Paired 2-Disjoint Path Covers and Strongly Hamiltonian Laceability of Bipartite Hypercube-Like Graphs

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Abstract

A *paired many-to-many k-disjoint path cover* (*paired k-DPC* for short) of a graph is a set of k vertex-disjoint paths joining k distinct source-sink pairs that altogether cover every vertex of the graph. We consider the problem of constructing paired 2-DPC's in an m-dimensional bipartite HL-graph, X_m, and its application in finding the longest possible paths. It is proved that every X_m, $m \geq 4$, has a fault-free paired 2-DPC if there are at most $m-3$ faulty edges and the set of sources and sinks is balanced in the sense that it contains the same number of vertices from each part of the bipartition. Furthermore, every X_m, $m \geq 4$, has a paired 2-DPC in which the two paths have the same length if each source-sink pair is balanced. Using 2-DPC properties, we show that every X_m, $m \geq 3$, with either at most $m-2$ faulty edges or one faulty vertex and at most $m-3$ faulty edges is strongly Hamiltonian-laceable.

Keywords: Disjoint path, strongly Hamiltonian-laceable, Hamiltonian path, bipartite HL-graphs, graph theory, fault tolerance.

1. Introduction

Since node and/or link failure is inevitable in a large network, fault tolerance is essential to the network performance. An interconnection network is frequently modeled as a graph, where vertices and edges respectively represent nodes and communication links in the network. The connectivity of

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the underlying graph has been a primary measure of fault tolerance [13, 24], and the connectivity of a graph is closely related to the existence of disjoint paths in the graph. Menger's theorem states the connectivity of a graph in terms of the number of disjoint paths of *one-to-one type* joining a pair of source and sink, whereas the Fan Lemma states the connectivity of a graph in terms of the number of disjoint paths of *one-to-many type* joining a source to a set of sinks [2]. Moreover, a graph is k-connected if and only if it has k disjoint paths of *many-to-many type*, respectively connecting arbitrary k distinct sources and arbitrary k distinct sinks, where, if a source coincides with a sink, then such source itself is regarded as a path.

One of the central issues in the study of interconnection networks is finding parallel paths, which is naturally related to routing among nodes and fault tolerance of the network [13, 24]. Parallel paths correspond to disjoint paths of the graph. If each copy of a message is routed along a different path of the disjont paths, whichever type they are, then at least one copy eventually arrives at its sink provided the total number of node and link faults is less than the number of disjoint paths. Here, the unique source of one-to-one type and of one-to-many type is assumed to be fault-free, as well as the unique sink of one-to-one type.

A *k-disjoint path cover* (k-DPC for short) of a graph is a set of k (internally) disjoint paths that altogether cover every vertex of the graph. The disjoint path cover problem finds applications in many areas such as software testing, database design, and code optimization [1, 25]. In addition, the problem is concerned with applications where full utilization of network nodes is important [29]. For example, basic communication problems for the dissemination of information, such as broadcasting (to send a message to all the nodes) and information gathering (to receive a message from each of the nodes), require visiting every node of the network at least once. Since visiting a node more than once results in unnecessary overhead, a disjoint path cover can be employed to avoid this unsatisfactory situation.

Disjoint path is one of the fundamental notions in graph theory from which many properties of a graph can be deduced [2, 24]. Our disjoint path cover problem, in the corresponding optimization version, is to find disjoint paths whose total length is the maximum possible (maxsum). Determining whether there exists a k-DPC in general graph was proven to be NP-complete for any fixed $k \geq 1$ [29, 30]. On the contrary, the shortest disjoint paths problem is to minimize the total length (minsum) or the length of the longest path (minmax). Much attention has been devoted to the shortest disjoint paths problem. For example, refer to [16, 18, 23] etc.

Suppose that there are a set of k sources $S = \{s_1, s_2, \ldots, s_k\}$ and a set of k sinks $T = \{t_1, t_2, \ldots, t_k\}$ in a graph G such that $S \cap T = \emptyset$. Sources and sinks are called *terminals* in general. A *many-to-many k-disjoint path cover* of G joining S and T is a set of k disjoint paths joining sources and sinks. It is called *paired* if each source s_i is joined to a specified sink t_i. If not, it is called *unpaired*. The other two possible k-disjoint path covers are of one-to-many type joining $S = \{s\}$ and $T = \{t_1, t_2, \ldots, t_k\}$, and of one-to-one type joining $S = \{s\}$ and $T = \{t\}$, which are clearly understandable. When a graph contains faulty elements, whether vertices or edges, its k-disjoint path cover naturally means a k-disjoint path cover of the graph with the faulty elements deleted. Some works on the construction of k-DPC's in hypercubes [3, 4, 7, 11] and hypercube-like graphs [15, 29, 30] can be found.

The embedding of a linear array or a ring into an interconnection network can be modeled as finding a long path or cycle, possibly a Hamiltonian path or cycle. A path or cycle in a graph is called *Hamiltonian* if it contains all the vertices of the graph. This problem has attracted much attention in the literature, such as works on faulty hypercubes [9, 10, 17]. The disjoint path cover problem is closely related to the Hamiltonian problem in that a Hamiltonian path joining a pair of vertices can be viewed as any type of 1-DPC joining them, and a Hamiltonian path joining a pair of vertices that passes through $k-1$ prescribed edges can be obtained directly from some paired k-DPC of the graph [29]. For the problem of Hamiltonian paths passing through prescribed edges, see [6, 32] for example. Furthermore, disjoint path coverability has been employed to establish some Hamiltonian properties, such as in hypercube-like graphs [21, 27, 29].

A class of hypercube-like interconnection networks, called *HL-graphs*, was introduced by Vaidya *et al.* [31]. It gives a unified perspective on many hypercube variants. It is well-known that twisting some edge pairs of a hypercube can reduce diameter while preserving attractive properties. Using this technique, many hypercube variants with smaller diameter have been proposed, such as twisted cube [12], crossed cube [8], and Möbius cube [5]. Most of the networks built in this way are nonbipartite HL-graphs. Furthermore, an interesting subclass of nonbipartite HL-graphs, called restricted HL-graphs, has been proposed and studied in [28, 29, 30].

Bipartite HL-graphs are *equitable*, that is, the two *parts* of the bipartition have the same number of vertices. There is a relative paucity of works on bipartite HL-graphs. Some notable studies are mentioned in the following. Every m-dimensional bipartite HL-graph X_m, $m \geq 2$, has a paired 2-DPC joining S and T if S and T are subsets of different parts [27]. Every X_m, $m \geq 2$, possessing at most $m-2$ faulty edges is Hamiltonian-laceable [21, 26]. An equitable bipartite graph is called *Hamiltonian-laceable* if each pair of vertices contained in different parts is joined by a Hamiltonian path. Recently, Lim *et al.* [20] presented a vertex-symmetric graph with a small diameter, called bicube, contained in the class of m-dimensional bipartite HL-graphs. Its diameter is $\lceil m/2 \rceil$, which is one of the smallest among the hypercube variants, thus providing further motivation for the study of bipartite HL-graphs.

In this paper, we investigate the problem of constructing paired 2-disjoint path covers in bipartite HL-graphs. It will be shown that every m-dimensional bipartite HL-graph X_m, $m \geq 4$, with $m-3$ or less faulty edges has a paired 2-DPC joining S and T provided $S \cup T$ is balanced. A vertex subset of an equitable bipartite graph is *balanced* if it contains the same number of vertices from each part. For an equitable bipartite graph to have a paired 2-DPC, it is necessary that $S \cup T$ is balanced. Furthermore, every X_m, $m \geq 4$, has a paired 2-DPC joining S and T in which the two paths are of the same length if S and T are subsets of different parts.

Using the result on paired 2-DPC, we will derive strongly Hamiltonian-laceability of bipartite HL-graphs. An equitable bipartite graph is *strongly Hamiltonian-laceable* if it is Hamiltonian-laceable and each pair of vertices in the same part is joined by a path of length $n-2$, where n is the number of vertices in the graph. We will show that every X_m with fault set F is strongly Hamiltonian-laceable if either F contains at most $m-2$ edges or F contains at most $m-3$ edges and one vertex. The definition of strongly Hamiltonian-laceability of an inequitable bipartite graph is provided in the next section.

In the rest of this paper, we use standard terminology in graph theory (see ref. [2]). This paper is organized as follows. In the next section, we present some definitions and related work. The paired 2-disjoint path cover problems are investigated in Section 3, and strongly Hamiltonian-laceability is derived in Section 4. Finally, we give concluding remarks in Section 5.

2. Definitions and Related Work

For graphs G_0 and G_1 having the same number of vertices, we denote by $G_0 \oplus G_1$ an arbitrary graph whose vertex set is $V(G_0) \cup V(G_1)$ and whose edge set is $E(G_0) \cup E(G_1) \cup E_2$, where $E_2 = \{(u, \phi(u)) : u \in V(G_0), \phi : V(G_0) \to V(G_1)$ is a bijection$\}$. Here, $V(G)$ and $E(G)$ denote the vertex set and edge set of a graph G, respectively. The class of HL-graphs can be defined by repeatedly applying the $\oplus$ operation as follows: $HL_0 = \{K_1\}$, where K_1 is the trivial graph; for $m \geq 1$, $HL_m = \{G_0 \oplus G_1 : G_0, G_1 \in HL_{m-1}\}$. A graph contained in HL_m is called an *m-dimensional HL-graph*. Every m-dimensional HL-graph has 2^m vertices and is of degree m. This paper deals with bipartite HL-graphs, which are equitable by definition.

Let G be a bipartite graph. For convenience, we refer to vertices of one part of the bipartition as *black* and the vertices of the other part as *white*. We denote by $c(v)$ the color of the vertex v. Let G have n_b black vertices and n_w white vertices, and let $n = n_b + n_w$. Strongly Hamiltonian-laceability of a bipartite graph is defined in terms of L^{OPT}-path. If $n_b = n_w$, a path of length $n-1$ joining a balanced pair of vertices is called an *L^{OPT}-path*; for a pair of vertices with the same color, a path of length $n-2$ is called an L^{OPT}-path. If $n_b < n_w$, the length of an L^{OPT}-path is $2n_b$ for a pair of white vertices, $2n_b - 1$ for a balanced pair of vertices, and $2n_b - 2$ for a pair of black vertices. If $n_w < n_b$, an L^{OPT}-path is defined symmetrically.

A bipartite graph G is called *strongly Hamiltonian-laceable* if every pair of vertices is joined by an L^{OPT}-path. The graph G is called *f-edge-fault strongly Hamiltonian-laceable* if $G \setminus F$ is strongly Hamiltonian-laceable for any edge-fault set F with $|F| \leq f$. The graph G is called *f_v-vertex-fault and f_e-edge-fault strongly Hamiltonian-laceable* if $G \setminus (F_v \cup F_e)$ is strongly Hamiltonian-laceable for any vertex-fault set F_v with $|F_v| \leq f_v$ and any edge-fault set F_e with $|F_e| \leq f_e$.

Throughout this paper, let F denote a set of faulty elements (vertices and/or edges), which corresponds to the set of node and/or link failures. A path in a graph is represented as a sequence of vertices. An s-t path refers to a path from vertex s to t. Let P be an s-t path and u be a vertex on P. We denote by $prev_P(u)$ and $next_P(u)$ the vertices adjacent to u on P encountered just before and just after u, respectively, when we traverse P starting at s. Of course, $prev_P(s)$ and $next_P(t)$ are undefined.

A paired 2-DPC problem in bipartite HL-graphs without faults for balanced source-sink pairs was studied by Park *et al.* [27], as stated in Lemma 1. This lemma will be extended to Theorem 1 of the next section in that source-sink pairs may have the same color and a bounded number of faulty edges are allowed. X_m denotes an m-dimensional bipartite HL-graph.

Lemma 1. *[27] Every X_m, $m \geq 2$, has a paired 2-DPC joining $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$ if S and T are subsets of different parts of the bipartition.*

It is known that every X_m is Hamiltonian-laceable [27] and every X_m, $m \geq 3$, is $(m-3)$-edge-fault Hamiltonian-laceable [14]. Hamiltonian-laceability of X_m with at most $m-2$ faulty edges was studied independently by Park [26] and by Lin *et al.* [21], as shown in Lemma 2. In addition, a Hamiltonian property of X_m in the presence of a single vertex fault was reported in [22, 26], as shown in Lemma 3. We extend these two lemmas to the case that there exist at most one single vertex fault and at most $m-3$ faults in total, as will be shown in Theorem 3 of Section 4.

Lemma 2. *[21, 26] Every X_m, $m \geq 2$, is $(m-2)$-edge-fault Hamiltonian-laceable.*

Lemma 3. *[22, 26] There exists an s-t Hamiltonian path in $X_m \setminus u$ for any distinct vertices s, t, and u such that $c(s) = c(t) \neq c(u)$.*

3. Paired 2-Disjoint Path Covers in Bipartite HL-graphs

In this section, we consider a paired 2-DPC problem in bipartite HL-graphs. We show that every X_m, $m \geq 4$, with at most $m-3$ faulty edges has a paired 2-DPC joining S and T for given balanced $S \cup T$, and that every X_m, $m \geq 4$, has a paired 2-DPC in which the two paths, s_1-t_1 path and s_2-t_2 path, have the same length provided S and T are contained in different parts. The latter type of paired 2-DPC is known as a *2-equal-disjoint path cover* (*2-eq-DPC* for short) and studied by Lai *et al.* [19] in some nonbipartite hypercube-like graphs.

In a graph $G_0 \oplus G_1$, a vertex u of G_i has a unique neighbor in G_{1-i}. We denote it by $\bar{u}$ and call it a *mate* of u. We call an edge between a vertex and its mate a *bridge*. F_0 and F_1 denote the sets of faulty elements in G_0 and G_1, respectively, and F_2 denotes the set of faulty bridges, so that $F = F_0 \cup F_1 \cup F_2$. Recall F denotes the set of faulty elements. Hereafter in this paper, we denote by 2-DPC$[(s_1, t_1), (s_2, t_2)|G, F]$ a paired 2-DPC in a graph G with a fault set F joining a set of sources $S = \{s_1, s_2\}$ and a set of sinks $T = \{t_1, t_2\}$ such that $S \cap T = \emptyset$.

3.1. Paired 2-DPC in X_m with at most $m-3$ faulty edges

It is assumed that $S \cup T$ is balanced, where $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$. We begin by considering a paired 2-DPC problem in the 3-dimensional bipartite HL-graph, which is isomorphic to the 3-dimensional hypercube Q_3. By Lemma 1, Q_3 has a paired 2-DPC joining S and T if each source-sink pair is balanced. If not, then $\{s_1, t_1\}$ is contained in one part of the bipartition and is unique up to symmetry. There are two positions of $\{s_2, t_2\}$ such that no paired 2-DPC joining S and T exists, as shown in Fig. 1. There are only two *forbidden configurations* that do not allow any paired 2-DPC as proved below. We denote by $N_G(u)$, or just by $N(u)$ if no confusion can arise, the set of vertices adjacent to u in a graph G.

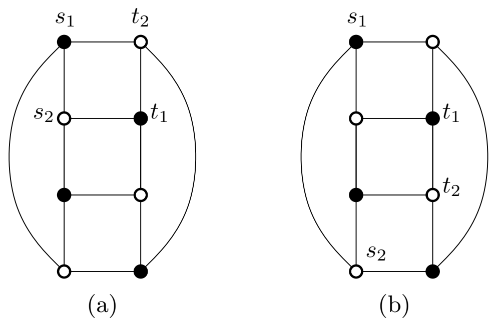


Figure 1: Forbidden configurations that do not allow any paired 2-DPC.

Lemma 4. *Let $\{s_1, t_1\}$ be a subset of one part of the bipartition of Q_3. Then, there exists a paired 2-DPC joining S and T if and only if $\{s_2, t_2\}$ is contained in the other part (so, $S \cup T$ is balanced) and it is equal to neither $N(s_1) \cap N(t_1)$ nor $N(s_1)$ XOR $N(t_1)$, where $N(s_1)$ XOR $N(t_1) = (N(s_1) \cup N(t_1)) \setminus (N(s_1) \cap N(t_1))$.*

Proof. For a fixed $\{s_1, t_1\}$, there are $\binom{4}{2} = 6$ possible positions of $\{s_2, t_2\}$. It can be easily verified that there exist four positions of $\{s_2, t_2\}$ (excluding the two forbidden configurations) that allow a paired 2-DPC joining S and T. $\square$

A vertex u is called *free* if u is fault-free and not a terminal. An edge (u, v) is called *free* if it is not faulty and both u and v are free. For a terminal $x \in V(G_i)$, we denote by x^* an arbitrary vertex in $R(x)$, where $R(x) = \{y : y \in V(G_{1-i}),\ c(y) = c(x), \text{ and } (y, \bar{y}) \text{ is free}\}$. We call x^* a *pseudo-terminal* of x.

Theorem 1. *Every m-dimensional bipartite HL-graph X_m, $m \geq 4$, with $m - 3$ or less faulty edges has a paired 2-DPC joining S and T provided $S \cup T$ is balanced.*

Proof. The proof will proceed by induction on m. Let $X_m = G_0 \oplus G_1$, where each G_i, $i = 0, 1$, is isomorphic to an $(m-1)$-dimensional bipartite HL-graph. Keep in mind that each G_i has 2^{m-1} vertices of degree $m - 1$ and is an equitable bipartite graph. The proof for the basis, i.e., when $m = 4$, is similar to that for the induction step, i.e., when $m \geq 5$, but there are additional subtleties. We combine these proofs to avoid repetition. For $m \geq 5$, we will assume that each G_i has a paired 2-DPC joining balanced terminals provided the number of faulty edges is at most $m - 4$. For $m = 4$, we can assume the same, except forbidden configurations, by Lemmas 1 and 4. Since a 2-DPC in X_m with a virtual fault set $F \cup F'$, where F' is a set of arbitrary $m-3-|F|$ fault-free edges, is also a 2-DPC in X_m with the fault set F, we can assume that $|F| = m-3$. It is assumed w.l.o.g. that the number of terminals in G_0 is greater than or equal to that in G_1. There are four cases depending on the positions of terminals.

Case 1: $s_1, t_1 \in V(G_0)$ and $s_2, t_2 \in V(G_1)$. (See Fig. 2a.)
If $c(s_1) \neq c(t_1)$, an s_1-t_1 Hamiltonian path in G_0 and an s_2-t_2 Hamiltonian path in G_1 constitute a paired 2-DPC, which is what we want. The existence of the two Hamiltonian paths relies on Lemma 2. Now, assume that $c(s_1) = c(t_1)$. W.l.o.g., we assume $|F_0| \geq |F_1|$. Then, $|F_1| \leq \lfloor (m-3)/2 \rfloor \leq m-4$ for every $m \geq 4$. We claim that it is possible to select two pseudo-terminals s_2^* and t_2^* in G_0 so that there exists a 2-DPC$[(s_1, t_1), (s_2^*, t_2^*)|G_0, F_0]$. The proof of this claim is deferred for a moment. Suppose that the claim holds true. Then there exists a 2-DPC$[(s_2, \bar{s_2^*}), (t_2, \bar{t_2^*})|G_1, F_1]$ by induction hypothesis when $m \geq 5$ and by Lemma 1 when $m = 4$. To obtain a paired 2-DPC in X_m, it suffices to merge the two 2-DPC's with bridges $(s_2^*, \bar{s_2^*})$ and $(t_2^*, \bar{t_2^*})$.

It remains to prove the claim. For the first case, let $|F_0| \leq m-4$. The number of candidates for s_2^* and t_2^* is at least $2^{m-2} - |F_2| \geq 2^{m-2} - (m-3) \geq 3$ for any $m \geq 4$, and thus picking up arbitrary two candidates is sufficient when $m \geq 5$. When $m = 4$, a care should be taken since arbitrary candidates may not allow a paired 2-DPC in G_0 since $c(s_1) = c(t_1) \neq c(s_2) = c(t_2)$. We have $|F| = |F_2| = 1$. To avoid forming a forbidden configuration, it suffices to pick up one in $N(s_1) \cap N(t_1)$ and the other in $N(s_1)$ XOR $N(t_1)$ by Lemma 4. Furthermore, we can assume both $(s_2^*, \bar{s_2^*})$ and $(t_2^*, \bar{t_2^*})$ are fault-free since both $N(s_1) \cap N(t_1)$ and $N(s_1)$ XOR $N(t_1)$ have two vertices. For the second case, assume that $|F_0| = m-3$ and $|F_1| = |F_2| = 0$. We can pick up pseudo-terminal t_2^* such that $(t_2^*, t_1) \notin E(G_0)$ since $2^{m-2} > m-1$ for every $m \geq 4$. There exists an s_1-t_2^* Hamiltonian path in G_0 by Lemma 2. We set $s_2^* = next_P(t_1)$. Clearly, $s_2^* \neq t_2^*$. If we remove (t_1, s_2^*) from P, there remain an s_1-t_1 path and an s_2^*-t_2^* path. These two paths are vertex-disjoint and cover all the vertices of G_0. Thus, the claim is proved.

Case 2: $s_1, s_2 \in V(G_0)$ and $t_1, t_2 \in V(G_1)$. (See Fig. 2b.)
Without loss of generality, we assume $|F_0| \geq |F_1|$. Similar to the proof of Case 1, we claim that there exist two pseudo-terminals t_1^* and t_2^* such that there exists a 2-DPC$[(s_1, t_1^*), (s_2, t_2^*)|G_0, F_0]$. Provided the claim holds true, we find a 2-DPC$[(\bar{t_1^*}, t_1), (\bar{t_2^*}, t_2)|G_1, F_1]$ and then merge the two DPC's to obtain a desired 2-DPC. Since $c(\bar{t_1^*}) \neq c(t_1)$ and $c(\bar{t_2^*}) \neq c(t_2)$, the 2-DPC in G_1 always exists by induction hypothesis and Lemma 1.

Let us prove the claim. Suppose $|F_0| \leq m-4$ for the first case. If either

$m \geq 5$ or both $m = 4$ and $c(s_1) \neq c(t_1)$, then picking up arbitrary pseudo-terminals is sufficient. In this case, the number of candidates for each of t_1^* and t_2^* is at least $2^{m-2} - (m-3) - 2 \geq 1$ for any $m \geq 4$. Now, assume that $m = 4$ and $c(s_1) = c(t_1)$, where $|F| = |F_2| = 1$. To avoid forbidden configurations, we will select pseudo-terminals so that the subgraph of G_0 induced by terminals and pseudo-terminals is isomorphic to a path of length three. Note that the subgraph induced by four terminals in a forbidden configuration is isomorphic to a cycle of length four (Fig. 1a) or a graph consisting of two paths of length one each (Fig. 1b).

If $(s_1, s_2) \notin E(G_0)$, there exists a vertex $x \in N(s_2)$ such that $x \in R(t_1)$ and $y \in R(t_2)$ for some $y \in N(x) \cap N(s_1)$. Assigning x and y to t_1^* and t_2^*, respectively, results in an induced path (s_1, t_2^*, t_1^*, s_2) since s_1 and s_2 are not adjacent. The case when $(s_1, s_2) \in E(G_0)$ remains to be considered. Let $X = N(s_2) \setminus s_1$ and $Y = N(s_1) \setminus s_2$. Let $\{z_1, z_2\} = V(G_0) \setminus (N(s_1) \cup N(s_2))$ with $c(z_1) = c(s_1)$. Then, z_1 and z_2 are adjacent to every element of Y and X, respectively. Observe that (i) $R(t_1) \subseteq X \cup \{z_1\}$, $R(t_2) \subseteq Y \cup \{z_2\}$, and (ii) $|R(t_1)|, |R(t_2)| \geq 1$, $|R(t_1)| + |R(t_2)| \geq 3$. If $X \cap R(t_1) = \emptyset$, then for some $t_2^* \in Y$, (t_1^*, t_2^*, s_1, s_2) forms an induced path, where $t_1^* = z_1$. A symmetric argument works when $Y \cap R(t_2) = \emptyset$. Now, assume $X \cap R(t_1), Y \cap R(t_2) \neq \emptyset$. If $X \subseteq R(t_1)$, then for some $t_2^* \in Y$, there exists $t_1^* \in X$ such that $(t_2^*, t_1^*) \notin E(G_0)$ and thus we have an induced path (t_2^*, s_1, s_2, t_1^*). In a symmetric manner, we can also construct an induced path (t_2^*, s_1, s_2, t_1^*) when $Y \subseteq R(t_2)$. Finally, suppose $|X \cap R(t_1)| = |Y \cap R(t_2)| = 1$. At least one of z_1 and z_2, say z_1 is contained in $R(t_1) \cup R(t_2)$. Then, we have an induced path (t_1^*, t_2^*, s_1, s_2), where $t_1^* = z_1$ and $t_2^* \in Y \cap R(t_2)$.

For the remaining case, let $|F_0| = m - 3$ and $|F_1| = |F_2| = 0$. If $c(s_1) \neq c(s_2)$, then there exists an s_1-s_2 Hamiltonian path P in G_0. We show that there exists an edge (u, v) with $u = prev_P(v)$ on P such that (i) $c(u) = c(t_1)$ and (ii) both $\bar{u}$ and $\bar{v}$ are free. There are at least $2^{m-2} - 1$ candidate edges satisfying (i), and there are two blocking elements, i.e. terminals in G_1. Since $2^{m-2} - 1 \geq 3$ for any $m \geq 4$, there exists at least one such edge (u, v). For our purpose, it suffices to remove (u, v) from P and set $t_1^* = u$ and $t_2^* = v$. If $c(s_1) = c(s_2)$, we pick up a pseudo-terminal t_2^* in G_0 and find an s_1-t_2^* Hamiltonian path P in G_0. It suffices to set $t_1^* = prev_P(s_2)$ and remove (t_1^*, s_2) from P. Obviously, $c(t_1^*) = c(t_1)$ and $\bar{t_1^*} \neq t_1, t_2$. The claim is proved.

Case 3: $s_1, t_1, s_2 \in V(G_0)$ and $t_2 \in V(G_1)$. (See Fig. 2c.)
We claim that (i) for some pseudo-terminal t_2^*, there exists a 2-DPC$[(s_1, t_1), (s_2, t_2^*)|G_0, F_0]$, or (ii) $G_0 \setminus (F_0 \cup \{s_2\})$ has an s_1-t_1 Hamiltonian path, $(s_2, \bar{s_2})$ is fault-free, and $c(\bar{s_2}) \neq c(t_2)$. Provided the claim holds true, a paired 2-

DPC in X_m is obtained by merging the 2-DPC of G_0 and a $\bar{t_2^*}$-t_2 Hamiltonian path in G_1 if (i) is satisfied, or by merging the s_1-t_1 Hamiltonian path of G_0 and an $\bar{s_2}$-t_2 Hamiltonian path in G_1 if (ii) is satisfied.

To prove the claim, assume $|F_0| \leq m-4$ first. It is possible to select t_2^* satisfying (i) as follows. When either $m \geq 5$ or both $m=4$ and $c(s_1) \neq c(t_1)$, it is sufficient to pick up an arbitrary vertex in $R(t_2)$. When $m=4$ and $c(s_1)=c(t_1)$, as shown in Fig. 1, there are two choices to avoid forming a forbidden configuration and anyone of them can be selected as t_2^*. Now, we assume $|F_0|=m-3$. Without loss of generality, we can assume that $c(s_1) \neq c(s_2)$. Then, there exists an s_1-s_2 Hamiltonian path P in G_0. If $next_P(t_1) \neq s_2$, it suffices to set $t_2^* = next_P(t_1)$ and remove (t_1, t_2^*) from P. Certainly, $c(t_2^*) = c(t_2)$ and $\bar{t_2^*} \neq t_2$. Thus, (i) is satisfied. If $next_P(t_1) = s_2$, removing (t_1, s_2) from P results in an s_1-t_1 Hamiltonian path in $G_0 \setminus (F_0 \cup \{s_2\})$. Furthermore, $(s_2, \bar{s_2})$ is fault-free and $c(\bar{s_2}) \neq c(t_2)$. Thus, (ii) is satisfied and the claim is proved.

Case 4: $s_1, t_1, s_2, t_2 \in V(G_0)$. (See Fig. 2d.)
Suppose that either $m \geq 5$ or $m=4$ and terminals do not form a forbidden configuration in G_0. We let $F_0' = F_0$ if $|F_0| \leq m-4$; otherwise, let $F_0' = F_0 \setminus e$ for an arbitrary faulty edge e in G_0. Then, $|F_0'| \leq m-4$. So, there exists a paired 2-DPC in $G_0 \setminus F_0'$ joining S and T. If no path in the DPC passes through e, let (u,v) be an edge on a path in the DPC such that $(u,\bar{u})$ and $(v,\bar{v})$ are fault-free; otherwise, let $(u,v)=e$. To obtain a paired 2-DPC in X_m, it suffices to replace (u,v) with a path (u, P_1, v), where P_1 is an $\bar{u}$-$\bar{v}$ Hamiltonian path in G_1.

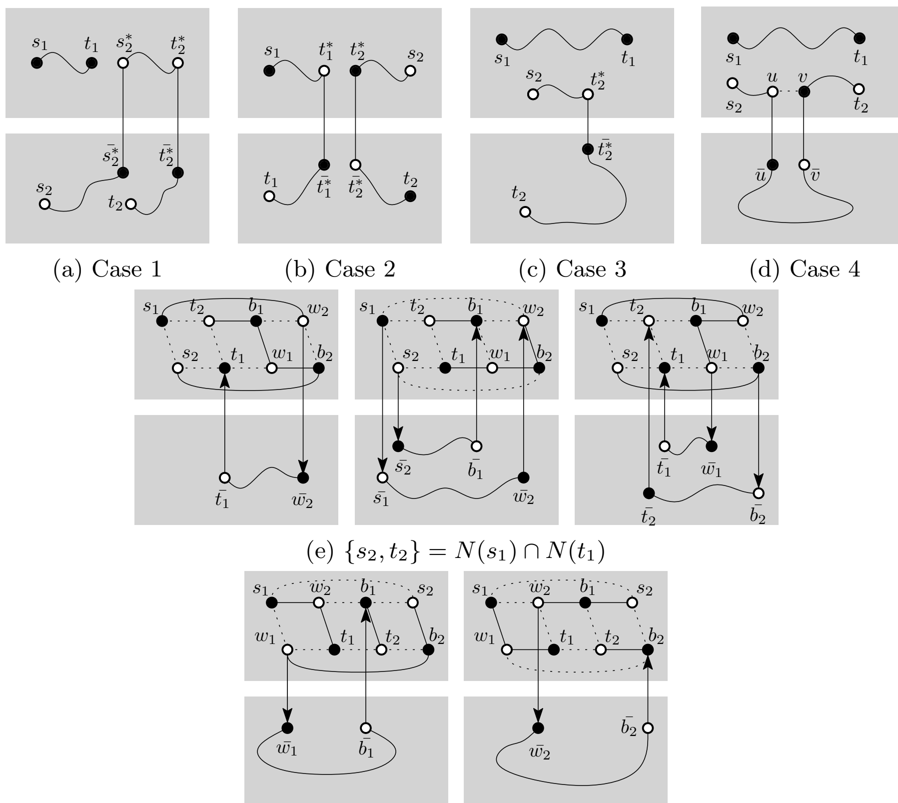
Finally, let $m=4$ and terminals form a forbidden configuration. Recall $|F|=1$. There are two forbidden configurations up to symmetry by Lemma 4. For the forbidden configuration of $\{s_2, t_2\} = N(s_1) \cap N(t_1)$, as shown in Fig. 2e, let w_1 and w_2 be vertices in $N(s_1)$ XOR $N(t_1)$ and let b_1 and b_2 be vertices in $N(s_2)$ XOR $N(t_2)$. We denote by $P[u,v]$ (resp. $P_h[u,v]$) an u-v path (resp. an u-v Hamiltonian path) in G_1. If $F_0 \cup F_2 = \emptyset$, there are two paths $(s_1, w_2, P_h[\bar{w_2}, \bar{t_1}], t_1)$ and $(s_2, b_2, w_1, b_1, t_2)$, which form a paired 2-DPC in X_m. If $F_1 = \emptyset$, we let

$P_1 = (s_1, P[\bar{s_1}, \bar{w_2}], w_2, b_2, w_1, t_1)$, $P_2 = (s_2, P[\bar{s_2}, \bar{b_1}], b_1, t_2)$,

$P_1' = (s_1, w_2, b_1, w_1, P[\bar{w_1}, \bar{t_1}], t_1)$, $P_2' = (s_2, b_2, P[\bar{b_2}, \bar{t_2}], t_2)$,

where $\{P[\bar{s_1}, \bar{w_2}], P[\bar{s_2}, \bar{b_1}]\}$ and $\{P[\bar{w_1}, \bar{t_1}], P[\bar{b_2}, \bar{t_2}]\}$ are 2-DPC's in G_1. Such 2-DPC's exist by Lemma 1. Since $\{P_1, P_2\}$ and $\{P_1', P_2'\}$ share neither bridges nor edges of G_0, at least one of the two sets is a desired 2-DPC in X_m.

For the forbidden configuration of $\{s_2, t_2\} = N(s_1)$ XOR $N(t_1)$, as shown in Fig. 2f, we let w_1 and w_2 be vertices in $N(s_1) \cap N(t_1)$ and let b_1 and b_2



(a) Case 1 (b) Case 2 (c) Case 3 (d) Case 4

(e) $\{s_2, t_2\} = N(s_1) \cap N(t_1)$

(f) $\{s_2, t_2\} = N(s_1)$ XOR $N(t_1)$.

Figure 2: Illustration of the proof of Theorem 1.

be vertices in $N(s_2) \cap N(t_2)$. We let

$P_1 = (s_1, w_2, t_1)$, $P_2 = (s_2, b_2, w_1, P_h[\bar{w_1}, \bar{b_1}], b_1, t_2)$,

$P_1' = (s_1, w_1, t_1)$, $P_2' = (s_2, b_1, w_2, P_h[\bar{w_2}, \bar{b_2}], b_2, t_2)$.

Similar to the proof of the previous case, we can conclude at least one of $\{P_1, P_2\}$ and $\{P_1', P_2'\}$ is a paired 2-DPC in X_m. This completes the proof of Theorem 1. □

Theorem 1 is an extension of the work stated in Lemma 1 by Park *et al.* in [27]. The number $m-3$ of faulty edges is the maximum possible in the sense that no m-dimensional bipartite HL-graph with $m-2$ faulty edges is guaranteed to have a paired 2-DPC joining S and T even if $S \cup T$ is balanced. Imagine the situation that $s_2, t_2 \in N(s_1)$ and all the edges that are incident to s_1 but not to s_2 or t_2 are faulty. The number of faulty edges is $m-2$. For any t_1 with $c(t_1) = c(s_1)$, $S \cup T$ is balanced. However, no paired 2-DPC joining S and T exists. The characterization of the existence of paired 2-DPC in X_m with $m-2$ or more faulty edges is an open problem.

3.2. Paired 2-eq-DPC in fault-free X_m

In an m-dimensional bipartite HL-graph X_m without faults, we consider the problem of constructing 2-eq-DPC's. It is necessary for X_m to have a paired 2-eq-DPC that $c(s_1) \neq c(t_1)$ and $c(s_2) \neq c(t_2)$. The reason for this is that each path of a paired 2-eq-DPC in X_m should contain 2^{m-1} vertices. It will be shown that X_m, $m \geq 4$, has a 2-eq-DPC if each source-sink pair is balanced. We begin by considering the 2-eq-DPC problem in the 3-dimensional bipartite HL-graph Q_3. Q_3 may not have a 2-eq-DPC even if each source-sink pair is balanced. A necessary and sufficient condition for Q_3 to have a 2-eq-DPC is derived as follows.

Lemma 5. *Suppose that S and T are subsets of different parts of the bipartition of Q_3. Then, Q_3 has a paired 2-eq-DPC joining S and T if and only if $(s_1, t_1), (s_2, t_2) \in E(Q_3)$ or $(s_1, t_1), (s_2, t_2) \notin E(Q_3)$.*

Proof. If $(s_1, t_1), (s_2, t_2) \in E(Q_3)$, then Q_3 can be divided into two Q_2's such that each subcube contains one source-sink pair. Each subcube is Hamiltonian-laceable, and thus there exists a 2-eq-DPC. If $(s_1, t_1), (s_2, t_2) \notin E(Q_3)$, an arbitrary paired 2-DPC is indeed a 2-eq-DPC since the two paths in the 2-DPC are of length at least three and their length sum should be six. The existence of a paired 2-DPC is guaranteed by Lemma 1. Suppose $(s_1, t_1) \in E(Q_3)$ and $(s_2, t_2) \notin E(Q_3)$. It is straightforward to check that for any s_2-t_2 path P of length 3, $Q_3 \setminus V(P)$ is isomorphic to a path of length 3. Since $(s_1, t_1) \in E(Q_3)$, no 2-eq-DPC can be constructed. The proof is completed. □

Lemma 6. *Suppose that S and T are subsets of different parts of the bipartition of Q_3.*
(a) If $(s_1, t_1) \in E(Q_3)$, there exists a paired 2-DPC joining S and T in which the s_1-t_1 path is of length 1 (and the s_2-t_2 path is of length 5) unless $\{s_2, t_2\} = V(Q_3) \setminus (N(s_1) \cup N(t_1))$.
(b) Let $(s_1, t_1), (s_2, t_2) \in E(Q_3)$. If $(t_1, s_2) \in E(Q_3)$ or $(t_2, s_1) \in E(Q_3)$, there exist three kinds of paired 2-DPC's joining S and T such that the s_1-t_1 path is of length 1, 3, and 5, respectively.

Proof. Let G' be the subgraph of Q_3 induced by $V(Q_3) \setminus \{s_1, t_1\}$. If $(s_1, t_1) \in E(Q_3)$, G' is isomorphic to the product of a path of length 1 and a path of length 2. It is straightforward to check that for any balanced pair of vertices s_2 and t_2, G' has an s_2-t_2 Hamiltonian path if and only if $\{s_2, t_2\} \neq V(Q_3) \setminus (N(s_1) \cup N(t_1))$. Thus, (a) is proved. The proof of (b) is a direct consequence of Lemmas 5 and 6(a). $\square$

Theorem 2. *Every m-dimensional bipartite HL-graph X_m, $m \geq 4$, has a paired 2-eq-DPC joining S and T if S and T are subsets of different parts of the bipartition.*

Proof. The proof is by an induction on m. Let $X_m = G_0 \oplus G_1$, where each G_i, $i = 0, 1$, is isomorphic to an X_{m-1}. The proofs for the basis, i.e., when $m = 4$, and the inductive step, i.e., when $m \geq 5$, are combined into one to avoid repetition. For the base case of $m = 4$, we will use Lemmas 5 and 6 to find 2-DPC's in G_0 and G_1 that can be merged to obtain a desired 2-eq-DPC. For $m \geq 5$, we will assume that each G_i, $i = 0, 1$, has a 2-eq-DPC joining balanced source-sink pairs. It is assumed w.l.o.g. that the number of terminals in G_0 is at least that in G_1. There are five cases depending on the positions of terminals.

Case 1: $s_1, t_1 \in V(G_0)$ and $s_2, t_2 \in V(G_1)$.
It suffices to construct an s_1-t_1 Hamiltonian path in G_0 and an s_2-t_2 Hamiltonian path in G_1.

Case 2: $s_1, s_2 \in V(G_0)$ and $t_1, t_2 \in V(G_1)$.
If $m \geq 5$, we pick up arbitrary pseudo-terminals t_1^* and t_2^* and then find a 2-eq-DPC$[(s_1, t_1^*), (s_2, t_2^*)|G_0, \emptyset]$ and a 2-eq-DPC$[(\bar{t}_1^*, t_1), (\bar{t}_2^*, t_2)|G_1, \emptyset]$. By merging them, we obtain a 2-eq-DPC in X_m. Let $m = 4$. Note that picking up arbitrary pseudo-terminals t_1^* and t_2^* is not sufficient since each G_i may not have a 2-eq-DPC even if all source-sink pairs are balanced. We pick up pseudo-terminals t_1^* and t_2^* from $N(s_1) \cap N(s_2)$. By Lemma 1, there exists a 2-DPC$[(\bar{t}_1^*, t_1), (\bar{t}_2^*, t_2)|G_1, \emptyset]$. Let l be the length of the $\bar{t}_1^*$-t_1 path in the DPC. Then, $l \in \{1, 3, 5\}$. For any l, by Lemma 6, there exists a 2-DPC$[(s_1, t_1^*), (s_2, t_2^*)|G_0, \emptyset]$ in which the s_1-t_1^* path is of length $6 - l$. To obtain a 2-eq-DPC in X_m, it suffices to merge the two DPC's.

Case 3: $s_1, t_2 \in V(G_0)$ and $t_1, s_2 \in V(G_1)$.
In a similar manner to Case 2, a 2-eq-DPC joining S and T will be constructed. If $m \geq 5$, it suffices to pick up arbitrary pseudo-terminals t_1^* and s_2^*, and then merge a 2-eq-DPC$[(s_1, t_1^*), (s_2^*, t_2)|G_0, \emptyset]$ and a 2-eq-DPC$[(\bar{t}_1^*, t_1), (s_2, \bar{s}_2^*)|G_1, \emptyset]$. Let $m = 4$. We select pseudo-terminals t_1^* and s_2^* so that there exist three kinds of 2-DPC$[(s_1, t_1^*), (s_2^*, t_2)|G_0, \emptyset]$ of Lemma 6(b). We first pick up t_1^* in $N(s_1)$. If $(s_1, t_2) \in E(G_0)$, then pick up s_2^* in $N(t_2)$; otherwise, pick up s_2^* in $N(t_2) \cap N(t_1^*)$. Finally, a 2-DPC$[(\bar{t}_1^*, t_1), (s_2, \bar{s}_2^*)|G_1, \emptyset]$ and one of the three 2-DPC's in G_0 are merged to obtain a 2-eq-DPC.

Case 4: $s_1, t_1, s_2 \in V(G_0)$ and $t_2 \in V(G_1)$.
If $m \geq 5$, we find a 2-eq-DPC$[(s_1,t_1),(s_2,t_2^*)|G_0,\emptyset]$ for an arbitrary pseudo-terminal t_2^*. For some edge (x,y) on the s_1-t_1 path such that $\bar{x},\bar{y} \neq t_2$, we find a 2-eq-DPC$[(\bar{x},\bar{y}),(\bar{t_2^*},t_2)|G_1,\emptyset]$. It suffices to merge the two DPC's. Let $m=4$. We pick up a pseudo-terminal t_2^* in $N(s_1)\cap N(s_2)$. Thus, there exists a 2-DPC$[(s_1,t_1),(s_2,t_2^*)|G_0,\emptyset]$ in which the s_2-t_2^* path is of length 1 by Lemma 6(a). If there exists an edge (x,y) on the s_1-t_1 path such that $\bar{x}$ and $\bar{y}$ are free, $(\bar{x},\bar{y}) \in E(G_1)$, and $\{\bar{t_2^*},t_2\} \neq V(G_1)\setminus(N(\bar{x})\cup N(\bar{y}))$, then there exists a 2-DPC$[(\bar{x},\bar{y}),(\bar{t_2^*},t_2)|G_1,\emptyset]$ in which the $\bar{t_2^*}$-t_2 path is of length 5 by Lemma 6(a). A 2-eq-DPC can be obtained by merging the two DPC's.

We must still show that such an edge (x,y) exists. The s_1-t_1 path has a path segment (u,v,w) such that $\bar{u},\bar{v},\bar{w} \neq t_2$. It suffices to prove a **claim** that for any path $P=(u,v,w)$ in G_0 and any $\alpha,\beta \in V(G_1)$ such that $c(\alpha)\neq c(\beta)$ and $\{\alpha,\beta\}\cap\{\bar{u},\bar{v},\bar{w}\}=\emptyset$, there exists an edge (x,y) on P such that (i) $(\bar{x},\bar{y})\in E(G_1)$ and (ii) $\{\alpha,\beta\}\neq V(G_1)\setminus(N(\bar{x})\cup N(\bar{y}))$. Recall $m=4$. Note that for each vertex p in Q_3, there is a unique vertex q such that $c(q)\neq c(p)$ and $(q,p)\notin E(Q_3)$. Since $c(\bar{v})\neq c(\bar{u})=c(\bar{w})$, $(\bar{u},\bar{v})$ or $(\bar{v},\bar{w})$ is an edge of G_1. If both $(\bar{u},\bar{v})$ and $(\bar{v},\bar{w})$ are edges of G_1, then at least one of (u,v) and (v,w) certainly satisfies (ii). Suppose that $(\bar{v},\bar{w})$ is not an edge of G_1 and (u,v) does not satisfy (ii). Since any vertex z of G_1 such that $z\neq\bar{w}$ and $c(z)=c(\bar{w})$ is adjacent to $\bar{v}$, α or β is adjacent to $\bar{v}$, which is a contradiction. Therefore, (u,v) satisfies (ii). Hence, the claim is true.

Case 5: $s_1,t_1,s_2,t_2 \in V(G_0)$.
If $m\geq 5$ or both $m=4$ and either $(s_1,t_1),(s_2,t_2)\in E(G_0)$ or $(s_1,t_1),(s_2,t_2)\notin E(G_0)$, there exists a 2-eq-DPC$[(s_1,t_1),(s_2,t_2)|G_0,\emptyset]$ by induction hypothesis and Lemma 5. We claim that there exist two edges, (u,v) on the s_1-t_1 path and (x,y) on the s_2-t_2 path such that a 2-eq-DPC$[(\bar{u},\bar{v}),(\bar{x},\bar{y})|G_1,\emptyset]$ exists. For $m\geq 5$, it is sufficient to pick up an arbitrary edge on each path. For $m=4$, due to the claim of Case 4, there exist (u,v) on the s_1-t_1 path and (x,y) on the s_2-t_2 path such that $(\bar{u},\bar{v}),(\bar{x},\bar{y})\in E(G_1)$. Then, by Lemma 5, the 2-eq-DPC in G_1 exists. It remains to merge the two 2-eq-DPC's.

Finally, let $m=4$ and $(s_1,t_1)\in E(G_0)$ and $(s_2,t_2)\notin E(G_0)$. By Lemma 6(a), there exists a 2-DPC$[(s_1,t_1),(s_2,t_2)|G_0,\emptyset]$ in which the s_1-t_1 path is of length 1 and the s_2-t_2 path is of length 5. By the claim of Case 4, there exists an edge (x,y) on the s_2-t_2 path such that $(\bar{x},\bar{y})\in E(G_1)$ and $\{\bar{s_1},\bar{t_1}\}\neq V(G_1)\setminus(N(\bar{x})\cup N(\bar{y}))$. Again by Lemma 6(a), there exists a 2-DPC$[(\bar{s_1},\bar{t_1}),(\bar{x},\bar{y})|G_1,\emptyset]$ in which the $\bar{x}$-$\bar{y}$ path is of length 1 and the $\bar{s_1}$-$\bar{t_1}$ path is of length 5. It suffices to merge the two DPC's. This completes the proof of Theorem 2. $\square$

4. Strongly Hamiltonian-Laceability of Bipartite HL-Graphs

In this section, it will be shown that if an m-dimensional bipartite-HL graph X_m, $m \geq 3$, has either at most $m-2$ faulty edges or one faulty vertex and at most $m-3$ faulty edges, the graph with the faulty elements removed is strongly Hamiltonian-laceable. Paired 2-disjoint path coverability of X_m studied in the previous section will play an important role in the construction of an L^{OPT}-path. We begin by considering strongly Hamiltonian-laceability of Q_3 having a unique vertex fault.

Lemma 7. *For a faulty vertex v_f in Q_3, $Q_3 \setminus v_f$ is strongly Hamiltonian-laceable.*

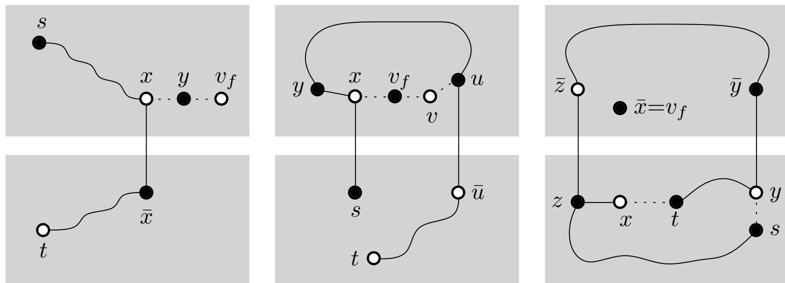
Proof. For a pair of fault-free vertices s and t with $c(s) = c(t) \neq c(v_f)$, an L^{OPT}-path joining them exists by Lemma 3. Now, at least one of s and t has the same color as v_f. There exist three induced subgraphs of $Q_3 \setminus v_f$ that are isomorphic to the product of a path of length 1 and a path of length 2. Among the three, at least two contain both s and t. It is immediately seen that at least one has an L^{OPT}-path joining s and t. The L^{OPT}-path is indeed an L^{OPT}-path of $Q_3 \setminus v_f$. Therefore, the lemma is proved. □

Theorem 3. *Let X_m be an m-dimensional bipartite HL-graph. For $m \geq 3$, (a) X_m is $(m-2)$-edge-fault strongly Hamiltonian-laceable, and (b) X_m is 1-vertex-fault and $(m-3)$-edge-fault strongly Hamiltonian-laceable.*

Proof. The proof is by induction on m. Let $X_m = G_0 \oplus G_1$, where each G_i, $i = 0, 1$, is isomorphic to an $(m-1)$-dimensional bipartite HL-graph. For any fault-free vertices s and t, an L^{OPT}-path in X_m joining them will be constructed. Since an L^{OPT}-path in X_m with a virtual fault set $F \cup F'$, where F' is a set of arbitrary $m-2-|F|$ fault-free edges, is also an L^{OPT}-path in X_m with the fault set F, we can assume $|F| = m-2$.

To prove (a), it is sufficient to consider the case when $c(s) = c(t)$ due to Lemma 2. Let (x, y) be an arbitrary faulty edge such that $c(x) \neq c(s) = c(t)$. Observe that an L^{OPT}-path joining s and t in $X_m \setminus F'$, where $F' = F \cup \{x\} \setminus (x, y)$, is also an L^{OPT}-path joining s and t in $X_m \setminus F$. This implies that it suffices to prove that X_m with one faulty vertex and $m-3$ faulty edges is strongly Hamiltonian-laceable, which corresponds to the (b) of Theorem 3.

Let us begin by proving (b). The base case when $m = 3$ is shown in Lemma 7. Let $m \geq 4$. It is assumed w.l.o.g. that the unique faulty vertex v_f is contained in G_0, so $|F_1| \leq m-3$. There are three cases depending on the positions of s and t. To construct an L^{OPT}-path joining s and t, in



(a) Case 2 (b) Case 3, $|F_0| = m-2$ (c) Case 3, $|F_1| = m-3$

Figure 3: Illustration of the proof of the Theorem 3.

almost all the cases, we merge an L^{OPT}-path of G_0 and either a Hamiltonian path or a paired 2-DPC of G_1. It is straightforward to see that the resultant path is always an L^{OPT}-path since a Hamiltonian path or a paired 2-DPC of G_1 covers all the vertices of G_1. Note that merging two L^{OPT}-paths, one in each G_i, may not result in an L^{OPT}-path of X_m.

Case 1: $s, t \in V(G_0)$.

We let $F'_0 = F_0$ if $|F_0| \leq m-3$; otherwise, let $F'_0 = F_0 \setminus e$ for an arbitrary faulty edge e in G_0. Then, $|F'_0| \leq m-3$. There exists an L^{OPT}-path P_0 in $G_0 \setminus F'_0$ joining s and t by induction hypothesis. If P_0 passes through the faulty edge e, let (u, v) be the faulty edge; otherwise, let (u, v) be an edge on P_0 such that $(u, \bar{u})$ and $(v, \bar{v})$ are fault-free. Such an edge (u, v) exists since $\lfloor (2^{m-1} - 1)/2 \rfloor > m-3$ for every $m \geq 4$. To construct a desired L^{OPT}-path of X_m, it suffices to replace (u, v) with a path (u, P_1, v), where P_1 is an $\bar{u}$-$\bar{v}$ Hamiltonian path in G_1.

Case 2: $s \in V(G_0)$ and $t \in V(G_1)$.

If $|F_0| \leq m-3$, we first pick up a pseudo-terminal t^*. There exist an L^{OPT}-path joining s and t^* in G_0 and a $\bar{t^*}$-t Hamiltonian path in G_1. Merging these paths results in a desired L^{OPT}-path. Let $|F_0| = m-2$. (See Fig. 3a.) Regarding v_f as a *virtual* fault-free vertex, we find an L^{OPT}-path P_0 joining s and v_f in $G_0 \setminus (F_0 \setminus v_f)$. Let $P_0 = (s, P_x, x, y, v_f)$ for some path segment P_x. Notice that (s, P_x, x, y) and (s, P_x, x) are L^{OPT}-paths in $G_0 \setminus F_0$. If $c(t) = c(y)$, a desired L^{OPT}-path is obtained by merging (s, P_x, x, y) and a $\bar{y}$-t Hamiltonian path in G_1; otherwise it is obtained by merging (s, P_x, x) and an $\bar{x}$-t Hamiltonian path in G_1.

Case 3: $s, t \in V(G_1)$.

If $|F_0| \leq m-3$ and $|F_1| \leq m-4$, we pick up two pseudo-terminals s^* and t^*. To obtain a desired L^{OPT}-path, it suffices to merge an L^{OPT}-

path joining s^* and t^* in $G_0 \setminus F_0$ and a 2-DPC$[(s, \bar{s^*}), (\bar{t^*}, t)|G_1, F_1]$. The existence of the 2-DPC is due to Theorem 1 and Lemma 1. Now, let $|F_0| = m - 2$ and $|F_1| = |F_2| = 0$ for the second case. (See Fig. 3b.) Regarding v_f as a virtual free vertex, we can find a Hamiltonian cycle $C = (x, y, P_y, u, v, v_f)$. The Hamiltonian cycle exists since G_0 is $(m-3)$-edge-fault strongly Hamiltonian-laceable. If we remove v_f from C, we have a Hamiltonian path $P_0 = (x, y, P_y, u, v)$ of $G_0 \setminus F_0$. Note that $c(x) = c(v) \neq c(v_f)$ and $c(y) = c(u) = c(v_f)$. It follows that $P_{x,v} = (x, y, P_y, u, v)$, $P_{x,u} = (x, y, P_y, u)$, $P_{y,v} = (y, P_y, u, v)$, and $P_{y,u} = (y, P_y, u)$ are all L^{OPT}-paths in $G_0 \setminus F_0$. When $c(s) = c(t) \neq c(v_f)$, a desired L^{OPT}-path is obtained by merging $P_{x,v}$ and a 2-DPC$[(s, \bar{x}), (\bar{v}, t)|G_1, \emptyset]$. When $c(s) = c(t) = c(v_f)$, a desired L^{OPT}-path is obtained from $P_{y,u}$ and a 2-DPC$[(s, \bar{y}), (\bar{u}, t)|G_1, \emptyset]$. Let $c(s) \neq c(t)$. We assume w.l.o.g. $c(s) = c(v_f)$. At least one of $\{\bar{x}, \bar{u}\}$ and $\{\bar{y}, \bar{v}\}$ is different from $\{s, t\}$, say $\{\bar{x}, \bar{u}\} \neq \{s, t\}$. If $\{\bar{x}, \bar{u}\} \cap \{s, t\} = \emptyset$, a desired L^{OPT}-path is obtained from $P_{x,u}$ and a 2-DPC$[(s, \bar{u}), (\bar{x}, t)|G_1, \emptyset]$. If $\{\bar{x}, \bar{u}\} \cap \{s, t\} \neq \emptyset$, then we have either $\bar{x} = s$ or $\bar{u} = t$, say $\bar{x} = s$. Then, a desired L^{OPT}-path is obtained from $P_{x,u}$ and a Hamiltonian path joining $\bar{u}$ and t in $G_1 \setminus s$.

Finally, let $|F_1| = m - 3$, $F_0 = \{v_f\}$, and $F_2 = \emptyset$ for the last case. If $c(s) \neq c(t)$, we find a Hamiltonian path joining s and t in $G_1 \setminus F_1$, and then for some edge (u, v) on the Hamiltonian path such that $\bar{u}, \bar{v} \neq v_f$, the edge (u, v) is replaced with (u, P_0, v), where P_0 is an L^{OPT}-path joining $\bar{u}$ and $\bar{v}$ in $G_0 \setminus F_0$. The path obtained is a desired L^{OPT}-path. Now, assume that $c(s) = c(t)$. (See Fig. 3c.) There exists a Hamiltonian cycle C in $G_1 \setminus F_1$. Let $C = (s, P_x, x, t, P_y, y)$ for some path segments P_x and P_y, which are possibly paths of length 0. Then, we have two paths $P_s = (s, P_x, x)$ and $P_t = (t, P_y, y)$ that cover all the vertices in G_1. If $v_f \notin \{\bar{x}, \bar{y}\}$, a desired L^{OPT}-path is obtained from P_s, P_t, and an L^{OPT}-path in $G_0 \setminus F_0$ joining $\bar{x}$ and $\bar{y}$. Suppose $v_f \in \{\bar{x}, \bar{y}\}$. Then, we have $c(s) = c(t) = c(v_f)$. Note that in this subcase, any L^{OPT}-path leaves out two vertices having colors different from v_f. We assume $v_f = \bar{x}$, and let $P_x = (P_z, z)$ so $P_s = (s, P_z, z, x)$; or $P_s = (s = z, x)$ if P_x is of length 0. An L^{OPT}-path joining s and t that does not contain x is obtained from $P_s \setminus x$, P_t, and an L^{OPT}-path in $G_0 \setminus F_0$ joining $\bar{z}$ and $\bar{y}$. This completes the proof of Theorem 3. □

Theorem 3 is an extension of the work in [21, 26], which is stated in Lemma 2. The total number $m - 2$ of faulty elements in Theorem 3 is the maximum possible regardless of whether there exists a faulty vertex or not. This can be verified as follows. Suppose that s and t are two vertices of X_m that are adjacent via a fault-free edge. If there are $m - 1$ faulty edges incident to s or $m-2$ faulty edges incident to s and one faulty vertex v_f adjacent to s, then there exists no L^{OPT}-path joining s and t.

5. Conclusions

In this paper, we investigated 2-disjoint path cover problems as an early stage in bipartite HL-graph research. It was proved that every X_m, $m \geq 4$, with $m-3$ or less faulty edges has a paired 2-DPC joining S and T provided $S \cup T$ is balanced. We also showed that every X_m, $m \geq 4$, has a paired 2-eq-DPC joining S and T if S and T are subsets of different parts of the bipartition. Using 2-DPC properties, it was proved that every X_m, $m \geq 3$, is $(m-2)$-edge-fault strongly Hamiltonian-laceable and is 1-vertex-fault and $(m-3)$-edge-fault strongly Hamiltonian-laceable. Both the number $m-3$ of allowed faulty edges for the paired 2-DPC and the number $m-2$ of faulty elements for the strongly Hamiltonian-laceability are the maximum possible.

As a result, a balanced pair of vertices s and t of X_m, $m \geq 4$, with $m-3$ or less faulty edges are joined by a Hamiltonian path passing through an arbitrary prescribed edge (x, y) such that $\{x, y\} \neq \{s, t\}$ (by Theorem 1 when $|\{x, y\} \cap \{s, t\}| = 0$ and by Theorem 3(b) when $|\{x, y\} \cap \{s, t\}| = 1$). We conjecture that for some constant $c \geq 4$, every X_m, $m \geq c$, possessing f or less faulty elements (vertices and/or edges) has a paired k-DPC for any f and $k \geq 2$ with $f + 2k \leq m + 1$ under the 'balancedness' condition of $k^b + 2f_v^b = k^w + 2f_v^w$, where k^b and k^w respectively are the numbers of black and white terminals and f_v^b and f_v^w respectively are the numbers of black and white vertex faults. It will be a challenging problem to verify this conjecture.

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