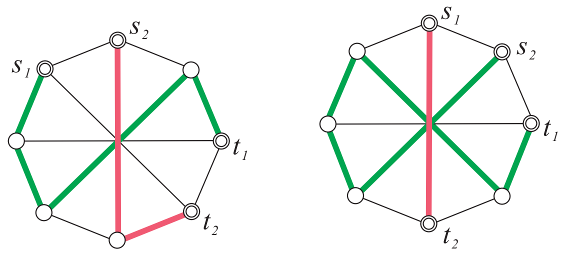


(a) Paired 2-DPC (b) Unpaired 2-DPC

Fig. 1: Examples of paired and unpaired DPCs. The configuration (b) admits no paired 2-DPC.

which every vertex of G is covered by exactly one path. The disjoint path cover problem finds applications in many areas such as software testing, database design, and code optimization [1, 23]. In addition, the problem is concerned with applications where full utilization of network nodes is important [30].

For a positive integer k, let $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ be two disjoint subsets of $V(G)$. Then, a disjoint path cover $\{P_1, \ldots, P_k\}$ of G is said to be a *paired many-to-many k-disjoint path cover* (*paired k-DPC* for short) between S and T if P_i is a path that runs from s_i to t_i for all i. The disjoint path cover is said to be an *unpaired many-to-many k-disjoint path cover* (*unpaired k-DPC* for short) if for some permutation σ on $\{1, \ldots, k\}$, P_i runs from s_i to $t_{\sigma(i)}$ for all i [30]. Refer to Fig. 1 for examples of paired and unpaired DPCs. Note that a paired k-DPC joining S and T is, by definition, an unpaired k-DPC joining them. Here, the vertices in S and T are often called *sources* and *sinks*, respectively, and *terminals* collectively.

Definition 1. (See [31].) A graph G is called *f-fault paired* (resp. *unpaired*) *k-disjoint path coverable* if $f + 2k \leq |V(G)|$ and G has a paired (resp. unpaired) k-DPC joining arbitrary disjoint set S of k sources and set T of k sinks in $G \setminus F$ for any fault set $F \subseteq V(G) \cup E(G)$ with $|F| \leq f$.

Simpler variants of the many-to-many disjoint path covers have also been investigated in previous literature. The *one-to-many k-DPC* for $S = \{s\}$ and $T = \{t_1, \ldots, t_k\}$ is a disjoint path cover made of k paths, each joining a pair of source s and sink t_j, $j \in \{1, \ldots, k\}$. The *one-to-one k-DPC* for $S = \{s\}$ and $T = \{t\}$ is a disjoint path cover, each of whose paths joins an identical pair of source s and sink t. The paths in the one-to-many k-DPC or in the one-to-one k-DPC may share a source and/or a sink and thus are pairwise internally disjoint. Readers are recommended to refer to the related literature, such as [15, 19, 26, 30], for more details.

In this paper, we deal with the paired many-to-many disjoint path coverability of Restricted Hypercube-Like graphs (RHL graphs for short) [29], which are a subset of non-bipartite hypercube-like graphs that have received much attention over the past few decades. The class includes most non-bipartite hypercube-like networks found in the literature, as the following examples: twisted cubes [11], crossed cubes [9], Möbius cubes [6], recursive circulant $G(2^m, 4)$ of odd m [25], multiply twisted cubes [8], Mcubes [33], and generalized twisted cubes [3]. An m-dimensional RHL graph (whose definition is deferred to the next section) has 2^m vertices of degree m. Its connectivity is also m. The paired many-to-many disjoint path coverability for RHL graphs has been studied, as summarized in Theorem 1.

Theorem 1. *(See [17, 30, 31].) (a) Every m-dimensional RHL graph, $m \geq 3$, is f-fault paired k-disjoint path coverable for any f and $k \geq 1$ subject to $f+2k \leq m-1$ [30].*
(b) Every m-dimensional RHL graph, $m \geq 4$, is f-fault paired k-disjoint path coverable for any f and $k \geq 2$ subject to $f + 2k \leq m$ [31].
(c) Every m-dimensional RHL graph, $m \geq 5$, is $(m-3)$-fault paired 2-disjoint path coverable [17].

On the other hand, a necessary condition for a general graph G to be f-fault paired k-disjoint path coverable has been derived in terms of the connectivity $\kappa(G)$ of G in [30].

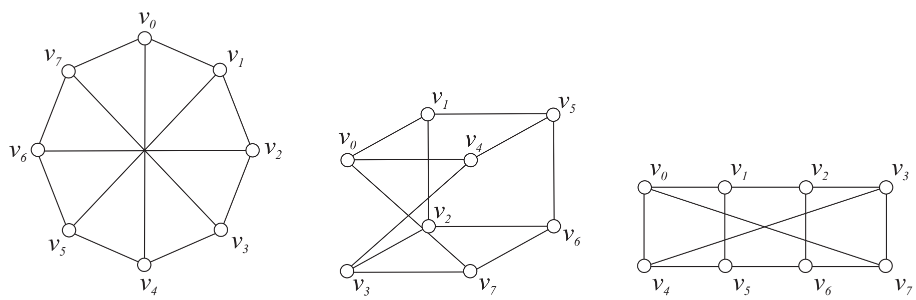
Lemma 1. *(See [30].) If a graph G is f-fault paired k-disjoint path coverable, then $f + 2k \leq \kappa(G) + 1$.*

For the specific $k = 2$, the bound on $f + 2k$ in Theorem 1(c) is equal to that of Lemma 1, and is thus optimal. For general $k \geq 2$, however, the bound on $f + 2k$ was suggested to be $m - 1$ in Theorem 1(a) and then improved to be m in Theorem 1(b), still one less than the bound, $m+1$, of the necessary condition in Lemma 1. Bridging the gap in this paper, we will achieve the optimal bound $m + 1$ on $f + 2k$ for every $k \geq 3$. In other words, we will prove our main theorem asserting that *every m-dimensional RHL graph, $m \geq 5$, is f-fault paired k-disjoint path coverable for any f and $k \geq 2$ subject to $f + 2k \leq m + 1$.*

The rest of this paper is organized as follows: in the next section, we address previous works and definitions. Sections 3 and 4 are devoted to proving our main theorem. Finally, we conclude our findings in Section 5.

2. Previous works and definitions

Given the disjoint source set S and sink set T in a general graph G, it is NP-complete to determine if there exists a one-to-one, one-to-many, or many-to-many k-DPC between S and T for any fixed $k \geq 1$ [30, 31]. The disjoint path cover problems have been studied for graphs such as hypercubes [4, 5, 7, 10, 14, 21], recursive circulants [15, 16, 30, 31], hypercube-like graphs [13, 17, 18, 30, 31], cubes of connected graphs [26, 27], k-ary n-cubes [32, 35], alternating group graphs [34], grid graphs [28], and tori [20]. In particular, the optimal construction of an unpaired k-DPC in an RHL graph has been established recently in [24], as shown in Theorem 2. Here, $\delta(G)$ denotes the minimum degree of a graph G.



(a) $G(8,4)$ (b) TQ_3 (c) Möbius ladder

Fig. 2: The 3-dimensional RHL graph.

Lemma 2. *(See [31].) Let G be an f-fault unpaired k-disjoint path coverable graph, where $k \geq 2$. Then, $\kappa(G) \geq f + k$. Furthermore, if G has $f + 2k + 1$ or more vertices, then $\delta(G) \geq f + k + 1$.*

Theorem 2. *(See [24].) Every m-dimensional RHL graph, $m \geq 5$, is f-fault unpaired k-disjoint path coverable for any f and $k \geq 2$ subject to $f + k \leq m - 1$.*

In addition, there is a class of graphs, called Recursive-Circulant-Like graphs (RCL graphs for short) [15, 16], that allows for the optimal constructions of paired and unpaired DPCs, achieving the bounds of the necessary conditions in Lemmas 1 and 2. That is, it was proven that for $m \geq 5$, every m-dimensional RCL graph, which has 2^m vertices of degree m (as an m-dimensional RHL graph does), is f-fault *paired* k-disjoint path coverable for any f and $k \geq 2$ subject to $f + 2k \leq m + 1$ [16], and is f-fault *unpaired* k-disjoint path coverable for any f and $k \geq 2$ subject to $f + k \leq m - 1$ [15].

A 3-dimensional RHL graph is isomorphic to recursive circulant $G(8,4)$ whose vertex and edge sets, respectively, are $\{v_i : 0 \leq i \leq 7\}$ and $\{(v_i, v_j) : i + 1 \text{ or } i + 4 \equiv j \pmod 8\}$. The 3-dimensional RHL graph is isomorphic to a 3-dimensional twisted cube TQ_3 and to a Möbius ladder with four spokes, as shown in Fig. 2. An m-dimensional RHL graph, $m \geq 4$, is recursively defined with a graph operation $\oplus$. Given two graphs, G_0 and G_1, with the same number of vertices and a bijection ϕ from $V(G_0)$ to $V(G_1)$, we denote by $G_0 \oplus_\phi G_1$ the graph whose vertex set is $V(G_0) \cup V(G_1)$ and edge set is $E(G_0) \cup E(G_1) \cup \{(v, \phi(v)) : v \in V(G_0)\}$. To simplify the notation, we often omit the bijection ϕ from $\oplus_\phi$.

Definition 2. (See [29].) A graph that belongs to RHL_m is called an *m-dimensional RHL graph*, where

- $RHL_3 = \{G(8,4)\}$, and
- $RHL_m = \{G_0 \oplus_\phi G_1 : G_0, G_1 \in RHL_{m-1}$, ϕ is a bijection from $V(G_0)$ to $V(G_1)\}$ for $m \geq 4$.

Lemma 3. *(See [17].) Let G be an m-dimensional RHL graph, where $m \geq 3$.*
(a) G has 2^m vertices of degree m. Moreover, it is non-bipartite.
(b) G has no triangle (cycle of length three).
(c) There are at most two common neighbors for any pair of vertices in G.

The disjoint path cover of a graph is closely related to its Hamiltonian properties. For instance, a Hamiltonian path between two distinct vertices in a graph G is in fact a 1-DPC, irrespective of its type, of G joining the vertices. The Hamiltonian properties of RHL graphs, shown in Lemma 4 (below) will be frequently referred to, where a graph G is said to be *f-fault Hamiltonian-connected* (resp. *Hamiltonian*) if any pair of vertices are joined by a Hamiltonian path (resp. there exists a Hamiltonian cycle) in $G \setminus F$ for any fault set $F \subseteq V(G) \cup E(G)$ with $|F| \leq f$.

Lemma 4. *(See [29].) Every m-dimensional RHL graph, $m \geq 3$, is $(m-3)$-fault Hamiltonian-connected and is $(m-2)$-fault Hamiltonian.*

Given two vertices u and v of a graph G, a *path P* in G from u to v is a sequence $(w_1, \ldots, w_n)$ of distinct vertices of G such that $w_1 = u$, $w_n = v$, and $(w_i, w_{i+1}) \in E(G)$ for all $i \in \{1, \ldots, n-1\}$. Hereafter, a *u–v path* refers to a path that runs from u to v. If $G = H_0 \oplus H_1$, subgraphs H_0 and H_1 are called the *components* of G. For a vertex v in a component H_i, we denote by $\bar{v}$ the neighbor of v contained in the other component H_{1-i}, for $i = 0, 1$. In addition, a vertex v is called to be *free* if it is neither a fault nor a terminal. An edge (u, v) is called to be *free* if it is nonfaulty and both u and v are free.

3. Paired many-to-many disjoint path covers

In this section, we will construct f-fault paired k-disjoint path covers in m-dimensional RHL graphs with $m \geq 5$ for f and $k \geq 2$ satisfying the optimal bound, $f + 2k \leq m + 1$, of Lemma 1. That is, we will establish the following theorem:

Theorem 3. *Every m-dimensional RHL graph, $m \geq 5$, is f-fault paired k-disjoint path coverable for any $f \geq 0$ and $k \geq 2$ subject to $f + 2k \leq m + 1$.*

Through the proof of Theorem 3, we assume $k \geq 3$ thanks to Theorem 1(c). The proof will proceed by induction on m. The base step of $m = 5$, however, will be deferred to Section 4. For the inductive step, let $m \geq 6$. Recall that an m-dimensional RHL graph G is isomorphic to $H_0 \oplus H_1$ for some $(m-1)$-dimensional RHL graphs H_0 and H_1. The induction hypothesis states that each component H_i, $i \in \{0, 1\}$, of G is f'-fault paired k'-disjoint path coverable if $k' \geq 2$ and $f' + 2k' \leq (m-1) + 1 = m$. Given F, S, and T in the graph G, a paired k-DPC joining S and T in $G \setminus (F \cup F')$, where F' is a set of arbitrary $m + 1 - 2k - |F|$ fault-free edges, is also a paired k-DPC joining them in $G \setminus F$. As a result, it can be assumed that

$$f = |F| \text{ and } f + 2k = m + 1. \qquad (1)$$

We will construct a paired k-DPC in $H_0 \oplus H_1 \setminus F$ for any given sets F, S, and T such that $|F| = f$, $|S| = |T| = k \geq 3$, and $f + 2k = m + 1$. Let F_0 and F_1, respectively, denote the sets of faults contained in H_0 and H_1, and F_2 denote the set of faulty edges bridging H_0 and H_1. Let $f_i = |F_i|$ for $i \in \{0, 1, 2\}$, so that $f = f_0 + f_1 + f_2$. We also denote by k_i the number of source-sink pairs in H_i where $i = 0, 1$, and by k_2 the number of source-sink pairs between H_0 and H_1. Then, $k = k_0 + k_1 + k_2$. We further assume w.l.o.g. that

$$k_0 \geq k_1, \text{ and if } k_0 = k_1, f_0 \geq f_1. \tag{2}$$

We have the following four cases to consider, according to the distribution of faults and terminals:

- $k_1 \geq 1$ or $f_0 \leq f - 1$;
- $f_0 = f$ and $k_0 = k$;
- $f_0 = f$, $k_1 = 0$, $k_0 \geq 1$, and $k_2 \geq 1$;
- $f_0 = f$ and $k_2 = k$.

The first three cases, and also the last one with an additional condition $f > 0$, can be proven if we utilize the Hamiltonian properties (Lemma 4) and the unpaired DPC properties (Theorem 2) of components, H_0 and H_1, as well as the induction hypothesis. Furthermore, the proofs for them are exactly the same as those for paired disjoint path coverability of RCL graphs given in [16]. This unexpected phenomenon occurs because the proofs given in [16] rely on the following five properties of an m-dimensional RCL graph G':

P1: G' is isomorphic to $H'_0 \oplus H'_1$ for some $(m-1)$-dimensional RCL graphs H'_0 and H'_1;
P2: G' has 2^m vertices of degree m;
P3: There are at most two common neighbors for any pair of vertices in G';
P4: G' is $(m-3)$-fault Hamiltonian-connected, i.e., $(m-3)$-fault 1-disjoint path coverable, where $m \geq 3$;
P5: G' is f-fault unpaired k-disjoint path coverable for any f and $k \geq 2$ subject to $f + k \leq m - 1$, where $m \geq 5$.

Note that an m-dimensional RHL graph G also possesses a property corresponding to P1, i.e., it is isomorphic to $H_0 \oplus H_1$ for some $(m-1)$-dimensional RHL graphs H_0 and H_1. Moreover, the graph G satisfies P2 through P5, by Lemmas 3 and 4, and Theorem 2. For the proofs excluding the last case with $f = 0$, we refer to Sections 3.1 through 3.4 of [16].

Now, let us concentrate on the remaining case, in which $f = 0$ and $k_2 = k$, which is the hardest one among the others. Here, we assume w.l.o.g. that all the sources are contained in H_0 whereas all the sinks are contained in H_1. Moreover, there are no faults. By the assumption (1), we have $k = \frac{m+1}{2}$, so that

$$m \geq 7 \text{ and } k \geq 4. \tag{3}$$

For the proof, we will rely on the paired/unpaired DPC properties and the Hamiltonian properties of the components of H_0 and H_1, i.e., the *subcomponents*

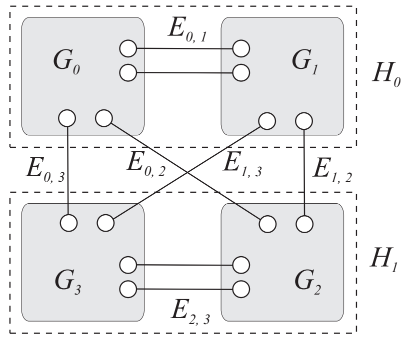


Fig. 3: Recursive structure of $H_0 \oplus H_1$, where there are two components, H_0 and H_1, and four subcomponents, G_0, G_1, G_2, and G_3.

of $H_0 \oplus H_1$ (instead of relying on these properties of the components). Note that H_0 and H_1 are $(m-1)$-dimensional RHL graphs, so that H_0 is isomorphic to $G_0 \oplus G_1$ for some $(m-2)$-dimensional RHL graphs G_0 and G_1; and also, H_1 is isomorphic to $G_2 \oplus G_3$ for some $(m-2)$-dimensional RHL graphs G_2 and G_3, as illustrated in Fig. 3. Contracting every subcomponent of $H_0 \oplus H_1$ to a single vertex results in a four-vertex graph isomorphic to a complete graph or to a cycle. (On the other hand, subcomponent contraction of an RCL graph always results in a four-vertex cycle. This is why the proof for paired disjoint path coverability of RCL graphs in this case cannot be recycled.)

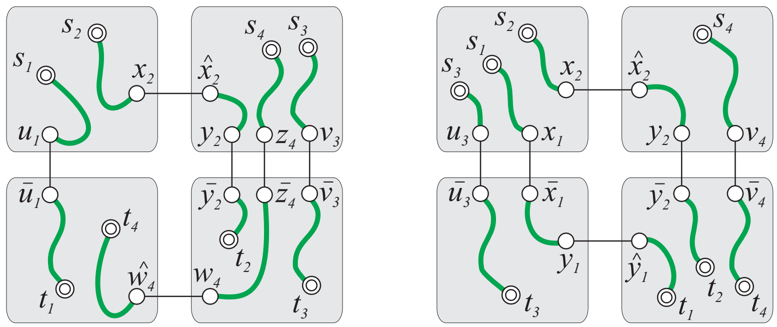
Let $E_{p,q}$ denote the set of edges of $H_0 \oplus H_1$ that connect two subcomponents G_p and G_q for $p, q \in \{0,1,2,3\}$, i.e., $E_{p,q} = \{(u,v) \in E(H_0 \oplus H_1) : u \in V(G_p),\, v \in V(G_q)\}$. Then, we have $|E_{0,1}| = |E_{2,3}| = 2^{m-2}$ and $|E_{0,3}| + |E_{0,2}| = |E_{1,2}| + |E_{1,3}| = 2^{m-2}$. Moreover, we have $|E_{0,3}| = |E_{1,2}|$ and $|E_{0,2}| = |E_{1,3}|$ since the union $E_{0,3} \cup E_{0,2} \cup E_{1,2} \cup E_{1,3}$ forms a perfect matching of $H_0 \oplus H_1$. Assume w.l.o.g. that

$$|E_{0,3}| = |E_{1,2}| \geq |E_{0,2}| = |E_{1,3}|, \tag{4}$$

so that $|E_{0,3}| = |E_{1,2}| \geq 2^{m-3}$. (Suppose otherwise, it suffices to switch G_2 and G_3.) In addition, we let I denote the index set $\{1, \ldots, k\}$ of sources (and sinks), and let $I_{p,q} = \{i \in I : s_i \in V(G_p), t_i \in V(G_q)\}$, so that $I = I_{0,2} \cup I_{0,3} \cup I_{1,2} \cup I_{1,3}$. If we let $k_{p,q} = |I_{p,q}|$, we have $|S \cap V(G_0)| = k_{0,2} + k_{0,3}$, $|S \cap V(G_1)| = k_{1,2} + k_{1,3}$, etc. We further assume w.l.o.g. that

$$\begin{gathered} |S \cap V(G_0)| \geq |S \cap V(G_1)|, |T \cap V(G_2)|, |T \cap V(G_3)|, \\ \text{i.e., } k_{0,2} + k_{0,3} \geq k_{1,2} + k_{1,3}, k_{0,2} + k_{1,2}, k_{0,3} + k_{1,3}. \end{gathered} \tag{5}$$

One of the tractable approaches for constructing a paired k-DPC of $H_0 \oplus H_1$ would be dividing the problem into subproblems of finding paired k'-DPCs of subcomponents for some $1 \leq k' < k$, and then merging the DPCs of subcomponents into a desired one. Note that the paired k'-DPC of a subcomponent exists



(a) $k_{0,2} + k_{0,3} \leq k - 2$ (b) $k_{0,2} + k_{0,3} = k - 1$ & $k_{0,3} \geq 1$

Fig. 4: Illustration of the Procedure Find-Paired-DPC, where $m = 7$ and $k = 4$.

for every $1 \leq k' < k$, by the induction hypothesis if $k' \geq 2$ and by Lemma 4 if $k' = 1$. In order to fully utilize the paired DPC properties of subcomponents, DPC paths will be chosen as follows (refer to Fig. 4):

- For $i \in I_{p,q}$, where $(p, q) = (0, 3)$ or $(1, 2)$, each s_i–t_i path passes through a unique inter-subcomponent edge from $E_{p,q}$.
- For $i \in I_{0,2}$, each s_i–t_i path passes through exactly two inter-subcomponent edges from either $E_{0,1} \cup E_{1,2}$ or $E_{0,3} \cup E_{3,2}$. Half of the s_i–t_i paths for $i \in I_{0,2}$ will take the first choice.
- For $i \in I_{1,3}$, each s_i–t_i path passes through exactly two inter-subcomponent edges from either $E_{1,2} \cup E_{2,3}$ or $E_{1,0} \cup E_{0,3}$. Half of the s_i–t_i paths for $i \in I_{1,3}$ will take the first choice.
- No edges from $E_{0,2} \cup E_{1,3}$, possibly an empty set, are used.

Based on the aforementioned approach, the following Procedure Find-Paired-DPC constructs a paired k-DPC joining S and T in an m-dimensional RHL graph $H_0 \oplus H_1$, leaving a few special cases inapplicable. For a vertex x in a subcomponent G_j of $H_0 \oplus H_1$, we denote by $\hat{x}$ the neighbor of x that is contained in the same component as x but in the other subcomponent than G_j. (If $x \in V(G_0)$, then $\hat{x} \in V(G_1)$ and vice versa. If $x \in V(G_2)$, then $\hat{x} \in V(G_3)$ and vice versa.) We let $I'_{0,2}$ and $I''_{0,2}$ be a partition of $I_{0,2}$ such that $|I'_{0,2}| = \lceil k_{0,2}/2 \rceil$ and $|I''_{0,2}| = \lfloor k_{0,2}/2 \rfloor$, and let $I'_{1,3}$ and $I''_{1,3}$ be a partition of $I_{1,3}$ such that $|I'_{1,3}| = \lceil k_{1,3}/2 \rceil$ and $|I''_{1,3}| = \lfloor k_{1,3}/2 \rfloor$. A paired k'-DPC joining $\{(x_1, y_1), \ldots, (x_{k'}, y_{k'})\}$ refers to a DPC composed of x_1–y_1, …, $x_{k'}$–$y_{k'}$ paths. For example, Fig. 4(a) shows a paired 2-DPC of G_0 joining $\{(s_1, u_1), (s_2, x_2)\}$, etc.

Procedure Find-Paired-DPC(S, T, $H_0 \oplus H_1$)
/* It is assumed that $m \geq 7$, $2k = m + 1$, $S \subseteq V(H_0)$ and $T \subseteq V(H_1)$. See Fig. 4. */

Pick up $\lceil k_{0,2}/2 \rceil + \lfloor k_{1,3}/2 \rfloor$ free edges from $E_{0,1}$, which are denoted by

- $(x_i, \hat{x_i})$, where $x_i \in V(G_0)$, for $i \in I'_{0,2}$ and
- $(z_j, \hat{z_j})$, where $z_j \in V(G_1)$, for $j \in I''_{1,3}$.

2: Pick up $k_{0,3} + \lfloor k_{0,2}/2 \rfloor + \lfloor k_{1,3}/2 \rfloor$ free edges from $E_{0,3}$ that are not adjacent to the free edges chosen in Step 1. The free edges are denoted by

- $(u_i, \bar{u_i})$, where $u_i \in V(G_0)$, for $i \in I_{0,3}$,
- $(x_j, \bar{x_j})$, where $x_j \in V(G_0)$, for $j \in I''_{0,2}$, and
- $(w_r, \bar{w_r})$, where $w_r \in V(G_0)$, for $r \in I''_{1,3}$.

3: Pick up $k_{1,2} + \lceil k_{1,3}/2 \rceil + \lceil k_{0,2}/2 \rceil$ free edges from $E_{1,2}$ that are not adjacent to the free edges chosen in Step 1. The free edges are denoted by

- $(v_i, \bar{v_i})$, where $v_i \in V(G_1)$, for $i \in I_{1,2}$,
- $(z_j, \bar{z_j})$, where $z_j \in V(G_1)$, for $j \in I'_{1,3}$, and
- $(y_r, \bar{y_r})$, where $y_r \in V(G_1)$, for $r \in I'_{0,2}$.

4: Pick up $\lfloor k_{0,2}/2 \rfloor + \lceil k_{1,3}/2 \rceil$ free edges from $E_{2,3}$ that are not adjacent to the free edges chosen in Steps 2 and 3. The free edges are denoted by

- $(y_i, \hat{y_i})$, where $y_i \in V(G_3)$, for $i \in I''_{0,2}$ and
- $(w_j, \hat{w_j})$, where $w_j \in V(G_2)$, for $j \in I'_{1,3}$.

5: Find a paired $(k_{0,3} + k_{0,2} + \lfloor k_{1,3}/2 \rfloor)$-DPC in G_0 joining $\{(s_i, u_i) : i \in I_{0,3}\} \cup \{(s_j, x_j) : j \in I_{0,2}\} \cup \{(\hat{z_r}, w_r) : r \in I''_{1,3}\}$.
6: Find a paired $(k_{1,2} + k_{1,3} + \lceil k_{0,2}/2 \rceil)$-DPC in G_1 joining $\{(s_i, v_i) : i \in I_{1,2}\} \cup \{(s_j, z_j) : j \in I_{1,3}\} \cup \{(\hat{x_r}, y_r) : r \in I'_{0,2}\}$.
7: Find a paired $(k_{1,2} + k_{0,2} + \lceil k_{1,3}/2 \rceil)$-DPC in G_2 joining $\{(\bar{v_i}, t_i) : i \in I_{1,2}\} \cup \{(y'_j, t_j) : j \in I_{0,2}\} \cup \{(\bar{z_r}, w_r) : r \in I'_{1,3},\}$, where $y'_j = \bar{y_j}$ if $j \in I'_{0,2}$; $y'_j = \hat{y_j}$ if $j \in I''_{0,2}$.
8: Find a paired $(k_{0,3} + k_{1,3} + \lfloor k_{0,2}/2 \rfloor)$-DPC in G_3 joining $\{(\bar{u_i}, t_i) : i \in I_{0,3}\} \cup \{(w'_j, t_j) : j \in I_{1,3}\} \cup \{(\bar{x_r}, y_r) : r \in I''_{0,2}\}$, where $w'_j = \hat{w_j}$ if $j \in I'_{1,3}$; $w'_j = \bar{w_j}$ if $j \in I''_{1,3}$.
9: Merge the four DPCs of subcomponents with the free edges, chosen in Steps 1 through 4, into a paired k-DPC.

Lemma 5. *The free edges in Steps 1 through 4 of the Procedure* FIND-PAIRED-DPC *exist.*

Proof. The number of free edges chosen in each of Steps 1 through 4 is at most k because at most one free edge is picked up for each source-sink pair (s_i, t_i) where $i \in I$. It will be shown that there exist at least k free edges in each step. The free edges of Step 1 exist because there are $|E_{0,1}|$ candidate edges whereas at most k of them could be blocked by the sources, for which $|E_{0,1}| - k = 2^{m-2} - \frac{m+1}{2} \geq \frac{m+1}{2} = k$ for $m \geq 7$. The free edges of Step 2 also exist because there are $|E_{0,3}|$ candidate edges whereas at most $3k$ of them could be blocked by terminals and the free edges chosen in Step 1, where $|E_{0,3}| - 3k \geq 2^{m-3} - 3 \cdot \frac{m+1}{2} \geq \frac{m+1}{2} = k$ for $m \geq 7$. Similarly, the free edges of Step 3 also exist. Finally, the free edges of Step 4 exist because there are $|E_{2,3}|$ candidate edges whereas at most $3k$ of them could be blocked by the sinks and the free edges chosen in Steps 2 and 3, where $|E_{2,3}| - 3k = 2^{m-2} - 3 \cdot \frac{m+1}{2} \geq \frac{m+1}{2} = k$ for $m \geq 7$. Thus, the lemma is proved. □

Lemma 6. *In Steps 5 through 8 of the Procedure* Find-Paired-DPC*, the four paired DPCs of the subcomponents exist if (i)* $k_{0,2} + k_{0,3} \leq k - 2$*, or (ii)* $k_{0,2} + k_{0,3} = k - 1$ *and* $k_{0,3} \geq 1$*.*

Proof. There is at least one terminal in each subcomponent from the assumption (5), since $k_{0,2} + k_{0,3} = |S \cap V(G_0)| < k$. So, it suffices to prove that the number of source-sink pairs in each of Steps 5 through 8 is at most $k - 1$. This is because each subcomponent, G_j, is paired k'-disjoint path coverable for every $k' \leq k - 1$ by the induction hypothesis if $k' \geq 2$, where $2k' \leq 2(k-1) = m - 1 = \delta(G_j) + 1$ for all j, and by Lemma 4 if $k' = 1$. For the paired DPC of G_0, we have $k_{0,3} + k_{0,2} + \lfloor k_{1,3}/2 \rfloor = k - (\lceil k_{1,3}/2 \rceil + k_{1,2}) \leq k - 1$ because $k_{1,3} + k_{1,2} = |S \cap V(G_1)| \geq 1$ from the hypothesis of this lemma. For the paired DPC of G_1, we have $k_{1,2} + k_{1,3} + \lceil k_{0,2}/2 \rceil = k - (\lfloor k_{0,2}/2 \rfloor + k_{0,3}) \leq k - 1$ because $k_{0,2} + k_{0,3} = |S \cap V(G_0)| \geq k/2 \geq 2$ by the assumption (5). For the paired DPC of G_2, we have $k_{1,2} + k_{0,2} + \lceil k_{1,3}/2 \rceil = k - (\lfloor k_{1,3}/2 \rfloor + k_{0,3}) \leq k - 1$ because $k_{0,3} + k_{1,3} = |T \cap V(G_3)| \geq 2$, from the assumption (5), if the condition (i) holds; $k_{0,3} \geq 1$ if the condition (ii) holds. For the paired DPC of G_3 finally, we have $k_{0,3} + k_{1,3} + \lfloor k_{0,2}/2 \rfloor = k - (\lceil k_{0,2}/2 \rceil + k_{1,2}) \leq k - 1$ because $k_{0,2} + k_{1,2} = |T \cap V(G_2)| \geq 1$. This completes the proof. □

The exceptional cases that are not covered by Procedure Find-Paired-DPC are as follows:

- $k_{0,2} + k_{0,3} = k$ $(k_{1,2} + k_{1,3} = 0)$;
- $k_{0,2} + k_{0,3} = k - 1$ and $k_{0,3} = 0$, or equivalently, $k_{0,2} = k - 1$ and $k_{1,3} = 1$ by the assumption (5).

In the exceptional cases, there may exist a subcomponent, G_j, such that every s_i–t_i path must pass through at least one vertex of G_j. (If $k_{0,2} + k_{0,3} = k$, G_0 will be such a subcomponent; If $k_{0,2} = k - 1$ and $k_{1,3} = 1$, G_0 or G_2 will be such a subcomponent in case of $E_{1,3} = \emptyset$.) Since G_j is not paired k-disjoint path coverable, we have no choice but to utilize the *unpaired* disjoint path coverability, or to introduce a *virtual* fault (instead of a source-sink pair) for some $i \in I$ and utilize the f'-fault paired k'-disjoint path coverability for some f' and k' with $f' + k' = k$ and $f' \geq 2$. The exceptional cases are dealt with in Lemmas 7 and 8.

Lemma 7. *If* $k_{0,2} + k_{0,3} = k$*, there exists a paired k-DPC between S and T in the m-dimensional RHL graph.*

Proof. Let $I' = \{i \in I : t_i \in V(G_3)\}$, $p = |I'|$, $I'' = \{j \in I : t_j \in V(G_2)\}$, and $q = |I''|$, so that $I = I' \cup I''$ and $p + q = k$. There are three cases depending on p, the number of sinks contained in G_3.

Case 1: *$1 \leq p < k$, so that $1 \leq q < k$.*

Procedure Paired-DPC-A(S, T, $H_0 \oplus H_1$) // See Fig. 5.

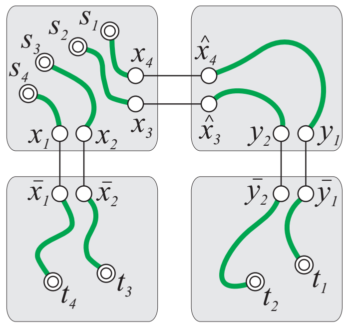


Fig. 5: Illustration of the Procedure Paired-DPC-A, where $m = 7$ and $k = 4$. In this example, the *unpaired* 4-DPC of G_0 joining $\{s_1, \ldots, s_4\}$ and $\{x_1, \ldots, x_4\}$ is composed of s_1–x_4, s_2–x_3, s_3–x_2, and s_4–x_1 paths, i.e., $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (4, 3, 2, 1)$.

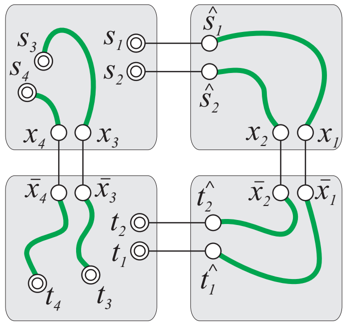
1: Pick up k free edges from $E_{0,3}$, which are denoted by $(x_i, \bar{x}_i)$, where $x_i \in V(G_0)$, for $i \in I$.
2: Find an *unpaired* k-DPC joining S and $\{x_1, \ldots, x_k\}$ in G_0. Let the paths in the unpaired DPC be s_i–x_{σ_i} paths, $i \in I$, for some permutation σ on I. (Here, $\sigma(i)$ is abbreviated as σ_i for convenience.)
3: Find a paired p-DPC in G_3 joining $\{(\bar{x_{\sigma_i}}, t_i) : i \in I'\}$.
4: Pick up q free edges from $E_{1,2}$ that are incident with no vertices in $\{\hat{x_{\sigma_i}} : i \in I''\}$. The free edges are denoted by $(y_i, \bar{y_i})$, where $y_i \in V(G_1)$, for $i \in I''$.
5: Find a paired q-DPC in G_1 joining $\{(\hat{x_{\sigma_i}}, y_i) : i \in I''\}$.
6: Find a paired q-DPC in G_2 joining $\{(\bar{y_i}, t_i) : i \in I''\}$.
7: Merge the four DPCs of subcomponents with the free edges $\{(x_{\sigma_i}, \bar{x_{\sigma_i}}) : i \in I'\} \cup \{(x_{\sigma_i}, \hat{x_{\sigma_i}}) : i \in I''\}$ as well as the free edges chosen in Step 4 into a paired k-DPC.

There exist k free edges in Step 1 since $|E_{0,3}| - 2k \geq 2^{m-3} - 2 \cdot \frac{m+1}{2} \geq \frac{m+1}{2} = k$ for $m \geq 7$. Similarly, the free edges in Step 4 also exist. The unpaired k-DPC in Step 2 exists by Theorem 2, where $k = \frac{m+1}{2} \leq m - 3 = \delta(G_0) - 1$ for $m \geq 7$. Also, the paired DPCs in Steps 3, 5, and 6 exist by the induction hypothesis or by Lemma 4. Thus, the correctness of the Procedure Paired-DPC-A is proved.

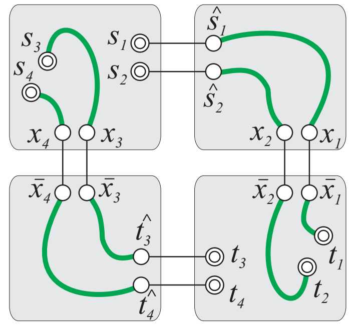
Case 2: *$p = k$, so that $q = 0$.* Let $J' = \{3, \ldots, k\}$ and $J'' = \{1, 2\}$, so that $J' \cup J'' = I$.

Procedure Paired-DPC-B(S, T, $H_0 \oplus H_1$) // See Fig. 6(a).

1: Pick up $k - 2$ free edges from $E_{0,3}$, which are denoted by $(x_i, \bar{x}_i)$, where $x_i \in V(G_0)$, for $i \in J'$.
2: Regarding s_1 and s_2 as *virtual* faults, find a 2-fault paired $(k-2)$-DPC in G_0 joining $\{(s_i, x_i) : i \in J'\}$.
3: Regarding t_1 and t_2 as *virtual* faults, find a 2-fault paired $(k-2)$-DPC in G_3 joining $\{(\bar{x}_i, t_i) : i \in J'\}$.
4: Pick up 2 free edges from $E_{1,2}$ that are incident with no vertex in $\{\hat{s}_i, \hat{t}_i : i \in J''\}$. The free edges are denoted by $(x_i, \bar{x}_i)$, where $x_i \in V(G_1)$, for $i \in J''$.



(a) Procedure Paired-DPC-B



(b) Procedure Paired-DPC-C

Fig. 6: Illustrations of the Procedures Paired-DPC-B and Paired-DPC-C, where $m = 7$ and $k = 4$.

5: Find a paired 2-DPC in G_1 joining $\{(\hat{s_i}, x_i) : i \in J''\}$.
6: Find a paired 2-DPC in G_2 joining $\{(\bar{x_i}, \hat{t_i}) : i \in J''\}$.
7: Merge the four DPCs of subcomponents with the edges $\{(s_i, \hat{s_i}), (t_i, \hat{t_i}) : i \in J''\}$ as well as the free edges chosen in Steps 1 and 4 into a paired k-DPC.

The existence of free edges in Steps 1 and 4 is straightforward from the fact that $2^{m-3} - 2k = 2^{m-3} - 2 \cdot \frac{m+1}{2} \geq \frac{m+1}{2} = k$ for $m \geq 7$. The 2-fault paired $(k-2)$-DPCs in Steps 2 and 3 exist by the induction hypothesis since $2 + 2(k-2) = (m-2) + 1$. The existence of paired 2-DPCs in Steps 5 and 6 is, again, by the induction hypothesis. Thus, the Procedure Paired-DPC-B is correct.

Case 3: *$p = 0$, so that $q = k$.* Again, let $J' = \{3, \ldots, k\}$ and $J'' = \{1, 2\}$. Similar to Case 2, a paired k-DPC can be constructed.

Procedure Paired-DPC-C(S, T, $H_0 \oplus H_1$) // See Fig. 6(b).

1: Pick up $k-2$ free edges from $E_{0,3}$ that are incident with no vertex in $\{\hat{t_i} : i \in J'\}$. The free edges are denoted by $(x_i, \bar{x_i})$, where $x_i \in V(G_0)$, for $i \in J'$.
2: Regarding s_1 and s_2 as *virtual* faults, find a 2-fault paired $(k-2)$-DPC in G_0 joining $\{(s_i, x_i) : i \in J'\}$.
3: Find a paired $(k-2)$-DPC in G_3 joining $\{(\bar{x_i}, \hat{t_i}) : i \in J'\}$.
4: Pick up 2 free edges from $E_{1,2}$ that are incident with no vertex in $\{\hat{s_1}, \hat{s_2}\}$. The free edges are denoted by $(x_i, \bar{x_i})$, where $x_i \in V(G_1)$, for $i \in J''$.
5: Find a paired 2-DPC in G_1 joining $\{(\hat{s_i}, x_i) : i \in J''\}$.
6: Regarding vertices in $\{t_i : i \in J'\}$ as *virtual* faults, find a $(k-2)$-fault paired 2-DPC in G_2 joining $\{(\bar{x_i}, t_i) : i \in J''\}$.
7: Merge the four DPCs of subcomponents with the edges $\{(s_i, \hat{s_i}) : i \in J''\} \cup \{(t_i, \hat{t_i}) : i \in J'\}$ as well as the free edges chosen in Steps 1 and 4 into a paired k-DPC.

The existence of free edges in Steps 1 and 4 is straightforward. The 2-fault paired $(k-2)$-DPC in Step 2 exists by the induction hypothesis since $2+2(k-2) = (m-2)+1$. The $(k-2)$-fault paired 2-DPC in Step 6 exists again by the induction hypothesis since $(k-2)+2\cdot 2 \le (k-2)+k = 2k-2 = (m-2)+1$. Recall $k \ge 4$. Also, the paired DPCs in Steps 3 and 5 exist by the induction hypothesis. Thus, the Procedure Paired-DPC-C is correct. This completes the entire proof. □

For the second exceptional case where $k_{0,2} = k-1$ and $k_{1,3} = 1$, we assume w.l.o.g. that $s_k \in V(G_1)$ and $t_k \in V(G_3)$, so that $S \setminus s_k \subseteq V(G_0)$ and $T \setminus t_k \subseteq V(G_2)$. For a vertex $v \in V(G_0)$ with $\bar{v} \in V(G_3)$, the three-vertex path $(\hat{v}, v, \bar{v})$ is called a *bend* associated with v. The bend is said to be *free* if both $(\hat{v}, v)$ and $(v, \bar{v})$ are free.

Lemma 8. *If $k_{0,2} = k-1$ and $k_{1,3} = 1$, there exists a paired k-DPC between S and T in the m-dimensional RHL graph.*

Proof. Let $J = \{1, \ldots, k-1\}$ and $J' = J \setminus 1$, so that $I = J' \cup \{1, k\}$.

Procedure Paired-DPC-D(S, T, $H_0 \oplus H_1$) // See Fig. 7.

1: Pick up a free bend $(\hat{v}, v, \bar{v})$ associated with $v \in V(G_0)$.
2: Pick up $k-1$ free vertices, $x_1, \ldots, x_{k-1}$, other than v such that the bends associated with them are all free, i.e., $(x_i, \hat{x_i})$ and $(x_i, \bar{x_i})$ are free edges for all $i \in J$.
3: Regarding v as a *virtual* fault, find a 1-fault *unpaired* $(k-1)$-DPC joining $\{s_i : i \in J\}$ and $\{x_i : i \in J\}$ in G_0. Let the paths in the unpaired DPC be s_i–x_{σ_i} paths, $i \in J$, for some permutation σ on J. (Again, $\sigma(i)$ is abbreviated as σ_i.)
4: Pick up $k-2$ free edges from $E_{1,2}$ that are incident with no vertices in $\{\hat{v}\} \cup \{\hat{x_{\sigma_i}} : i \in J'\}$. The free edges are denoted by $(y_i, \bar{y_i})$, where $y_i \in V(G_1)$, for $i \in J'$.
5: Find a paired $(k-1)$-DPC in G_1 joining $\{(s_k, \hat{v})\} \cup \{(\hat{x_{\sigma_i}}, y_i) : i \in J'\}$.
6: Pick up a free edge from $E_{2,3}$ that is incident with no vertex in $\{\bar{x_{\sigma_1}}, \bar{v}\} \cup \{\bar{y_i} : i \in J'\}$. The free edge is denoted by $(y_1, \hat{y_1})$, where $y_1 \in V(G_3)$.
7: Find a paired 2-DPC in G_3 joining $\{(\bar{v}, t_k), (\bar{x_{\sigma_1}}, y_1)\}$.
8: Find a paired $(k-1)$-DPC in G_2 joining $\{(y'_i, t_i) : i \in J\}$, where $y'_1 = \hat{y_1}$ and $y'_i = \bar{y_i}$ for $i \in J'$.
9: Merge the four DPCs of subcomponents with the free bend and the free edges $\{(x_{\sigma_1}, \bar{x_{\sigma_1}})\} \cup \{(x_{\sigma_i}, \hat{x_{\sigma_i}}) : i \in J'\}$ as well as the free edges chosen in Steps 4 and 6 into a paired k-DPC.

For Steps 1 and 2, it suffices to prove a claim asserting that there exist k free bends. There are $|E_{0,3}|$ candidate free bends whereas at most $k+1$ of them could be blocked by the sources and the sink t_k, for which $|E_{0,3}| - (k+1) \ge 2^{m-3} - (\frac{m+1}{2} + 1) \ge \frac{m+1}{2} = k$ for $m \ge 7$, thus proving the claim. The existence of free edges in Steps 4 and 6 is obvious from the simple counting arguments, as before. The 1-fault unpaired $(k-1)$-DPC of Step 3 exists by Theorem 2, where

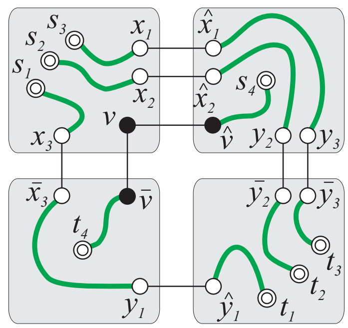


Fig. 7: Illustration of the Procedure Paired-DPC-D, where $m = 7$ and $k = 4$. In this example, we have $\sigma = (\sigma_1, \sigma_2, \sigma_3) = (3, 2, 1)$.

$1 + (k-1) = \frac{m+1}{2} \leq (m-2) - 1$ for $m \geq 7$. The paired DPCs in Steps 5, 7, and 8 exist by the induction hypothesis. Therefore, the lemma is proved. □

4. Paired DPCs in 5-dimensional RHL graphs

To establish the base step of our main theorem (Theorem 3), it is sufficient to prove Lemma 9 (below), due to Theorem 1(c). We denote by $N_G(v)$ the open neighborhood of a vertex $v \in V(G)$, i.e., $N_G(v) = \{u \in V(G) : (v,u) \in E(G)\}$. For a path P, $V(P)$ represents the vertex set of P.

Lemma 9. *Every 5-dimensional RHL graph G is (0-fault) paired 3-disjoint path coverable.*

Proof. Let $G = H_0 \oplus H_1$ for some 4-dimensional RHL graphs H_0 and H_1. We assume $(s_i, t_i) \notin E(G)$ for all $i \in \{1,2,3\}$ again by Theorem 1(c), because suppose otherwise, a paired 3-DPC can be obtained from the 2-vertex path (s_i, t_i) and a paired 2-DPC joining $S \setminus s_i$ and $T \setminus t_i$ in $G \setminus \{s_i, t_i\}$. Remember that each component H_j is paired 2-disjoint path coverable, by Theorem 1(b), and is 1-fault Hamiltonian-connected, by Lemma 4. There are six cases, up to symmetry, depending on the distribution of terminals.

Case 1: $S \cup T \subseteq V(H_0)$. There exists a source-sink pair, say (s_3, t_3), such that $N_{H_0}(s_3) \nsubseteq S \cup T$ and $N_{H_0}(t_3) \nsubseteq S \cup T$. (Suppose otherwise, i.e., $N_{H_0}(s_3) = \{s_1, s_2, t_1, t_2\}$ or $N_{H_0}(t_3) = \{s_1, s_2, t_1, t_2\}$, it suffices to pick up the (s_1, t_1) pair by Lemma 3(b).) Regarding s_3 and t_3 as *virtual* free vertices, we find a paired 2-DPC composed of an s_1–t_1 path, P_1, and an s_2–t_2 path, P_2, in H_0. If some s_i–t_i path in the 2-DPC passes through both s_3 and t_3, represented as $(s_i, P_x, x, s'_3, P', t'_3, y, P_y, t_i)$ where $\{s'_3, t'_3\} = \{s_3, t_3\}$, x is the immediate predecessor of s'_3, and y is the immediate successor of t'_3, it suffices to let $P_3 = (s'_3, P', t'_3)$ and merge (s_i, P_x, x), a Hamiltonian $\bar{x}$–$\bar{y}$ path of H_1, and (y, P_y, t_i) into a new s_i–t_i path.

Now, assume w.l.o.g. that P_1 in the 2-DPC passes through s_3 and P_2 passes through t_3. We first claim that there is an edge (α, β) of H_0 such that $\alpha \in$

$V(P_1) \setminus \{s_1, t_1\}$ and $\beta \in V(P_2) \setminus \{s_2, t_2\}$. Suppose otherwise, $\{s_1, s_2, t_1, t_2\}$ would be a vertex cut of H_0 separating s_3 and t_3, which must be the open neighborhood of some vertex v, because H_0 is *super-connected* (by Corollary 7.3 of [12]), i.e., $\kappa(H_0) = \delta(H_0)$ and every minimum vertex cut is $N_{H_0}(u)$ for some vertex u. The vertex v is neither a terminal, s_3 or t_3, by the choice of the pair (s_3, t_3), nor a nonterminal (suppose v is a nonterminal, P_1 would be (s_1, v, t_1) or P_2 would be (s_2, v, t_2), which contradicts the hypothesis that $s_3 \in V(P_1)$ and $t_3 \in V(P_2)$). Thus, the claim is proved. Then, we divide P_1 into three s_1–x, s_3–α, and y–t_1 subpaths for some $x, y \in V(P_1)$, and divide P_2 into s_2–z, β–t_3, and w–t_2 subpaths for some $z, w \in V(P_2)$. It suffices to merge the s_3–α and β–t_3 subpaths into an s_3–t_3 path, and combine the remaining four subpaths with the paired 2-DPC of H_1 made of $\bar{x}$–$\bar{y}$ and $\bar{z}$–$\bar{w}$ paths into s_1–t_1 and s_2–t_2 paths.

Case 2: $\{s_1, s_2, s_3, t_1, t_2\} \subseteq V(H_0)$ *and* $\{t_3\} \subseteq V(H_1)$. Firstly, suppose $\bar{t_3}$ is a nonterminal. We regard s_3 as a *virtual* free vertex and find a paired 2-DPC of H_0 composed of an s_1–t_1 path, P_1, and an s_2–t_2 path, P_2. Assume w.l.o.g. that P_1 passes through s_3, represented as $(s_1, P_x, x, s_3, y, P_y, t_1)$. If $\bar{x}, \bar{y} \neq t_3$, it suffices to divide the path P_1 into (s_1, P_x, x), one-vertex path (s_3), and (y, P_y, t_1) subpaths, and then combine the subpaths with a paired 2-DPC of H_1 made of $\bar{x}$–$\bar{y}$ and $\bar{s_3}$–t_3 paths into new s_1–t_1 and s_3–t_3 paths. Now, $\bar{x}$ or $\bar{y}$ is t_3, say $\bar{x} = t_3$. Then, P_1 may be represented as $(s_1, P_z, z, x, s_3, y, P_y, t_1)$ since $x = \bar{t_3} \neq s_1$. It suffices to divide P_1 into (s_1, P_z, z), (x, s_3), and (y, P_y, t_1) subpaths and combine them with one-vertex path (t_3) and a Hamiltonian $\bar{z}$–$\bar{y}$ path of $H_1 \setminus t_3$. Secondly, suppose $\bar{t_3}$ is a terminal, say s_2 (note that $\bar{t_3} \neq s_3$ from the fact $(s_3, t_3) \notin E(G)$). There exists a Hamiltonian s_1–t_1 path in $H_0 \setminus s_3$, represented as $(s_1, P_x, x, s_2', P', t_2', y, P_y, t_1)$ where $\{s_2', t_2'\} = \{s_2, t_2\}$. It suffices to divide the Hamiltonian path into (s_1, P_x, x), (s_2', P', t_2'), and (y, P_y, t_1) subpaths, and then combine the subpaths (s_1, P_x, x), (y, P_y, t_1) and one-vertex path (s_3) with a paired 2-DPC of H_1 made of $\bar{x}$–$\bar{y}$ and $\bar{s_3}$–t_3 paths.

Case 3: $\{s_1, s_2, t_1, t_2\} \subseteq V(H_0)$ *and* $\{s_3, t_3\} \subseteq V(H_1)$. A desired 3-DPC is obtained from a paired 2-DPC of H_0 joining $\{s_1, s_2\}$ and $\{t_1, t_2\}$ and a Hamiltonian s_3–t_3 path of H_1.

Case 4: $\{s_1, s_2, s_3, t_1\} \subseteq V(H_0)$ *and* $\{t_2, t_3\} \subseteq V(H_1)$. We claim that for some vertices $x, y \in V(H_0)$ such that $\bar{x} \neq t_3$ & $\bar{y} \neq t_2$ & $\{\bar{x}, \bar{y}\} \neq \{t_2, t_3\}$, there exist three disjoint paths, an s_1–t_1 path, s_2–x path (possibly, $s_2 = x$), and s_3–y path (possibly, $s_3 = y$), that altogether cover every vertex of H_0. Provided the claim is proved, a desired 3-DPC can be obtained by combining the three paths of H_0 with two paths of H_1, $\bar{x}$–t_2 and $\bar{y}$–t_3 paths, that cover all the vertices of H_1. Here, the two paths of H_1 form a paired 2-DPC if $\bar{x}$ and $\bar{y}$ are free vertices; They are one-vertex path (t_2) and a Hamiltonian $\bar{y}$–t_3 path of $H_1 \setminus t_2$ if $\bar{x} = t_2$; They are (t_3) and a Hamiltonian $\bar{x}$–t_2 path of $H_1 \setminus t_3$ if $\bar{y} = t_3$.

It remains to prove the claim. Let $H_0 = G_0 \oplus G_1$, where G_0 and G_1 are 3-dimensional RHL graphs, isomorphic to $G(8, 4)$ shown in Fig. 2. We will utilize a DPC property of $G(8, 4)$ asserting that given $\{s_1', t_1', s_2'\}$ in $G(8, 4)$, there exist two vertices z_i, $i \in \{1, 2\}$, such that for each z_i, there exist two disjoint paths,

s'_1–t'_1 and s'_2–z_i paths (possibly, $s'_2 = z_i$), that altogether cover every vertex of the graph. Its proof is by an immediate inspection and omitted here. Remember that $G(8,4)$ is Hamiltonian-connected and 1-fault Hamiltonian by Lemma 4. Firstly, suppose $s_1, t_1 \in V(G_0)$ for the proof of the claim. If $s_2, s_3 \in V(G_0)$, we divide a Hamiltonian s_2–s_3 path of G_0 into s_1–t_1, s_2–u, and s_3–v subpaths for some $u, v \in V(G_0)$, and combine the latter two subpaths with a Hamiltonian $\hat{u}$–$\hat{v}$ path of G_1 into an s_2–s_3 path. It suffices to delete an appropriate edge (x, y) from the s_2–s_3 path. If $s_2 \in V(G_0)$ and $s_3 \in V(G_1)$, there exist two disjoint s_1–t_1 and s_2–x paths in G_0, by the aforementioned DPC property of $G(8,4)$, such that $\bar{x} \neq t_3$. It suffices to find a Hamiltonian s_3–y path of G_1 for some $y \in V(G_1)$ with $\bar{y} \neq t_2, t_3$. If $s_2, s_3 \in V(G_1)$, it suffices to find a Hamiltonian s_1–t_1 path of G_0 and divide a Hamiltonian s_2–s_3 path of G_1 w.r.t. an appropriate edge (x, y).

Secondly, suppose $s_1 \in V(G_0)$ and $t_1 \in V(G_1)$. If $s_2 \in V(G_0)$ and $s_3 \in V(G_1)$, we pick up a free edge $(u, \hat{u})$ for $u \in V(G_0)$ and find disjoint s_1–u and s_2–x paths in G_0 for some $x \in V(G_0)$, from the DPC property of $G(8,4)$, such that $\bar{x} \neq t_3$ and moreover, $\bar{x} \neq t_2$ if possible. Similarly, we find disjoint $\hat{u}$–t_1 and s_3–y paths in G_1 for some $y \in V(G_1)$ such that $\bar{y} \neq t_2$ and moreover, $\bar{y} \neq t_3$ if possible. It follows that $\{\bar{x}, \bar{y}\} \neq \{t_2, t_3\}$. It suffices to merge the s_1–u and $\hat{u}$–t_1 paths into an s_1–t_1 path. If $s_2, s_3 \in V(G_0)$ finally, assuming w.l.o.g. $\hat{s_3} \neq t_1$, we find disjoint s_1–s_3 and s_2–x paths in G_0 for some $x \in V(G_0)$ such that $\bar{x} \neq t_3$ and moreover, $\bar{x} \neq t_2$ if possible. Let the s_1–s_3 path be represented as (s_1, P', w, s_3), where w is the immediate predecessor of s_3. If $\hat{w} \neq t_1$, we find disjoint $\hat{w}$–t_1 and $\hat{s_3}$–y paths in G_1 for some $y \in V(G_1)$ such that $\bar{y} \neq t_2$ and moreover, $\bar{y} \neq t_3$ if possible; If $\hat{w} = t_1$, we also have two disjoint paths in G_1, a one-vertex path (t_1) and an $\hat{s_3}$–y path for some $y \in V(G_1)$ obtained by deleting an edge incident with $\hat{s_3}$ from a Hamiltonian cycle of $G_1 \setminus t_1$, such that $\bar{y} \neq t_2$ and moreover, $\bar{y} \neq t_3$ if possible. Then, we have $\{\bar{x}, \bar{y}\} \neq \{t_2, t_3\}$. It suffices to combine the two paths, (s_1, P', w) and (s_3), of G_0 with the two paths of G_1 into s_1–t_1 and s_3–y paths. Thus, the claim is proved.

Case 5: *$\{s_1, s_2, t_1\} \subseteq V(H_0)$ and $\{s_3, t_2, t_3\} \subseteq V(H_1)$.* It suffices to pick up a free edge $(x, \bar{x})$, where $x \in V(H_0)$, and then merge the two DPCs: a paired 2-DPC of H_0 composed of s_1–t_1 and s_2–x paths and a paired 2-DPC of H_1 composed of $\bar{x}$–t_2 and s_3–t_3 paths.

Case 6: *$\{s_1, s_2, s_3\} \subseteq V(H_0)$ and $\{t_1, t_2, t_3\} \subseteq V(H_1)$.* By Lemmas 10 and 11 (below), there exist three edges $(x_i, \bar{x_i})$, where $x_i \in V(H_0)$, for $i \in \{1, 2, 3\}$ such that (i) $\{s_1, s_2, s_3\}$ and $\{x_1, x_2, x_3\}$ are not necessarily disjoint, (ii) for some permutation σ on $\{1, 2, 3\}$, there exist three disjoint s_i–$x_{\sigma(i)}$ paths for $i \in \{1, 2, 3\}$ that collectively cover every vertex of H_0, and (iii) there exists a paired 3-DPC of H_1 composed of $\bar{x_j}$–$t_{\sigma^{-1}(j)}$ paths for $j \in \{1, 2, 3\}$. Note that $432 + 183 > \binom{16}{3} = 560$, where $|V(H_0)| = |V(H_1)| = 16$. Combining the two DPCs results in a desired 3-DPC. This completes the entire proof. □

It is sometimes useful if we extend the notion of an *unpaired* k-disjoint path cover on not necessarily disjoint sets, S and T, of sources and sinks in a way that a vertex that belongs to both sets is considered as a valid, one-vertex path. A *generalized k-disjoint path cover* [24] joining S and T in a graph G is defined as a set of k pairwise disjoint paths of G composed of

- $|S \cap T|$ one-vertex paths for terminals in $S \cap T$, and
- $k - |S \cap T|$ paths that form an unpaired $(k - |S \cap T|)$-DPC joining $S \setminus (S \cap T)$ and $T \setminus (S \cap T)$ in $G \setminus (S \cap T)$.

Lemmas 10 and 11, concerning DPC properties of the 4-dimensional RHL graphs, were verified from computer programs that exhaustively search for DPCs. The source codes may be downloaded from `http://tcs.catholic.ac.kr/~jhpark/papers/Lemmas_10_and_11.zip`.

Lemma 10. *Given $S = \{s_1, s_2, s_3\}$ in a 4-dimensional RHL graph, there exists a 3-vertex set X that admits a generalized 3-DPC joining S and X. Furthermore, the number of such 3-vertex sets X is at least 432.*

Lemma 11. *Given $S = \{s_1, s_2, s_3\}$ in a 4-dimensional RHL graph, there exists a 3-vertex set $X = \{x_1, x_2, x_3\}$ such that for every permutation σ on $\{1, 2, 3\}$, there exists a paired 3-DPC composed of s_1–$t_{\sigma(1)}$, s_2–$t_{\sigma(2)}$, and s_3–$t_{\sigma(3)}$ paths. Furthermore, the number of such 3-vertex sets X is at least 183.*

5. Conclusion

In this paper, we have proven that every m-dimensional RHL graph, $m \geq 5$, is f-fault paired k-disjoint path coverable for any $f \geq 0$ and $k \geq 2$ subject to $f + 2k \leq m + 1$, achieving the optimal bound $m + 1$ on $f + 2k$ of the necessary condition, in Lemma 1. The proofs given in this paper are constructive, hence, can be used effectively to design an efficient algorithm for finding a paired many-to-many disjoint path cover in a restricted hypercube-like graph with vertex and/or edge faults.

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