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Paired many-to-many 3-disjoint path covers in bipartite toroidal grids

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Abstract

Given two disjoint vertex-sets, $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ in a graph, a *paired many-to-many k-disjoint path cover* joining S and T is a set of pairwise vertex-disjoint paths $\{P_1, \ldots, P_k\}$ that altogether cover every vertex of the graph, in which each path P_i runs from s_i to t_i. In this paper, we first study the disjoint-path-cover properties of a bipartite cylindrical grid. Based on the findings, we prove that every bipartite toroidal grid, excluding the smallest one, has a paired many-to-many 3-disjoint path cover joining $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ if and only if the set $S \cup T$ contains the equal numbers of vertices from different parts of the bipartition.

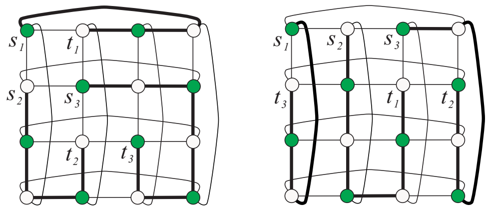
Category: Algorithms and Complexity

Keywords: Disjoint path; path cover; path partition; cylindrical grid; torus

I. INTRODUCTION

Let G be a finite, simple undirected graph whose vertex and edge sets are denoted by $V(G)$ and $E(G)$, respectively. A *path* from $v \in V(G)$ to $w \in V(G)$, referred to as a v–w path, is a sequence $\langle u_1, \ldots, u_l \rangle$ of distinct vertices of G such that $u_1 = v$, $u_l = w$, and $(u_i, u_{i+1}) \in E(G)$ for all $i \in \{1, \ldots, l-1\}$. If $l \geq 3$ and $(u_l, u_1) \in E(G)$, the sequence is called a *cycle*. A path that visits each vertex exactly once is a *Hamiltonian path*; a cycle that visits each vertex exactly once is a *Hamiltonian cycle*. A *path cover* of a graph G is a set of paths in G such that every vertex of G is contained in at least one path. A *disjoint path cover* (DPC for short) of G is a set of disjoint paths that altogether cover every vertex of G. This paper is concerned with a DPC in which each path runs from a prescribed source to a prescribed sink.

Given disjoint subsets $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ of $V(G)$ for a positive integer k, a *many-to-many k-disjoint path cover* is a DPC composed of k paths that collectively join S and T; if each source $s_i \in S$ must be joined to a specific sink $t_i \in T$, the DPC is called *paired*, and it is *unpaired* if no such constraint is imposed. Refer to Fig. 1 for examples.



(a) Paired 3-DPC (b) Unpaired 3-DPC

Fig. 1: Examples of many-to-many disjoint path covers.

There are two other DPC types: A *one-to-many k-disjoint path cover* for $S = \{s\}$ and $T = \{t_1, \ldots, t_k\}$ is a DPC made of k paths, each of which joins a pair of source s and sink t_i, $i \in \{1, \ldots, k\}$; when $S = \{s\}$ and $T = \{t\}$, a DPC composed of k paths, each of which joins s and t, is named a *one-to-one k-disjoint path cover*. As intuitively clear, we will call the vertices in S and in T *sources* and *sinks*, respectively, which together form a set of *terminals*.

The existence of a disjoint path cover in a graph is closely related to the Hamiltonian properties, as well as the concept of vertex connectivity which was characterized in terms of the minimum number of

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disjoint paths. For instance, a Hamiltonian cycle forms a one-to-one 2-DPC joining $\{s\}$ and $\{t\}$ for every pair of distinct vertices s and t. The disjoint path cover problems are applicable in many areas such as software testing, database design, and code optimization [1, 2]. In addition, the problem is concerned with applications where full utilization of network nodes is important [3]. The problems have been studied for various classes of graphs, such as interval graphs [4, 5], hypercubes [6, 7, 8], torus networks [9, 10, 11, 12], dense graphs [13], and cubes of connected graphs [14, 15].

In the context of the Hamiltonian path problem, the rectangular grid first appeared in the literature in [16]. In the formal definition of the $m\times n$ rectangular grid, the vertices are often chosen from the points of the Euclidean plane with integer coordinates so that the vertices and edges form a rectangular grid with n vertices appearing in each of m rows and m vertices in each of n columns.

Definition 1 (Rectangular grid). *The* $m \times n$ rectangular grid G *is a graph such that* $V(G) = \{v^i_j : 0 \le i \le m-1, 0 \le j \le n-1\}$ *and* $E(G) = \{(v^i_j, v^{i'}_{j'}) : |i-i'| + |j-j'| = 1\}$.

Besides the rectangular grid graph, there are two related classes of grid graphs: The $m \times n$ cylindrical grid is constructed from the $m\times n$ rectangular grid by adding horizontal wrap-around edges (v^i_{n-1}, v^i_0) for $i \in \{0, \dots, m-1\}$; the toroidal grid can be generated from the $m \times n$ cylindrical grid by adding vertical wrap-around edges (v^{m-1}_j, v^0_j) for $j \in \{0, \dots, n-1\}$.

Definition 2 (Cylindrical grid). *The* $m \times n$ cylindrical grid G *is a graph such that* $V(G) = \{v^i_j : 0 \le i \le m-1, 0 \le j \le n-1\}$ *and* $E(G) = \{(v^i_j, v^{i'}_{j'}) : (j = j'\ \&\ |i-i'| = 1)$ *or* $(i = i'\ \&\ j' \equiv j+1 \pmod{n})\}$*, where* $n \ge 3$.

Definition 3 (Toroidal grid). *The* $m\times n$ toroidal grid G *is a graph such that* $V(G) = \{v^i_j : 0 \le i \le m-1, 0 \le j \le n-1\}$ *and* $E(G) = \{(v^i_j, v^{i'}_{j'}) : (j = j'\ \&\ i' \equiv i+1 \pmod{m})$ *or* $(i = i'\ \&\ j' \equiv j+1 \pmod{n})\}$*, where* $m, n \ge 3$.

The rectangular grid is a bipartite graph and thus its vertices may be colored in two colors, green and white, in such a way that every pair of adjacent vertices is colored differently (hereafter, we will denote the color of vertex v by $c(v)$). In contrast, the $m\times n$ cylindrical grid is bipartite if and only if n is even; the $m\times n$ toroidal grid is bipartite if and only if both m and n are even. The bipartite cylindrical and toroidal grids each is *balanced* in a sense that its two color classes have equal cardinality. We will also call a subset of $V(G)$ *balanced* if the number of vertices in the subset that belong to each of the two color classes is equal.

The existence of a paired (many-to-many) 2-DPC in a bipartite toroidal grid was studied, as shown below:

Theorem 1 (Makino [17]). *An* $m\times n$ *toroidal grid with* $m, n \ge 4$*, both even, has a paired 2-DPC for a pair of terminal sets* S *and* T *if and only if their union is balanced.*

Theorem 2 (Park and Ihm [18]). *For an* $m \times n$ *toroidal grid* G *with* $m, n \ge 4$*, both even, and an arbitrary edge* e_f *of* G*, the subgraph,* $G - e_f$*, of* G *with* e_f *being deleted has a paired 2-DPC joining* S *and* T *if and only if* $S \cup T$ *is balanced.*

Theorem 3 (Kim and Park [19]). *For an* $m \times n$ *toroidal grid* G *with* $m, n \ge 4$*, both even, and an arbitrary vertex* v_f *of* G*, the subgraph,* $G - v_f$*, of* G *with* v_f *being deleted has a paired 2-DPC joining* S *and* T *if and only if one of the four terminals in* $S\cup T$ *has the same color as* v_f *and the other three have a different color from* v_f*.*

In this paper, we prove that an $m \times n$ bipartite toroidal grid with $(m, n) \ne (4, 4)$ has a paired 3-DPC joining $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ if and only if $S \cup T$ is balanced. The proof is based on some disjoint-path-cover properties of a bipartite cylindrical grid (investigated in Section III), as well as the necessary and sufficient condition for a bipartite cylindrical grid to have a paired 2-DPC joining S and T (established in [18]).

II. NOTATION AND PREVIOUS WORKS

For an $m \times n$ grid graph, whether rectangular, cylindrical, or toroidal, R_i denotes the vertex set $\{v^i_j : 0 \le j \le n-1\}$ of row i, whereas C_j denotes the vertex set $\{v^i_j : 0 \le i \le m-1\}$ of column j, implying v^i_j is the vertex both in row i and in column j. Based on these notations, we indicate multiple rows and columns respectively as $R_{i,i'} = \bigcup_{i\le r\le i'} R_r$ if $i \le i'$; $R_{i,i'} = \emptyset$ otherwise, and $C_{j,j'} = \bigcup_{j\le r\le j'} C_r$ if $j \le j'$; $C_{j,j'} = \emptyset$ otherwise. All arithmetic on the indices of vertices of the cylindrical and toroidal grids is done modulo n or m as needed.

Hamiltonian properties of the rectangular and cylindrical grids have been revealed in previous studies, some of which will be effectively used for deriving our results. A bipartite graph that is balanced is called *Hamiltonian-laceable* if there is a Hamiltonian path between any two vertices from different color classes [20]. The concept of Hamiltonian-laceability has often been extended in such a way that a bipartite graph whose color classes may differ in cardinality by exactly one is also *Hamiltonian-laceable* if every pair of vertices from the same major color class can be joined by a Hamiltonian path. Finally, a bipartite graph G is called 1-*fault Hamiltonian-laceable* if G remains Hamiltonian-laceable even if a single vertex or edge is deleted from G.

Lemma 1 (Chen and Quimpo [21]). *Let G be an $m \times n$ rectangular grid with $m, n \geq 2$. (a) If mn is even, then G has a Hamiltonian path from a corner vertex, i.e., a vertex of degree two, to any other vertex in the different color class. (b) If mn is odd, then G has a Hamiltonian path from a corner vertex to any other vertex in the same color class.*

Lemma 2 (Tsai, Tan, Chuang, and Hsu [22]). *An $m \times n$ cylindrical grid with $m \geq 2$ and even $n \geq 4$ is 1-fault Hamiltonian-laceable.*

A necessary and sufficient condition was established by Park and Ihm [18] for an $m \times n$ bipartite cylindrical grid to have a paired 2-DPC joining disjoint terminal sets $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$; furthermore, *inadmissible configurations* of the four terminals which would not permit a paired 2-DPC in the cylindrical grid were classified in four cases: (i) $m \geq 4$ & even $n \geq 6$, (ii) $n = 4$, (iii) $m = 2$ & even $n \geq 6$, and (iv) $m = 3$ & even $n \geq 6$, as shown in Lemmas 3 through 6.

Lemma 3. *For $m \geq 4$ and even $n \geq 6$, an $m \times n$ cylindrical grid G has a paired 2-DPC joining $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$ if and only if $S \cup T$ is balanced, and the four terminals in $S \cup T$ do not form an inadmissible configuration equivalent to A0, B0, or C0:*

A0: *$s_1 = v_i^0$, $s_2 = v_p^0$, $t_1 = v_j^0$, and $t_2 = v_q^0$ for some i, j, p, and q such that $i < p < j < q$;*
B0: *$s_1 = v_i^r$, $t_1 = v_{i+1}^{r+1}$, $s_2 = v_{i+1}^r$, and $t_2 = v_i^{r+1}$ for some i and r;*
C0: *$s_1 = v_i^0$, $t_1 = v_{i+1}^1$, $t_2 = v_{i+2}^1$, and $s_2 = v_{i+3}^0$ for some i.*

Lemma 4. *For $m \geq 2$, an $m \times 4$ cylindrical grid G has a paired 2-DPC joining $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$ if and only if $S \cup T$ is balanced, and the four terminals in $S \cup T$ do not form an inadmissible configuration equivalent to A1, B0, or C1:*

A1: *$s_1, t_1 \in R_{r_1}$, $s_2, t_2 \in R_{r_2}$, and $c(s_1) = c(t_1) \neq c(s_2) = c(t_2)$ for some r_1 and r_2;*
C1: *$s_1 = v_i^r$, $t_1 = v_{i+1}^{r+1}$, $t_2 = v_{i+2}^{r+1}$, and $s_2 = v_{i+3}^r$ for some i and r.*

Lemma 5. *For even $n \geq 6$, a $2 \times n$ cylindrical grid G has a paired 2-DPC joining $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$ if and only if $S \cup T$ is balanced, and the four terminals in $S \cup T$ do not form an inadmissible configuration equivalent to A0, B2, C2, or D2:*

B2: *$S \cup T = \{v_i^0, v_i^1, v_j^0, v_j^1\}$ and $c(s_1) = c(t_1) \neq c(s_2) = c(t_2)$ for some i and j with $i \neq j$;*
C2: *$s_1 = v_i^0$, $t_1 = v_j^1$, $s_2 = v_p^0$, $t_2 = v_q^1$, and $c(s_1) = c(t_1) \neq c(s_2) = c(t_2)$ for some i, j, p, and q such that $\max\{i, j\} < \min\{p, q\}$;*
D2: *$s_1 = v_i^0$, $s_2 = v_p^0$, $t_1 = v_j^1$, $t_2 = v_q^1$, and $c(s_1) = c(s_2) \neq c(t_1) = c(t_2)$ for some i, j, p, and q such that $i < p < j < q$.*

Lemma 6. *For even $n \geq 6$, a $3 \times n$ cylindrical grid G has a paired 2-DPC joining $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$ if and only if $S \cup T$ is balanced, and the four terminals in $S \cup T$ do not form an inadmissible configuration equivalent to A0, B0, C3, D3, E3, or F3:*

C3: *$s_1 = v_i^0$, $t_1 = v_j^1$, $t_2 = v_q^1$, $s_2 = v_p^0$, and $c(s_1) = c(t_1) \neq c(s_2) = c(t_2)$ for some i, j, p, and q such that $i < j < q < p$, $q = j + 1$, and $(n - 1 - p) + i \geq 2$;*
D3: *$s_1 = v_i^1$, $s_2 = v_p^1$, $t_1 = v_j^1$, $t_2 = v_q^1$, and $c(s_1) = c(t_2) \neq c(t_1) = c(s_2)$ for some i, j, p, and q such that $i < p < j < q$, $p = i + 1$, and $q = j + 1$;*
E3: *$s_1 = v_i^0$, $s_2 = v_p^0$, $t_2 = v_q^2$, $t_1 = v_j^2$, and $c(s_1) = c(s_2) \neq c(t_1) = c(t_2)$ for some i, j, p, and q such that $i < p < q < j$, $q - p - 1 \geq 2$, and $(n - 1 - j) + i \geq 2$;*
F3: *$s_1 = v_i^0$, $t_2 = v_q^2$, $s_2 = v_p^0$, $t_1 = v_j^2$, and $c(s_1) = c(t_2) \neq c(s_2) = c(t_1)$ for some i, j, p, and q such that $q' < j'$, $j' - q' - 1 \geq 2$, and $(n - 1 - p') + i' \geq 2$, where $i' = \min\{i, q\}$, $q' = \max\{i, q\}$, $j' = \min\{j, p\}$, and $p' = \max\{j, p\}$.*

Remark 1. The four terminals in $S \cup T$ form an inadmissible configuration in a bipartite cylindrical grid only if each row contains an even number of terminals.

III. DISJOINT PATH COVERS IN BIPARTITE CYLINDRICAL GRIDS

Suppose that disjoint source and sink sets $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ are given in an $m \times n$ bipartite toroidal grid. If we divide the toroidal grid into two cylindrical grids, $m_1 \times n$ and $m_2 \times n$ cylindrical grids for some $m_1, m_2 \geq 2$ with $m_1 + m_2 = m$, then each cylindrical grid may have an "incomplete" terminal set in a sense that s_i is contained in its terminal set but t_i is not for some $i \in \{1, 2, 3\}$, and vice versa. In this section, we derive some useful properties of a disjoint path cover in a bipartite cylindrical grid with an incomplete terminal set, where the notion of a disjoint path cover is "generalized" in a way that a one-vertex path is allowed. (Note that a disjoint path cover joining disjoint terminal sets S and T contains no one-vertex path.) A *boundary* row in an $m \times n$ cylindrical grid hereafter refers to the row 0 or row $m - 1$.

Theorem 4. *Let G be an $m \times n$ cylindrical grid with $m \geq 2$ and even $n \geq 4$, in which three distinct terminals $s_1, s_2 \in S$ and $t_1 \in T$ are given such that not all the three are of the same color. Then, there exist two disjoint paths, s_1–t_1 and s_2–x paths, possibly $x = s_2$, that altogether cover all the vertices of G*

- *for every vertex x in one boundary row and for at least one vertex x in the other boundary row*

such that $\{s_1, t_1, s_2, x\}$ is balanced, or
- *for every vertex x except one in one boundary row and for at least two vertices x in the other boundary row such that $\{s_1, t_1, s_2, x\}$ is balanced.*

Proof. Suppose we are given three distinct terminals s_1, t_1, and s_2 in G such that the three are not of the same color. Then, there is a terminal with a different color from the other two, so $\{s_1, t_1, s_2, x\}$ is balanced if and only if x has the same color as the terminal. In addition, inspection of the inadmissible configurations in each of the four cases, where (i) $m \geq 4$ & even $n \geq 6$, (ii) $n = 4$, (iii) $m = 2$ & even $n \geq 6$, and (iv) $m = 3$ & even $n \geq 6$, can reveal that there exists an inadmissible configuration Z such that for every vertex $x \in V(G) \setminus \{s_1, t_1, s_2\}$, the four terminals in $\{s_1, t_1, s_2, x\}$ do not form an inadmissible configuration or form an inadmissible configuration equivalent to Z only, i.e., the four terminals do not form an inadmissible configuration not equivalent to Z.

Firstly, suppose $m \geq 4$ & even $n \geq 6$. From Lemma 3, there exists a paired 2-DPC, made of s_1–t_1 and s_2–x paths, in G for every vertex $x \in (R_0 \cup R_{m-1}) \setminus \{s_1, t_1, s_2\}$ such that $\{s_1, t_1, s_2, x\}$ is balanced and the four terminals in $\{s_1, t_1, s_2, x\}$ do not form an inadmissible configuration equivalent to A0, B0, or C0. Also, if $c(s_1) = c(t_1)$ and $s_2 \in R_0 \cup R_{m-1}$, then there exist two disjoint s_1–t_1 and s_2–x paths that cover all the vertices of G for $x = s_2$, because G is 1-fault Hamiltonian-laceable by Lemma 2. An inspection of the three inadmissible configurations each leads to that two disjoint s_1–t_1 and s_2–x paths exist, provided $\{s_1, t_1, s_2, x\}$ is balanced, for every vertex x in one boundary row and at least one vertex x in the other boundary row, as required. Analogously, we can prove the theorem in each of the remaining three cases from Lemmas 4 through 6, and Lemma 2. Note that if the inadmissible configuration Z is not equal to F3 (where $m = 3$ & even $n \geq 6$), there exist required disjoint paths, s_1–t_1 and s_2–x paths, for every vertex x in one boundary row and at least one vertex x in the other boundary row such that $\{s_1, t_1, s_2, x\}$ is balanced; otherwise, the required disjoint paths exist for every vertex x except one in one boundary row and at least two vertices x in the other boundary row such that $\{s_1, t_1, s_2, x\}$ is balanced. This completes the proof. □

REMARK 2. The number of such vertices x in Theorem 4 is at least $\frac{n}{2} + 1$.

THEOREM 5. *For distinct terminals $s_1, s_2, s_3 \in S$ and $t_1 \in T$ in an $m \times n$ cylindrical grid G with $m \geq 2$ and even $n \geq 4$ such that not all the four are of the same color, there exist vertices x and y in the boundary rows, possibly $x = s_2$ and/or $y = s_3$, such that G has three disjoint paths, s_1–t_1, s_2–x, and s_3–y paths, that altogether cover all the vertices of G.*

Proof. The proof will proceed by induction on m. Let $m = 2$ for the base step, where the two rows of G are both boundary ones. If $c(s_2) \neq c(s_3)$, then a Hamiltonian s_2–s_3 path exists in G since G is 1-fault Hamiltonian-laceable by Lemma 2. It suffices to divide the Hamiltonian path, represented as $\langle s_2, \ldots, x, s_1', \ldots, t_1', y \ldots, s_3 \rangle$, where $\{s_1', t_1'\} = \{s_1, t_1\}$, x is the predecessor of s_1', and y is the successor of t_1', into three subpaths: $\langle s_2, \ldots, x \rangle$, $\langle s_1', \ldots, t_1' \rangle$,$\langle y, \ldots, s_3 \rangle$. If $c(s_2) = c(s_3)$, then $c(s_1) \neq c(s_2)$ or $c(t_1) \neq c(s_2)$, so we assume w.l.o.g. $c(s_1) \neq c(s_2)$. Then, there exists a Hamiltonian s_2–s_3 path in $G - s_1$ by Lemma 2. For a neighbor v of s_1 other than s_2 and s_3, the Hamiltonian path can be represented as $\langle s_2, \ldots, x, v', \ldots, t_1', y \ldots, s_3 \rangle$, where $\{v', t_1'\} = \{v, t_1\}$. It suffices to divide the Hamiltonian path into three subpaths, $\langle s_2, \ldots, x \rangle$, $\langle v', \ldots, t_1' \rangle$, $\langle y \ldots, s_3 \rangle$, and combine the one-vertex path $\langle s_1 \rangle$ with the second subpath through the edge (s_1, v).

Let $m \geq 3$ for the inductive step. We assume w.l.o.g. that R_0 contains no fewer terminals than R_{m-1}, i.e., $|R_0 \cap (S \cup T)| \geq |R_{m-1} \cap (S \cup T)|$. There are several cases depending on the distribution of terminals.

Case 1: *There is a boundary row that contains no terminal, i.e., $R_{m-1} \cap (S \cup T) = \emptyset$.* By the induction hypothesis, there are two vertices $x, y \in R_0 \cup R_{m-2}$ that admit three disjoint s_1–t_1, s_2–x, and s_3–y paths that cover all the vertices of the subgraph $G[R_{0,m-2}]$ induced by $R_{0,m-2}$. If exactly one of x and y is contained in R_{m-2}, say $x \in R_0$ and $y \in R_{m-2}$, it suffices to extend the s_3–y path to cover the vertices of R_{m-1}, i.e., concatenate the s_3–y path and a Hamiltonian w–y' path of the subgraph $G[R_{m-1}]$ induced by R_{m-1} for the neighbor $w \in R_{m-1}$ of y and a neighbor $y' \in R_{m-1}$ of w. If $x, y \in R_{m-2}$, then it suffices to extend the s_2–x and s_3–y paths to cover the vertices of R_{m-1}. That is, for the neighbor $u \in R_{m-1}$ of x and the neighbor $w \in R_{m-1}$ of y, we extract two disjoint u–x' and w–y' paths from a Hamiltonian cycle of $G[R_{m-1}]$, and then concatenate the s_2–x and u–x' paths and concatenate again the s_3–y and w–y' paths.

Finally, suppose $x, y \notin R_{m-2}$, i.e., $x, y \in R_0$. If there is a nonterminal vertex v in R_{m-2}, i.e., $v \notin \{s_1, t_1, s_2, s_3\}$, then one of the three disjoint paths, s_1–t_1, s_2–x, and s_3–y paths, of $G[R_{0,m-2}]$ passes through v, hence passes through an edge (v, w) of $G[R_{m-2}]$. It suffices to reroute the path, instead of passing through the edge (v, w), to traverse a Hamiltonian v'–w' path of $G[R_{m-1}]$ for the neighbors $v', w' \in R_{m-1}$ of v and w, respectively. Now, let every vertex in R_{m-2} be a terminal, i.e., $R_{m-2} = \{s_1, t_1, s_2, s_3\}$ and $n = 4$. For the neighbors $s_1', t_1', s_2' \in R_{m-3}$, respectively, of s_1, t_1, and s_2, there are two disjoint s_1'–t_1' and s_2'–x paths for some $x \in R_0$ that cover $G[R_{0,m-3}]$. (The existence is by Theorem 4 if $m \geq 4$; the existence is obvious if

$m = 3$.) It suffices to concatenate the one-vertex path $\langle s_1 \rangle$, the s'_1–t'_1 path, and $\langle t_1 \rangle$ into an s_1–t_1 path, concatenate again the one-vertex path $\langle s_2 \rangle$ and the s'_2–x path, and extend $\langle s_3 \rangle$ to cover R_{m-1}.

Case 2: *There is a boundary row, say R_{m-1}, that contains a single terminal in $\{s_2, s_3\}$, say s_3, whose color is the same as at least one of the other terminals.* That is, $R_{m-1} \cap (S \cup T) = \{s_3\}$ and the three terminals $s_1, t_1, s_2 \in R_{0,m-2}$ are not of the same color. Then, for some $x \in R_0$, there exist disjoint s_1–t_1 and s_2–x paths that cover $G[R_{0,m-2}]$ by Theorem 4. It suffices to build a Hamiltonian s_3–y path of $G[R_{m-1}]$ for some y.

Case 3: $R_0 \cap (S \cup T) = \{s_1, s_2, s_3\}$. Assume w.l.o.g. that the three terminals in $\{s_1, t_1, s_2\}$ are not of the same color. It suffices to divide the Hamiltonian cycle $\langle s_1, \ldots, u, s_2, \ldots, x, s_3, \ldots, v \rangle$ of $G[R_0]$ into three paths $\langle s_1, \ldots, u \rangle$, $\langle s_2, \ldots, x \rangle$, and $\langle s_3, \ldots, v \rangle$, and then build two disjoint u'–t_1 and v'–y paths that cover $G[R_{1,m-1}]$ for some $y \in R_{m-1}$, where $u', v' \in R_1$ are the neighbors of u and v, respectively. Note that $c(u') = c(s_2)$ and $c(v') = c(s_1)$, meaning the three vertices of $\{u', v', t_1\}$ are not of the same color.

Case 4: $R_0 \cap (S \cup T) = \{s_1, t_1, s_2\}$. From the hypotheses of Cases 1 and 2, we can assume that $s_3 \in R_{m-1}$ and $c(s_1) = c(t_1) = c(s_2) \neq c(s_3)$. The proof is similar to that of Case 3. Dividing the Hamiltonian cycle $\langle s_1, \ldots, u, t_1, \ldots, v, s_2, \ldots, x \rangle$ of $G[R_0]$ into $\langle s_1, \ldots, u \rangle$, $\langle t_1, \ldots, v \rangle$, $\langle s_2, \ldots, x \rangle$ paths and building two disjoint u'–v' and s_3–y paths that cover $G[R_{1,m-1}]$ for some $y \in R_{m-1}$ leads to required three paths, where $u', v' \in R_1$ are the neighbors of u and v, respectively.

Case 5: $R_0 \cap (S \cup T) = \{s_2, s_3\}$. Similar to Case 3, assume w.l.o.g. that the three terminals in $\{s_1, t_1, s_2\}$ are not of the same color. It suffices to divide the Hamiltonian cycle of $G[R_0]$, represented as $\langle s_2, \ldots, x, s_3, \ldots, u \rangle$ with $(u, s_1), (u, t_1) \notin E(G)$, into two paths $\langle s_2, \ldots, x \rangle$ and $\langle s_3, \ldots, u \rangle$, and then build two disjoint s_1–t_1 and u'–y paths that cover $G[R_{1,m-1}]$ for some $y \in R_{m-1}$, where $u' \in R_1$ is the neighbor of u. (Note that R_1 contains at most one terminal from the hypothesis of Case 1.)

Case 6: $R_0 \cap (S \cup T) = \{s_1, s_2\}$. Unless $c(s_1) \neq c(t_1) = c(s_2) = c(s_3)$, it suffices to divide the Hamiltonian cycle $\langle s_1, \ldots, u, s_2, \ldots, x \rangle$ of $G[R_0]$, represented in a way that the neighbor $u' \in R_1$ of u is not a terminal, into s_1–u and s_2–x paths, and then build two disjoint u'–t_1 and s_3–y paths that cover $G[R_{1,m-1}]$ for some $y \in R_{m-1}$. Suppose $c(s_1) \neq c(t_1) = c(s_2) = c(s_3)$ now. If $R_{m-1} \cap (S \cup T) = \{t_1, s_3\}$, then we can also build the required three paths symmetrically, so we assume that R_{m-1} contains a single terminal. If $(s_1, s_2) \in E(G)$, it suffices to divide the Hamiltonian cycle of $G[R_0]$ into $\langle s_2, x \rangle$ and s_1–u paths for some $x, u \in R_0$, and then build two disjoint u'–t_1 and s_3–y paths that cover $G[R_{1,m-1}]$ for some $y \in R_{m-1}$, where $u' \in R_1$ is the neighbor of u. If $(s_1, s_2) \notin E(G)$, it suffices to divide the Hamiltonian cycle $\langle s_1, \ldots, u, x, s_2, y, \ldots, v \rangle$ of $G[R_0]$ into three paths $\langle s_1, \ldots, u \rangle$, $\langle x, s_2 \rangle$, and $\langle y, \ldots, v \rangle$, and then build a paired 2-DPC of $G[R_{1,m-1}]$, made of u'–t_1 and s_3–v' paths, where $u', v' \in R_1$ are the neighbors of u and v, respectively. The paired 2-DPC exists because R_{m-1} contains an odd number of terminals.

Case 7: $R_0 \cap (S \cup T) = \{s_1, t_1\}$. From the hypotheses of Cases 1, 2, and 5, we can assume that $R_{m-1} \cap (S \cup T) = \{s_3\}$ and $c(s_1) = c(t_1) = c(s_2) \neq c(s_3)$. From the Hamiltonian cycle of $G[R_0]$, we extract two disjoint paths, s_1–t_1 and u–v paths, that cover $G[R_0]$ for some $u, v \in R_0$, such that the neighbor $u' \in R_1$ of u is different from s_2. It suffices to build two disjoint s_2–u' and s_3–y paths that cover $G[R_{1,m-1}]$ for some $y \in R_{m-1}$.

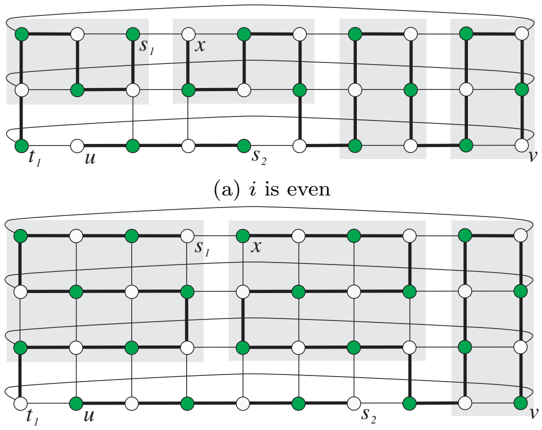
Case 8: $R_0 \cap (S \cup T) = \{s_2\}$ *and* $R_{m-1} \cap (S \cup T) = \{s_3\}$. This case is reduced to Case 2.

Case 9: $R_0 \cap (S \cup T) = \{s_1\}$ *and* $R_{m-1} \cap (S \cup T) = \{s_3\}$. We assume $c(s_1) = c(t_1) = c(s_2) \neq c(s_3)$ from the hypothesis of Case 2. Let $t_1 \in R_i$ and $s_2 \in R_j$ for some $i, j \in \{1, \ldots, m-2\}$. If $i < j$, then for some edge (u, v) with $u \in R_i$, $v \in R_{i+1}$, and $c(u) = c(s_3)$, it suffices to build two disjoint s_1–t_1 and u–x paths that cover $G[R_{0,i}]$ for some $x \in R_0$, and build two disjoint s_2–v and s_3–y paths that cover $G[R_{i+1,m-1}]$ for some $y \in R_{m-1}$. Analogously, if $j < i$, for some edge (u, v) with $u \in R_j$, $v \in R_{j+1}$, and $c(u) = c(s_3)$, we can build two disjoint s_1–u and s_2–x paths that cover $G[R_{0,j}]$ for some $x \in R_0$, and build two disjoint v–t_1 and s_3–y paths that cover $G[R_{j+1,m-1}]$ for some $y \in R_{m-1}$.

Finally, suppose $i = j$. Let s_1, t_1, and s_2, respectively, be contained in columns C_p, C_q, and C_r. Assume w.l.o.g. $q \le p \le r$ and $q = 0$.

Claim 1. There exist three disjoint s_1–t_1, s_2–u, and v–x paths that cover $G[R_{0,i}]$, where $u = v^i_1$, $v = v^i_{n-1}$, and $x = v^0_{p+1}$. Furthermore, each of the $\frac{n}{2} - 1$ edges (v^i_a, v^i_{a+1}) for odd $a \in \{1, 3, \ldots, n-3\}$ is visited by one of the three paths.

Proof of Claim 1. It holds true that $c(u) = c(v) = c(x) \neq c(s_1) = c(t_1) = c(s_2)$. If i is even, then $R_{0,i}$ has an odd number of rows, so possibly $p \in \{0, r\}$; if i is odd, then $R_{0,i}$ has an even number of rows, so $0 < p < r$. (Refer to Fig. 2.) An s_1–t_1 path is obtained by concatenating a Hamiltonian s_1–v^{i-1}_0 path of $G[R_{0,i-1} \cap C_{0,p}]$ and the one-vertex path $\langle t_1 \rangle$; set an s_2–u path be $\langle v^i_r, v^i_{r-1}, \ldots, v^i_1 \rangle$; in addition, a v–x path is obtained from concatenating a Hamiltonian v^i_{n-1}–v^i_{n-2} path of $G[R_{0,i} \cap C_{n-2,n-1}]$, …, a Hamiltonian v^i_{r+3}–v^i_{r+2} path of $G[R_{0,i} \cap C_{r+2,r+3}]$, the one-vertex path $\langle v^i_{r+1} \rangle$, and a Hamiltonian v^{i-1}_{r+1}–v^0_{p+1} path of $G[R_{0,i-1} \cap C_{p+1,r+1}]$. The existence of the Hamiltonian paths



(a) i is even

(b) i is odd

Fig. 2: Three disjoint s_1–t_1, s_2–u, and v–x paths in $G[R_{0,i}]$.

in the induced subgraphs that are isomorphic to rectangular grids is due to Lemma 1(a). Thus, the claim is proven. □

Let $u', v' \in R_{i+1}$ be the neighbors of u and v, respectively. If $i \le m-3$, it suffices to build two disjoint u'–v' and s_3–y paths that cover $G[R_{i+1,m-1}]$ for some $y \in R_{m-1}$, which exist by Theorem 4, and combine them with the three disjoint paths of Claim 1. So, let $i = m-2$ now, where $u' = v_1^{m-1}$, $v' = v_{n-1}^{m-1}$, and $s_3 = v_b^{m-1}$ for some even $b \in \{0, \dots, n-2\}$ because $c(s_3) \ne c(u') = c(v')$. If $b = 0$, it suffices to set s_3–y and u'–v' paths be $\langle v_0^{m-1} \rangle$ and $\langle v_1^{m-1}, v_2^{m-1}, \dots, v_{n-1}^{m-1} \rangle$, respectively, and combine the two with the three paths of Claim 1. If $b \ge 2$, we set an s_3–y path be $\langle v_b^{m-1}, v_{b-1}^{m-1}, \dots, v_2^{m-1} \rangle$ and set a u'–v' path be $\langle u', v_0^{m-1}, v' \rangle$. To deal with the vertices $v_{b+1}^{m-1}, \dots, v_{n-2}^{m-1}$ not visited till now, we use the fact shown in Claim 1 that every edge (v_a^i, v_{a+1}^i) for odd $a \in \{1, 3, \dots, n-3\}$ is visited by one of the three disjoint paths of $G[R_{0,i}]$. To cover each pair of unvisited vertices v_c^{m-1} and v_{c+1}^{m-1} for odd $c \in \{b+1, \dots, n-3\}$, it suffices to reroute the path that visits the edge $(v_c^{m-2}, v_{c+1}^{m-2})$ to traverse $\langle v_c^{m-2}, v_c^{m-1}, v_{c+1}^{m-1}, v_{c+1}^{m-2} \rangle$.

Case 10: *$R_0 \cap (S \cup T) = \{s_1\}$ and $R_{m-1} \cap (S \cup T) = \{t_1\}$.* Let $s_2 \in R_i$ and $s_3 \in R_j$ for some $i, j \in \{1, \dots, m-2\}$. Assume w.l.o.g. that the three terminals t_1, s_2, and s_3 are not of the same color. If $i < j$, we first pick up an edge (u, v) with $u \in R_i$ and $v \in R_{i+1}$ such that $c(u) \ne c(s_2)$ and $v \ne s_3$. Then, the three vertices of $\{s_1, s_2, u\}$ are not of the same color; also, the three vertices of $\{t_1, s_3, v\}$ are not of the same color because $c(v) = c(s_2)$. It suffices to build two disjoint s_1–u and s_2–x paths that cover $G[R_{0,i}]$ for some $x \in R_0$, and combine them with the two disjoint v–t_1 and s_3–y paths that cover $G[R_{i+1,m-1}]$ for some $y \in R_{m-1}$. The case where $j < i$ is symmetric to the case where $i < j$, so we consider the remaining case where $i = j$ hereafter.

Claim 2. There exists an edge (u, v) of $G[R_i]$ with $(v, s_1) \notin E(G)$ such that for some $y \in R_{m-1}$, the subgraph $G[R_{i,m-1}]$ contains three disjoint paths, composed of

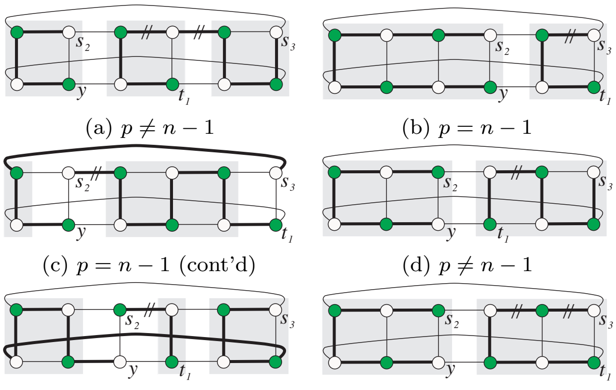
either u–t_1, s_2–v, and s_3–y paths, or u–t_1, s_3–v, and s_2–y paths,

that cover all the vertices of $G[R_{i,m-1}]$.

Proof of Claim 2. If $i \le m-3$, then $G[R_{i,m-1}]$ contains three or more rows. For an edge (u, v) of $G[R_i]$ with $u \notin \{s_2, s_3\}$ and $(v, s_1) \notin E(G)$, it suffices to decompose the Hamiltonian cycle of $G[R_i]$, represented as $\langle u, \dots, w, s_3, \dots, z, s_2, \dots, v \rangle$, into three paths $\langle u, \dots, w \rangle$, $\langle s_3, \dots, z \rangle$, and $\langle s_2, \dots, v \rangle$, and then build disjoint w'–t_1 and z'–y paths that cover $G[R_{i+1,m-1}]$ for some $y \in R_{m-1}$, where $w', z' \in R_{i+1}$ are the neighbors of w and z, respectively. Note that $c(w') = c(s_3)$ and $c(z') = c(s_2)$, so the vertices of $\{t_1, w', z'\}$ are not of the same color. Now, suppose $i = m-2$, where $G[R_{i,m-1}]$ contains exactly two rows. Let $t_1 \in C_p$, $s_2 \in C_q$, and $s_3 \in C_r$ for some $p, q, r \in \{0, \dots, n-1\}$.

For the first case, suppose $c(s_2) = c(s_3)$, so $c(s_2) = c(s_3) \ne c(t_1)$ from our assumption. We further assume w.l.o.g. that $q < p \le r$ and $r = n-1$. (See Fig. 3 (a) through (c).) If $p \ne n-1$, it suffices to set an s_2–y path be a Hamiltonian s_2–v_q^{m-1} path of $G[R_{m-2,m-1} \cap C_{0,q}]$, and then decompose the s_3–t_1 path, built by concatenating a Hamiltonian s_3–v_{p+1}^{m-2} path of $G[R_{m-2,m-1} \cap C_{p+1,n-1}]$ and a Hamiltonian v_p^{m-2}–t_1 path of $G[R_{m-2,m-1} \cap C_{q+1,p}]$, by deleting an edge $(u, v) = (v_{p-1}^{m-2}, v_p^{m-2})$ or $(v_p^{m-2}, v_{p+1}^{m-2})$ so that $(v, s_1) \notin E(G)$. If $p = n-1$, the required three paths are obtained in one of the following two ways: (i) set an s_2–y path be a Hamiltonian s_2–v_q^{m-1} path of $G[R_{m-2,m-1} \cap C_{0,q}]$, and then decompose the Hamiltonian s_3–t_1 path of $G[R_{m-2,m-1} \cap C_{q+1,n-1}]$ through $(u, v) = (v_{n-2}^{m-2}, v_{n-1}^{m-2})$; or (ii) concatenate $\langle s_3 \rangle$, a Hamiltonian v_0^{m-2}–v_{q-1}^{m-1} path of $G[R_{m-2,m-1} \cap C_{0,q-1}]$, and $\langle v_q^{m-1} \rangle$ into an s_3–y path, and then decompose the s_2–t_1 path, built by concatenating $\langle s_2 \rangle$, a Hamiltonian v_{q+1}^{m-2}–v_{n-2}^{m-1} path of $G[R_{m-2,m-1} \cap C_{q+1,n-2}]$, and $\langle t_1 \rangle$, through $(u, v) = (v_{q+1}^{m-2}, v_q^{m-2})$.

For the second case, suppose $c(s_2) \ne c(s_3)$. Assume w.l.o.g. that $c(t_1) = c(s_2) \ne c(s_3)$ and moreover, $q < p \le r = n-1$. (See Fig. 3 (d) through (f).) If $p \ne n-1$, the required three paths are obtained in one of the following two ways: (i) set an s_2–y path be a Hamiltonian s_2–v_{p-1}^{m-1} path of $G[R_{m-2,m-1} \cap C_{0,p-1}]$, and then decompose the Hamiltonian s_3–t_1 path of $G[R_{m-2,m-1} \cap C_{p,n-1}]$ through $(u, v) = (v_p^{m-2}, v_{p+1}^{m-2})$; or (ii) concatenate a Hamiltonian s_3–v_{q-1}^{m-1} path of $G[R_{m-2,m-1} \cap (C_{0,q-1} \cup C_{p+1,n-1})]$ and $\langle v_q^{m-1} \rangle$ into an s_3–y path, and then decompose the s_2–t_1 path, built by concatenating $\langle s_2 \rangle$ and a Hamiltonian v_{q+1}^{m-2}–t_1



(a) $p \neq n-1$ (b) $p = n-1$ (c) $p = n-1$ (cont'd) (d) $p \neq n-1$ (e) $p \neq n-1$ (cont'd) (f) $p = n-1$

Fig. 3: Three disjoint u–t_1, s_a–v, and s_b–y paths that cover $G[R_{m-2,m-1}]$ for some $(a,b) \in \{(2,3),(3,2)\}$, where $c(s_2) = c(s_3)$ for (a), (b), and (c); $c(s_2) \neq c(s_3)$ for (d), (e), and (f).

path of $G[R_{m-2,m-1} \cap C_{q+1,p}]$, through $(u,v) = (v_{q+1}^{m-2}, v_q^{m-2})$. If $p = n-1$, assuming w.l.o.g. $q \neq n-2$, it suffices to set an s_2–y path be a Hamiltonian s_2–v_q^{m-1} path of $G[R_{m-2,m-1} \cap C_{0,q}]$ and then decompose the Hamiltonian s_3–t_1 path of $G[R_{m-2,m-1} \cap C_{q+1,n-1}]$ through an edge $(u,v) = (v_{n-3}^{m-2}, v_{n-2}^{m-2})$ or $(v_{n-2}^{m-2}, v_{n-1}^{m-2})$. Thus, the claim is proven. □

Let $u', v' \in R_{i-1}$ be the neighbors of u and v, respectively. It remains to build two disjoint s_1–u' and v'–x paths that cover $G[R_{0,i-1}]$ for some $x \in R_0$. If $i \geq 2$, the two disjoint paths exist by Theorem 4; if $i = 1$, dividing the Hamiltonian cycle $\langle s_1, \ldots, u', v', \ldots, x \rangle$, where $v' \neq s_1$, of $G[R_0]$ results in two paths $\langle s_1, \ldots, u' \rangle$ and $\langle v', \ldots, x \rangle$, as required. If we combine the two paths of $G[R_{0,i-1}]$ with the three paths of Claim 2, we obtain the required three paths that cover G. This completes the entire proof. □

Remark 3. If distinct terminals $s_1, s_2 \in S$ and $t_1, t_2 \in T$ (instead of $s_1, s_2, s_3 \in S$ and $t_1 \in T$) are given in an $m \times n$ cylindrical grid with $m \geq 2$ and even $n \geq 4$, then there exist three disjoint paths, s_1–t_1, s_2–x, and t_2–y paths (instead of s_1–t_1, s_2–x, and s_3–y paths), that altogether cover all the vertices.

IV. PAIRED 3-DPC IN BIPARTITE TOROIDAL GRIDS

In this section, we will show that every $m \times n$ bipartite toroidal grid with $(m,n) \neq (4,4)$ has a paired 3-DPC joining S and T for any disjoint source and sink sets $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ such that $S \cup T$ is balanced. The 6×4 and 6×6 toroidal grids admit a paired 3-DPC joining S and T for any such terminal sets S and T, while the 4×4 toroidal grid does not, as shown in Fig. 4. Lemma 7 below was verified from a computer program that exhaustively searches for DPCs. The source code may

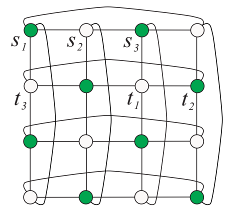


Fig. 4: A configuration that does not admit a paired 3-DPC. Every s_i–t_i path that does not pass through a terminal as an intermediate vertex contains at least 6 vertices, whereas the toroidal grid has less than 18 vertices.

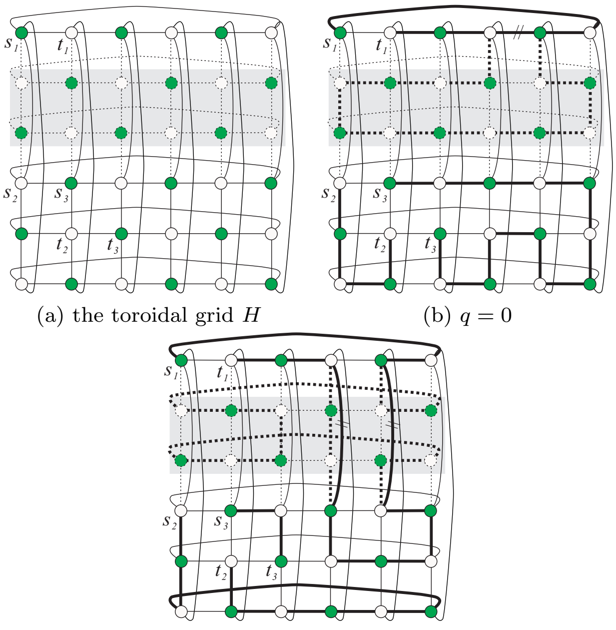
be downloaded from `http://tcs.catholic.ac.kr/~jhpark/papers/toroidal_grid.zip`.

Lemma 7. *Let G be a 6×4 or 6×6 toroidal grid, in which disjoint source and sink sets $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ are given. Then, G has a paired 3-DPC joining S and T if $S \cup T$ is balanced.*

One of the natural approaches would be reduction of our problem to a problem on a smaller bipartite toroidal grid. This is possible if there are two consecutive rows that contain no terminal as follows:

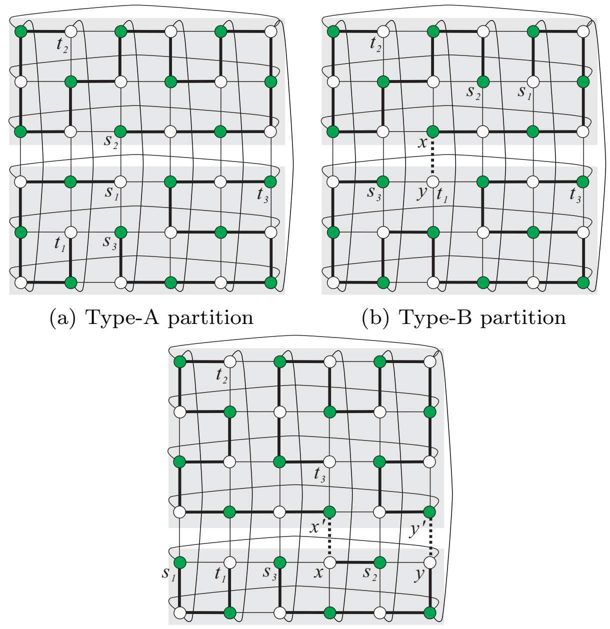
Lemma 8 (Row reduction). *An $m \times n$ bipartite toroidal grid G with $m \geq 6$ has a paired 3-DPC joining $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ if (i) $S \cup T$ is balanced, (ii) there are two consecutive rows R_p and R_{p+1} that contain no terminal, and (iii) an $(m-2) \times n$ toroidal grid has a paired 3-DPC joining S' and T' for any disjoint terminal sets S' and T' such that $S' \cup T'$ is balanced.*

Proof. Let H denote the $(m-2) \times n$ toroidal grid, obtained from G by deleting the vertices of $R_{p,p+1}$ and adding n *virtual* edges (v_j^{p-1}, v_j^{p+2}) for $j \in \{0, \ldots, n-1\}$, as shown in Fig. 5(a). Then, by the hypothesis (iii) of the lemma, H has a paired 3-DPC joining S and T. If none of the virtual edges is passed through by a path in the 3-DPC of H (see Fig. 5(b)), then for an edge in row $p-1$ or $p+2$, say $(v_j^{p-1}, v_{j+1}^{p-1})$ w.l.o.g., that is covered by the 3-DPC of H, replacing the edge with a path obtained by concatenating $\langle v_j^{p-1} \rangle$, a Hamiltonian v_j^p–v_{j+1}^p path of $G[R_{p,p+1}]$, and $\langle v_{j+1}^{p-1} \rangle$ results in a paired 3-DPC of G. Now, suppose that there is a virtual edge that is covered by the 3-DPC of H (see Fig. 5(c)). Let $\{(v_j^{p-1}, v_j^{p+2}) : j \in \{j_1, \ldots, j_q\}\}$ be the set of such virtual edges, and assume $j_1 < \cdots < j_q$. A paired 3-DPC of G can be built by replacing the virtual edge $(v_{j_i}^{p-1}, v_{j_i}^{p+2})$ with a path obtained by concatenating $\langle v_{j_i}^{p-1} \rangle$, a Hamiltonian $v_{j_i}^p$–$v_{j_i}^{p+1}$ path of $G[R_{p,p+1} \cap C_{j_i, j_{i+1}-1}]$, and $\langle v_{j_i}^{p+2} \rangle$ if $i < q$; with a path obtained by concatenating $\langle v_{j_q}^{p-1} \rangle$, a Hamiltonian $v_{j_q}^p$–$v_{j_q}^{p+1}$ path of $G[R_{p,p+1} \cap (C_{j_q,n-1} \cup C_{0,j_1-1})]$, and $\langle v_{j_q}^{p+2} \rangle$ if $i = q$. Thus, the lemma is proven. □



(a) the toroidal grid H (b) $q = 0$ (c) $j_1 = 3$ and $j_2 = 4$

Fig. 5: Illustrations of the row reduction, where $R_{1,2}$ contains no terminal.



(a) Type-A partition (b) Type-B partition (c) Type-C partition

Fig. 6: Three types of partitions of a toroidal grid into two cylindrical grids.

An $m \times n$ bipartite toroidal grid with $m \geq 6$ is said to be *row-reducible* if there are two consecutive rows R_p and R_{p+1} that contain no terminal. Besides the row reduction of Lemma 8, we can try a partition of the $m \times n$ toroidal grid into two cylindrical grids each having at least two rows, so as to build a paired 3-DPC in the toroidal grid. Three types of such partitions are investigated in Lemmas 9, 10, and 11 below and illustrated in Fig. 6.

Lemma 9 (Type-A partition). *An $m \times n$ bipartite toroidal grid G has a paired 3-DPC joining $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ if $S \cup T$ is balanced and there are r, $2 \leq r \leq m-2$, consecutive rows $R_p, \ldots, R_{p+r-1}$ that contain four terminals s_a, t_a, s_b, and t_b for some $a, b \in \{1, 2, 3\}$ in total such that the subgraph $G[R_{p,p+r-1}]$ induced by $R_{p,p+r-1}$ has a paired 2-DPC composed of s_a–t_a and s_b–t_b paths.*

Proof. The subgraph $G - R_{p,p+r-1}$ contains two terminals s_c and t_c with $c(s_c) \neq c(t_c)$, so there exists a Hamiltonian s_c–t_c path in the subgraph by Lemma 2. A paired 2-DPC of $G[R_{p,p+r-1}]$ along with the Hamiltonian path form a paired 3-DPC of G. □

Lemma 10 (Type-B partition). *An $m \times n$ bipartite toroidal grid G has a paired 3-DPC joining $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ if $S \cup T$ is balanced and there are r, $2 \leq r \leq m-2$, consecutive rows $R_p, \ldots, R_{p+r-1}$ that contain three terminals s_a, t_a, and s_b for some $a, b \in \{1, 2, 3\}$ in total such that the three are not of the same color.*

Proof. In the subgraph $G[R_{p,p+r-1}]$, there are two disjoint s_a–t_a and s_b–x paths for some $x \in R_p \cup R_{p+r-1}$ that cover all the vertices of the subgraph; moreover, the number of such vertices x is at least $\frac{n}{2} + 1$ by Theorem 4. Consider the subgraph H of G induced by $R_{0,p-1} \cup R_{p+r,n-1}$ now (i.e., $H = G - R_{p,p+r-1}$), in which there are three terminals s_c, t_c, and t_b for some $c \in \{1, 2, 3\}$ with $c \neq a, b$. Also, the three terminals of H are not of the same color, so there exist two disjoint s_c–t_c and t_b-y paths that cover H for at least $\frac{n}{2} + 1$ choices of $y \in R_{p-1} \cup R_{p+r}$ by Theorem 4 again. It follows that there is an edge (x, y) of G, where $x \in R_p \cup R_{p+r-1}$ and $y \in R_{p-1} \cup R_{p+r}$, that admits not only a 2-DPC, made of s_a–t_a and s_b–x paths, of $G[R_{p,p+r-1}]$ but also a 2-DPC, made of s_c–t_c and t_b-y paths, of H, because $c(x) \neq c(y)$ and there are at least $\frac{n}{2} + 1$ choices of x and y each. It suffices to combine the s_b–x path with the t_b-y path into an s_b–t_b path through the edge (x, y), completing the proof. □

Lemma 11 (Type-C partition). *An $m \times n$ bipartite toroidal grid G has a paired 3-DPC joining $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ if $S \cup T$ is balanced, G is not row-reducible, and there are r, $2 \leq r \leq m-2$, consecutive rows $R_p, \ldots, R_{p+r-1}$ that contain two terminals α and β in total such that*

- *$c(\alpha) = c(\beta)$ or $\alpha, \beta \notin R_p \cup R_{p+r-1}$* ***when*** *$r \geq 4$,*
- *$c(\alpha) = c(\beta)$ & $|\{\alpha, \beta\} \cap R_{p+1}| = 1$*
  or $c(\alpha) = c(\beta)$ & $(\alpha, \beta) \in \mathcal{K}$ & $|\{\alpha, \beta\} \cap R_p| = |\{\alpha, \beta\} \cap R_{p+2}| = 1$
  or $c(\alpha) = c(\beta)$ & $(\alpha, \beta) \notin \mathcal{K}$ & $\alpha, \beta \in R_{p+1}$
  or $c(\alpha) \neq c(\beta)$ & $(\alpha, \beta) \notin \mathcal{K}$ & $\alpha, \beta \in R_{p+1}$ & $(\alpha, \beta) \notin E(G)$ ***when*** *$r = 3$,*

- $c(\alpha) = c(\beta)$ & $(\alpha, \beta) \notin \mathcal{K}$ & $|\{\alpha, \beta\} \cap R_p| = 1$ ***when*** $r = 2$,

where $\mathcal{K} = \{(s_1, t_1), (s_2, t_2), (s_3, t_3)\}$.

Proof. Let H be the subgraph $G - R_{p,p+r-1}$ induced by $R_{0,p-1} \cup R_{p+r,n-1}$, in which there are four terminals, say s_a, t_a, α', and β' for some $a \in \{1, 2, 3\}$, so that $S \cup T = \{s_a, t_a, \alpha, \alpha', \beta, \beta'\}$, where $(\alpha, \alpha'), (\beta, \beta') \in \mathcal{K}$, or $(\alpha, \beta), (\alpha', \beta') \in \mathcal{K}$, or $(\alpha, \beta'), (\alpha', \beta) \in \mathcal{K}$. The four terminals of H are not of the same color since $S \cup T$ is balanced. So, from Theorem 5, there exist three disjoint s_a–t_a, α'–x, and β'–y paths that cover H for some $x, y \in R_{p-1} \cup R_{p+r}$. Let $x', y' \in R_p \cup R_{p+r-1}$ be the neighbors of x and y, respectively.

Claim 3. For the two terminals α and β of $G[R_{p,p+r-1}]$ satisfying the hypothesis of the lemma, (i) $\{x', y'\} \cap \{\alpha, \beta\} = \emptyset$; moreover, (ii) $G[R_{p,p+r-1}]$ has three kinds of a paired 2-DPC, a DPC made of α–x' and β–y' paths, a DPC made of α–y' and β–x' paths, and a DPC made of α–β and x'–y' paths.

Proof of Claim 3. Within the scope of this proof, x' and y' as well as α and β are said to be terminals. Observing that $\{\alpha, \beta, x', y'\}$ is balanced, we prove the assertion (i) first. If $c(\alpha) = c(\beta)$, then $c(x') = c(y') \neq c(\alpha) = c(\beta)$, so $\{x', y'\} \cap \{\alpha, \beta\} = \emptyset$; if $\alpha, \beta \notin R_p \cup R_{p+r-1}$, then $\{x', y'\} \cap \{\alpha, \beta\} = \emptyset$ obviously. An inspection of the hypothesis of the lemma leads to $c(\alpha) = c(\beta)$ or $\alpha, \beta \notin R_p \cup R_{p+r-1}$, proving (i). For the proof of the assertion (ii), let $\alpha \in R_i$ and $\beta \in R_j$ for some $i, j \in \{p, \ldots, p + r - 1\}$. Firstly, let $r \geq 4$. It follows that $i \neq j$ and $\{i, j\} \neq \{p, p + r - 1\}$; suppose otherwise, G would be row-reducible. This leads to that there is a (non-boundary) row that contains a single terminal, meaning the required 2-DPCs exist by Lemmas 3 and 4 (also, by Remark 1). Secondly, let $r = 3$. If $|\{\alpha, \beta\} \cap R_{p+1}| = 1$, then R_{p+1} contains a single terminal, so the required 2-DPCs exist. If $|\{\alpha, \beta\} \cap R_p| = |\{\alpha, \beta\} \cap R_{p+2}| = 1$, $c(\alpha) = c(\beta)$, and $(\alpha, \beta) \in \mathcal{K}$, then the four terminals in $\{\alpha, \beta, x', y'\}$ cannot form an inadmissible configuration of Lemmas 4 and 6, so the required 2-DPCs exist. Analogously, we can see that the required 2-DPCs exist for the remaining two cases where $\alpha, \beta \in R_{p+1}$. Finally, let $r = 2$. If $c(\alpha) = c(\beta)$, $(\alpha, \beta) \notin \mathcal{K}$, and $i \neq j$ (i.e., $|\{\alpha, \beta\} \cap R_p| = |\{\alpha, \beta\} \cap R_{p+1}| = 1$), then the four terminals in $\{\alpha, \beta, x', y'\}$ cannot form an inadmissible configuration of Lemmas 4 and 5, so the required 2-DPCs exist. Thus, the claim is proven. □

Combining the α'–x and β'–y paths of H with one of the three paired 2-DPCs of $G[R_{p,p+r-1}]$ through the edges (x, x') and (y, y') leads to a paired 3-DPC of G, as required. This completes the proof. □

Now, we are ready to prove our main theorem.

Theorem 6. *An $m \times n$ bipartite toroidal grid G with $(m, n) \neq (4, 4)$ has a paired 3-DPC joining disjoint terminal sets $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ if and only if $S \cup T$ is balanced.*

Proof. The necessity part is straightforward from the fact that the two color classes of G are always the same in size. The sufficiency proof will proceed by induction on $m + n$, where m and n are both even integers with $m, n \geq 4$ and $m + n \geq 10$. Assume w.l.o.g. $m \geq n$. The base step of $(m, n) = (6, 4)$ is due to Lemma 7. Moreover, the theorem holds true for the case of $(m, n) = (6, 6)$ by Lemma 7 again, so we assume $m \geq 8$ for the inductive step. Keep in mind that if G is row-reducible, then G has a paired 3-DPC joining S and T by Lemma 8 because by the induction hypothesis, an $(m - 2) \times n$ bipartite toroidal grid has a paired 3-DPC joining any disjoint terminal sets S' and T' of size 3 each such that $S' \cup T'$ is balanced. We assume w.l.o.g. that R_0 contains as many terminals as the other rows, i.e., $|R_0 \cap (S \cup T)| \geq |R_i \cap (S \cup T)|$ for all $i \in \{1, \ldots, m-1\}$. There are three cases according to the size of $R_0 \cap (S \cup T)$.

Case 1: $|R_0 \cap (S \cup T)| \geq 3$. The $m - 1$ (≥ 7) rows other than R_0 contain 3 or less terminals in total, so (i) G is row-reducible, or (ii) $m = 8$ and the three rows R_2, R_4, and R_6 each contains a single terminal. For possibility (i), G has a paired 3-DPC joining S and T by the induction hypothesis and Lemma 8; for possibility (ii), G admits a type-C partition w.r.t. $R_{1,5}$, and hence G has a paired 3-DPC joining S and T by Lemma 11.

Case 2: $|R_0 \cap (S \cup T)| = 2$.

Case 2.1: $|R_i \cap (S \cup T)| = 2$ *for some* $i \in \{1, \ldots, m - 1\}$. In this case, there are at most three rows other than R_0 each of which contains a terminal. It follows that G is row-reducible, or $m = 8$ and the three rows R_2, R_4, and R_6 each contains a terminal. If G is row-reducible, we are done by the induction hypothesis and Lemma 8. If $i = 2$, i.e., R_2 contains two terminals, then G has a paired 3-DPC joining S and T by Lemma 11 because G admits a type-C partition w.r.t. $R_{3,7}$; symmetrically in case of $i = 6$, G is also type-C-partitionable. Let $i = 4$ now. There are two possibilities: (i) $R_0 \cap (S \cup T) = \{s_a, t_a\}$ for some a, and (ii) $R_0 \cap (S \cup T) \neq \{s_a, t_a\}$ for all a.

For the first possibility, suppose $s_a, t_a \in R_0$. If $c(s_a) \neq c(t_a)$, then G admits a type-A partition w.r.t. $R_{2,7}$, hence G has a required 3-DPC by Lemma 9. (Note that the four terminals in $(S \cup T) \setminus \{s_a, t_a\}$ do not form an inadmissible configuration in the induced subgraph $G[R_{2,7}]$ since there is a row, say R_2, that contains an odd number of terminals.) If $c(s_a) = c(t_a)$, then there is a terminal α in R_2 or in R_6 such that $c(\alpha) \neq c(s_a) = c(t_a)$, hence, assuming w.l.o.g. $\alpha \in R_2$, G admits a type-B partition w.r.t. $R_{0,2}$ and has a required 3-DPC by Lemma 10.

For the second possibility, suppose $s_a, s_b \in R_0$ for some $a, b \in \{1, 2, 3\}$ with $a \neq b$ (or symmetrically,

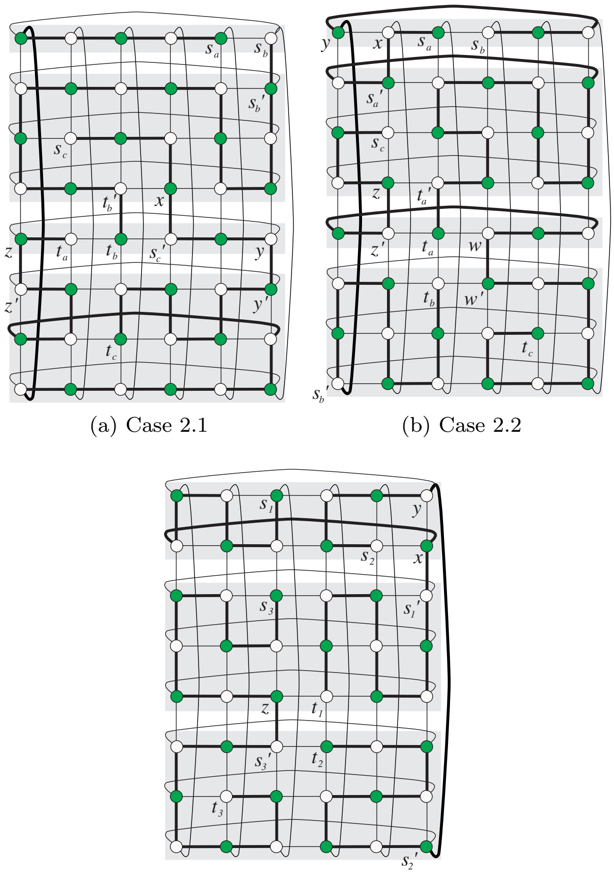
$s_a, t_b \in R_0$). For the two terminals, denoted α and β, in R_4, if $\{\alpha, \beta\} = \{s_c, t_c\}$ for some $c \in \{1, 2, 3\}$ with $c \neq a, b$, then a paired 3-DPC can be constructed in a symmetric way to the first possibility where $s_a, t_a \in R_0$. So, we assume $\{\alpha, \beta\} \neq \{s_c, t_c\}$. If either $c(s_a) = c(s_b)$ or $c(s_a) \neq c(s_b)$ & $(s_a, s_b) \notin E(G)$, then G admits a type-C partition w.r.t. $R_7 \cup R_{0,1}$, hence G has a required 3-DPC by Lemma 11. Similarly, if either $c(\alpha) = c(\beta)$ or $c(\alpha) \neq c(\beta)$ & $(\alpha, \beta) \notin E(G)$, then G is type-C-partitionable w.r.t. $R_{3,5}$ and has a required 3-DPC. So, we further assume $(s_a, s_b), (\alpha, \beta) \in E(G)$ ($c(s_a) \neq c(s_b)$ and $c(\alpha) \neq c(\beta)$). If $t_a \in R_2$ or $t_b \in R_2$, then G is type-B-partitionable w.r.t. $R_{0,2}$ and thus G has a required 3-DPC by Lemma 10; also, G is type-B-partitionable w.r.t. $R_{6,7} \cup R_0$ if $t_a \in R_6$ or $t_b \in R_6$.

Finally, there remains a case where $t_a, t_b \in R_4$ and $s_c, t_c \in R_2 \cup R_6$, say $s_c \in R_2$ and $t_c \in R_6$, and moreover $(s_a, s_b), (t_a, t_b) \in E(G)$ and $c(s_c) \neq c(t_c)$. None of the three types of a partition can be applied in this case, so we will devise a direct construction of a paired 3-DPC joining S and T. We assume w.l.o.g. that $c(s_b) = c(s_c)$, $s_a = v^0_{n-2}$, and $s_b = v^0_{n-1}$, and let $t_b = v^4_j$ for some j. The construction will be completed in five steps as follows (see Fig. 7(a)):

1: Find a Hamiltonian s_a–v^0_0 path, $\langle v^0_{n-2}, \ldots, v^0_0 \rangle$, in $G[R_0] - s_b$.
2: Let $x = v^3_{j+1}$ if $t_a \neq v^4_{j+1}$; let $x = v^3_{j-1}$ otherwise. For $s'_b = v^1_{n-1}$ and $t'_b = v^3_j$, find a paired 2-DPC composed of s'_b–t'_b and s_c–x paths in $G[R_{1,3}]$.
3: Let s'_c be the neighbor of x in R_4. Divide the Hamiltonian s'_c–t_a path of $G[R_4] - t_b$ into s'_c–y and z–t_a paths, by deleting an arbitrary edge (y, z) of the Hamiltonian path.
4: Let y' and z', respectively, be the neighbors of y and z in R_5. Find a paired 2-DPC composed of y'–t_c and v^7_0–z' paths in $G[R_{5,7}]$.
5: Concatenating the s_a–v^0_0, v^7_0–z', and z–t_a paths results in an s_a–t_a path; concatenating the one-vertex path $\langle s_b \rangle$, the s'_b–t'_b path, and $\langle t_b \rangle$ leads to an s_b–t_b path; finally, concatenating the s_c–x, s'_c–y, and y'–t_c paths leads to an s_c–t_c path.

The paired 2-DPCs in Steps 2 and 4 exist due to Lemmas 4 and 6 (also, due to Remark 1).

Case 2.2: $|R_i \cap (S \cup T)| \leq 1$ *for all* $i \in \{1, \ldots, m-1\}$. There are exactly four rows other than R_0 each of which contains a terminal, so G is row-reducible (and we are done) or $m \leq 10$. If $m = 10$, then the four rows R_2, R_4, R_6, and R_8 each contains a single terminal, hence G admits a type-C partition w.r.t. $R_{1,5}$ and has a required 3-DPC by Lemma 11. Suppose $m = 8$ from now on. Let r be the maximum number of consecutive rows, including R_0, each of which contains a terminal; also, let $R_p, \ldots, R_q$ denote the remaining $8-r$ consecutive rows. (Note that $R_{p,q}$ contains $5 - r$ terminals; but R_p and R_q contain no terminal.) It follows that $r \leq 3$ because G is not row-reducible. If $r = 3$, then R_{p+1} and R_{p+3} each contains a single terminal, hence G admits a type-C-partition w.r.t. $R_{p,p+4}$ and has a required 3-DPC. If $r = 2$, then R_{p+1} and R_{p+4} each contains a single terminal; also, either R_{p+2} or R_{p+3} contains a single terminal. This leads to that G is type-C-partitionable (w.r.t. $R_{p,p+3}$ for the former case and w.r.t. $R_{p+2,p+5}$ for the latter case) and has a required 3-DPC. Finally, if $r = 1$, then R_2 and R_6 each contains a single terminal; also, two of the three R_3, R_4, and R_5 each contains a single terminal. If R_3 and R_5 each contains a single terminal (but R_4 does not), then G is type-C-partitionable w.r.t. $R_{1,4}$. So, we assume w.l.o.g. R_4 and R_5 each contains a single terminal, i.e., $|R_j \cap (S \cup T)| = 1$ for $j \in \{2, 4, 5, 6\}$.



(a) Case 2.1 (b) Case 2.2

(c) Case 3

Fig. 7: Illustrations of the proof of Theorem 6 for the cases to which none of the three types of a partition is applicable.

Let α and β denote the two terminals in R_0. Firstly, suppose $c(\alpha) = c(\beta)$. If $\{\alpha, \beta\} = \{s_a, t_a\}$ for some $a \in \{1, 2, 3\}$, then assuming w.l.o.g. that the terminal in R_2 has a color different from $c(\alpha)$, G is type-B-partitionable w.r.t. $R_{0,2}$. If $\{\alpha, \beta\} \neq \{s_a, t_a\}$ for all a, then G is type-C-partitionable w.r.t. $R_7 \cup R_{0,1}$. Secondly, suppose $c(\alpha) \neq c(\beta)$. If $\{\alpha, \beta\} = \{s_a, t_a\}$ for some a, then G is type-A-partitionable w.r.t. $R_{2,7}$. If $\{\alpha, \beta\} \neq \{s_a, t_a\}$ for all

a, and moreover $(\alpha, \beta) \notin E(G)$, then G is type-C-partitionable w.r.t. $R_7 \cup R_{0,1}$. So, we further assume $\{\alpha, \beta\} = \{s_a, s_b\}$ for some $a, b \in \{1, 2, 3\}$ with $a \neq b$, and $(s_a, s_b) \in E(G)$. If R_2 contains t_a or t_b, then G is type-B-partitionable w.r.t. $R_{0,2}$; if R_6 contains t_a or t_b, then G is also type-B-partitionable w.r.t. $R_{6,7} \cup R_0$. There remains a case where $(R_2 \cup R_6) \cap (S \cup T) = \{s_c, t_c\}$ for some $c \in \{1, 2, 3\}$ with $c \neq a, b$. Assume w.l.o.g. $s_c \in R_2$ and $t_c \in R_6$, and moreover $t_a \in R_4$ and $t_b \in R_5$. If $c(t_a) = c(t_b)$, then G is type-C-partitionable w.r.t. $R_{3,5}$; also, if $c(t_b) = c(t_c)$, then G is type-C-partitionable w.r.t. $R_{5,7}$. Under the condition $c(t_a) = c(t_c) \neq c(t_b) = c(s_c)$, we give a direct construction of a paired 3-DPC below for the remaining case (see Fig. 7(b)).

1: Find a Hamiltonian s_a–s_b path in $G[R_0]$. Let the Hamiltonian path be represented as $\langle s_a, \ldots, x, y, \ldots, s_b \rangle$, possibly $x = s_a$, for some x with $c(x) = c(s_c)$.
2: For the neighbor $s'_a \in R_1$ of x, the neighbor $t'_a \in R_3$ of t_a, and a neighbor $z \in R_3$ of t'_a, find a paired 2-DPC made of s'_a–t'_a and s_c–z paths in $G[R_{1,3}]$.
3: For the neighbor $z' \in R_4$ of z and the neighbor $w \in R_4$ of t_a other than z', find a Hamiltonian z'–w path in $G[R_4] - t_a$.
4: For the neighbor $s'_b \in R_7$ of y and the neighbor $w' \in R_5$ of w, find a paired 2-DPC composed of s'_b–t_b and w'–t_c paths in $G[R_{5,7}]$.
5: Concatenating the s_a–x path, the s'_a–t'_a path, and $\langle t_a \rangle$ results in an s_a–t_a path; concatenating the s_b–y and s'_b–t_b paths leads to an s_b–t_b path; finally, concatenating the s_c–z, z'–w, and w'–t_c paths leads to an s_c–t_c path.

Case 3: $|R_0 \cap (S \cup T)| = 1$. Let r denote the maximum number of consecutive rows each of which contains a terminal; assume w.l.o.g. that $R_0, \ldots, R_{r-1}$ are such consecutive rows. Firstly, suppose $r = 1$. Then, G is type-C-partitionable w.r.t. $R_{m-1} \cup R_{0,q+1}$ for some $q \geq 1$ such that R_q contains a terminal but R_j does not for all $j \in \{1, \ldots, q-1\}$. Secondly, suppose $r = 2$. Then, G is also type-C-partitionable w.r.t. $R_{m-1} \cup R_{0,2}$. Thirdly, suppose $r = 3$. Then, G is row-reducible or $m \leq 10$. If $m = 10$, then R_4, R_6, and R_8 each contains a single terminal, so G is type-C-partitionable w.r.t. $R_{3,7}$. Let $m = 8$ now. The rows R_3 and R_7 contain no terminal, so each of R_4, R_5, R_6 contains a terminal, i.e., $|R_j \cap (S \cup T)| = 1$ iff $j \in \{0, 1, 2, 4, 5, 6\}$. Let α_i denote the terminal in R_i. If $c(\alpha_0) = c(\alpha_1)$, then G is type-C-partitionable; if $c(\alpha_1) = c(\alpha_2)$, then G is also type-C-partitionable; so, $c(\alpha_0) = c(\alpha_2) \neq c(\alpha_1)$. A similar argument leads to $c(\alpha_4) = c(\alpha_6) \neq c(\alpha_5)$. It follows that $c(\alpha_0) = c(\alpha_2) = c(\alpha_5) \neq c(\alpha_1) = c(\alpha_4) = c(\alpha_6)$. Furthermore, if $\{\alpha_0, \alpha_1, \alpha_2\}$ contains s_a, t_a for some a, then G is type-B-partitionable; if $\{\alpha_1, \alpha_2, \alpha_4\}$ contains s_a, t_a for some a, then G is also type-B-partitionable, and so on. Thus, we can assume w.l.o.g. that $s_1 \in R_0$, $s_2 \in R_1$, $s_3 \in R_2$, $t_1 \in R_4$, $t_2 \in R_5$, and $t_3 \in R_6$. A paired 3-DPC for the remaining case can be constructed as follows (see Fig. 7(c)):

1: For a vertex $x \in R_1$ with $c(x) = c(s_1)$, there exists a vertex $y \in R_0$ that admits a disjoint path cover composed of s_1–x and s_2–y paths in $R_{0,1}$.
2: For the neighbor $s'_1 \in R_2$ of x, there exists a vertex $z \in R_4$ that admits a disjoint path cover composed of s'_1–t_1 and s_3–z paths in $R_{2,4}$.
3: For the neighbor $s'_3 \in R_5$ of z and the neighbor $s'_2 \in R_7$ of y, there exists a paired 2-DPC composed of s'_2–t_2 and s'_3–t_3 paths in $R_{5,7}$.
4: Concatenating the s_1–x and s'_1–t_1 paths results in an s_1–t_1 path; concatenating the s_2–y and s'_2–t_2 paths leads to an s_2–t_2 path; finally, concatenating the s_3–z and s'_3–t_3 paths leads to an s_3–t_3 path.

The vertices y in Step 1 and z in Step 2 exist due to Theorem 4. The paired 2-DPC in Step 3 exists by Lemmas 4 and 6 (also, by Remark 1).

Finally, suppose $r \geq 4$. Then, G is row-reducible, or $m = 8$ and $r \in \{4, 5\}$. Let $m = 8$. If $r = 4$, then R_4 and R_7 contain no terminal, but R_5 and R_6 each contains a single terminal, hence G is type-C-partitionable w.r.t. $R_{4,7}$. If $r = 5$, then R_6 contains a terminal but R_5 and R_7 does not. Let α_i denote the terminal in R_i again. If $c(\alpha_3) = c(\alpha_4)$, then G is type-C-partitionable w.r.t. $R_{3,5}$; also, if $c(\alpha_4) = c(\alpha_6)$, then G is type-C-partitionable w.r.t. $R_{4,7}$; in addition, if $c(\alpha_6) = c(\alpha_0)$, then G is type-C-partitionable w.r.t. $R_{5,7} \cup R_0$; finally, if $c(\alpha_0) = c(\alpha_1)$, then G is type-C-partitionable w.r.t. $R_7 \cup R_{0,1}$. It follows that $c(\alpha_3) \neq c(\alpha_4) \neq c(\alpha_6) \neq c(\alpha_0) \neq c(\alpha_1)$, and thus $c(\alpha_0) = c(\alpha_2) = c(\alpha_4) \neq c(\alpha_1) = c(\alpha_3) = c(\alpha_6)$. Furthermore, if $\{\alpha_0, \alpha_1, \alpha_2\}$ contains s_a, t_a for some a, then G is type-B-partitionable; if $\{\alpha_1, \alpha_2, \alpha_3\}$ contains s_a, t_a for some a, then G is also type-B-partitionable, and so on. Thus, we can assume w.l.o.g. that $s_1 \in R_0$, $s_2 \in R_1$, $s_3 \in R_2$, $t_1 \in R_3$, $t_2 \in R_4$, and $t_3 \in R_6$. The construction, shown below, is almost the same as in the previous case where $r = 3$, $m = 8$, $s_1 \in R_0$, $s_2 \in R_1$, $s_3 \in R_2$, $t_1 \in R_4$, $t_2 \in R_5$, and $t_3 \in R_6$.

1: For a vertex $x \in R_1$ with $c(x) = c(s_1)$, there exists a vertex $y \in R_0$ that admits a disjoint path cover composed of s_1–x and s_2–y paths in $R_{0,1}$.
2: For the neighbor $s'_1 \in R_2$ of x, there exists a vertex $z \in R_3$ that admits a disjoint path cover composed of s'_1–t_1 and s_3–z paths in $R_{2,3}$.
3: For the neighbor $s'_3 \in R_4$ of z and the neighbor $s'_2 \in R_7$ of y, there exists a paired 2-DPC composed of s'_2–t_2 and s'_3–t_3 paths in $R_{4,7}$.
4: Concatenating the s_1–x and s'_1–t_1 paths results in an s_1–t_1 path; concatenating the s_2–y and s'_2–t_2 paths leads to an s_2–t_2 path; finally,

concatenating the s_3–z and s_3'–t_3 paths leads to an s_3–t_3 path.

This completes the entire proof. □

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