Paired many-to-many 3-disjoint path covers in bipartite toroidal grids

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Abstract
Given two disjoint vertex-sets, \( S = \{s_1, \ldots, s_k\} \) and \( T = \{t_1, \ldots, t_k\} \) in a graph, a paired many-to-many \( k \)-disjoint path cover joining \( S \) and \( T \) is a set of pairwise vertex-disjoint paths \( \{P_1, \ldots, P_k\} \) that altogether cover every vertex of the graph, in which each path \( P_i \) runs from \( s_i \) to \( t_i \). In this paper, we first study the disjoint-path-cover properties of a bipartite cylindrical grid. Based on the findings, we prove that every bipartite toroidal grid, excluding the smallest one, has a paired many-to-many 3-disjoint path cover joining \( S = \{s_1, s_2, s_3\} \) and \( T = \{t_1, t_2, t_3\} \) if and only if the set \( S \cup T \) contains the equal numbers of vertices from different parts of the bipartition.

Category: Algorithms and Complexity

Keywords: Disjoint path; path cover; path partition; cylindrical grid; torus

I. INTRODUCTION

Let \( G \) be a finite, simple undirected graph whose vertex and edge sets are denoted by \( V(G) \) and \( E(G) \), respectively. A path from \( v \in V(G) \) to \( w \in V(G) \), referred to as a \( v \rightarrow w \) path, is a sequence \( \langle u_1, \ldots, u_l \rangle \) of distinct vertices of \( G \) such that \( u_1 = v \), \( u_l = w \), and \( (u_i, u_{i+1}) \in E(G) \) for all \( i \in \{1, \ldots, l-1\} \). If \( l \geq 3 \) and \( (u_i, u_1) \in E(G) \), the sequence is called a cycle. A path that visits each vertex exactly once is a Hamiltonian path; a cycle that visits each vertex exactly once is a Hamiltonian cycle. A path cover of a graph \( G \) is a set of paths in \( G \) such that every vertex of \( G \) is contained in at least one path. A disjoint path cover (DPC for short) of \( G \) is a set of disjoint paths that altogether cover every vertex of \( G \). This paper is concerned with a DPC in which each path runs from a prescribed source to a prescribed sink.

Given disjoint subsets \( S = \{s_1, \ldots, s_k\} \) and \( T = \{t_1, \ldots, t_k\} \) of \( V(G) \) for a positive integer \( k \), a many-to-many \( k \)-disjoint path cover is a DPC composed of \( k \) paths that collectively join \( S \) and \( T \); if each source \( s_i \in S \) must be joined to a specific sink \( t_i \in T \), the DPC is called paired, and it is unpaired if no such constraint is imposed. Refer to Fig. 1 for examples.

There are two other DPC types: A one-to-many \( k \)-disjoint path cover for \( S = \{s\} \) and \( T = \{t_1, \ldots, t_k\} \) is a DPC made of \( k \) paths, each of which joins a pair of source \( s \) and sink \( t_i \), \( i \in \{1, \ldots, k\} \); when \( S = \{s\} \) and \( T = \{t\} \), a DPC composed of \( k \) paths, each of which joins \( s \) and \( t \), is named a one-to-one \( k \)-disjoint path cover. As intuitively clear, we will call the vertices in \( S \) and in \( T \) sources and sinks, respectively, which together form a set of terminals.

The existence of a disjoint path cover in a graph is closely related to the Hamiltonian properties, as well as the concept of vertex connectivity which was characterized in terms of the minimum number of
disjoint vertices. For instance, a Hamiltonian cycle forms a one-to-one 2-DPC joining \( s \) and \( t \) for every pair of distinct vertices \( s \) and \( t \). The disjoint path cover problem is applicable in many areas such as software testing, database design, and code optimization [1, 2]. In addition, the problem is concerned with applications where full utilization of network nodes is important [3]. The problems have been studied for various classes of graphs, such as interval graphs [4, 5], hypercubes [6, 7, 8], torus networks [9, 10, 11, 12], dense graphs [13], and cubes of connected graphs [14, 15].

In the context of the Hamiltonian path problem, the rectangular grid first appeared in the literature in [16]. In the formal definition of the \( m \times n \) rectangular grid, the vertices are often chosen from the points of the Euclidean plane with integer coordinates so that the vertices and edges form a rectangular grid with \( n \) vertices appearing in each of \( m \) rows and \( m \) vertices in each of \( n \) columns.

**Definition 1** (Rectangular grid). The \( m \times n \) rectangular grid \( G \) is a graph such that \( V(G) = \{v_i^j : 0 \leq i \leq m-1, 0 \leq j \leq n-1 \} \) and \( E(G) = \{(v_i^j, v_i^{j+1}) : |i - i'| + |j - j'| = 1 \} \).

Besides the rectangular grid graph, there are two related classes of grid graphs: The \( m \times n \) cylindrical grid is constructed from the \( m \times n \) rectangular grid by adding horizontal wrap-around edges \((v_i^{n-1}, v_i^0)\) for \( i \in \{0, \ldots, m-1\} \); the toroidal grid can be generated from the \( m \times n \) cylindrical grid by adding vertical wrap-around edges \((v_j^{m-1}, v_j^0)\) for \( j \in \{0, \ldots, n-1\} \).

**Definition 2** (Cylindrical grid). The \( m \times n \) cylindrical grid \( G \) is a graph such that \( V(G) = \{v_i^j : 0 \leq i \leq m-1, 0 \leq j \leq n-1 \} \) and \( E(G) = \{(v_i^j, v_i^{j+1}) : j = j' \& |i - i'| = 1 \} \) or \((i = i' \& j' \equiv j + 1 \pmod{n})\), where \( n \geq 3 \).

**Definition 3** (Toroidal grid). The \( m \times n \) toroidal grid \( G \) is a graph such that \( V(G) = \{v_i^j : 0 \leq i \leq m-1, 0 \leq j \leq n-1 \} \) and \( E(G) = \{(v_i^j, v_i^{j+1}) : j = j' \& i' \equiv i + 1 \pmod{m} \} \) or \((i = i' \& j' \equiv j + 1 \pmod{n})\), where \( m, n \geq 3 \).

The rectangular grid is a bipartite graph and thus its vertices may be colored in two colors, green and white, in such a way that every pair of adjacent vertices is colored differently (hereafter, we will denote the color of vertex \( v \) by \( c(v) \)). In contrast, the \( m \times n \) cylindrical grid is bipartite if and only if \( n \) is even; the \( m \times n \) toroidal grid is bipartite if and only if both \( m \) and \( n \) are even. The bipartite cylindrical and toroidal grids each is balanced in a sense that its two color classes have equal cardinality. We will also call a subset of \( V(G) \) balanced if the number of vertices in the subset that belong to each of the two color classes is equal.

The existence of a paired (many-to-many) 2-DPC in a bipartite toroidal grid was studied, as shown below:

**Theorem 1** (Makino [17]). An \( m \times n \) toroidal grid with \( m, n \geq 4 \), both even, has a paired 2-DPC for a pair of terminal sets \( S \) and \( T \) if and only if their union is balanced.

**Theorem 2** (Park and Ihm [18]). For an \( m \times n \) toroidal grid \( G \) with \( m, n \geq 4 \), both even, and an arbitrary edge \( e_f \) of \( G \), the subgraph, \( G - e_f \), of \( G \) with \( e_f \) being deleted has a paired 2-DPC joining \( S \) and \( T \) if and only if \( S \cup T \) is balanced.

**Theorem 3** (Kim and Park [19]). For an \( m \times n \) toroidal grid \( G \) with \( m, n \geq 4 \), both even, and an arbitrary vertex \( v_f \) of \( G \), the subgraph, \( G - v_f \), of \( G \) with \( v_f \) being deleted has a paired 2-DPC joining \( S \) and \( T \) if and only if one of the four terminals in \( S \cup T \) has the same color as \( v_f \) and the other three have a different color from \( v_f \).

In this paper, we prove that an \( m \times n \) bipartite toroidal grid with \((m, n) \neq (4, 4)\) has a paired 3-DPC joining \( S = \{s_1, s_2, s_3\} \) and \( T = \{t_1, t_2, t_3\} \) if and only if \( S \cup T \) is balanced.

**II. NOTATION AND PREVIOUS WORKS**

For an \( m \times n \) grid graph, whether rectangular, cylindrical, or toroidal, \( R_t \) denotes the vertex set \( \{v_i^j : 0 \leq j \leq n-1\} \) of row \( i \), whereas \( C_{ij} \) denotes the vertex set \( \{v_i^j : 0 \leq i \leq m-1\} \) of column \( j \), implying \( v_i^j \) is the vertex both in row \( i \) and in column \( j \). Based on these notations, we indicate multiple rows and columns respectively as \( R_i = \bigcup_{0 \leq j \leq n-1} v_i^j \) if \( i \leq i' \); \( R_{i'} = \emptyset \) otherwise, and \( C_{j'} = \bigcup_{0 \leq i \leq m-1} v_i^j \) if \( j \leq j' \); \( C_{j} = \emptyset \) otherwise. All arithmetic on the indices of vertices of the cylindrical and toroidal grids is done modulo 6 or 2 as needed.

Hamiltonian properties of the rectangular and cylindrical grids have been revealed in previous studies, some of which will be effectively used for deriving our results. A bipartite graph that is balanced is called *Hamiltonian-laceable* if there is a Hamiltonian path between any two vertices from different color classes [20]. The concept of Hamiltonian-laceability has often been extended in such a way that a bipartite graph whose color classes may differ in cardinality by exactly one is also *Hamiltonian-laceable* if every pair of vertices from the same major color class can be joined by a Hamiltonian path. Finally, a bipartite graph \( G \) is called 1-fault *Hamiltonian-laceable* if \( G \) remains Hamiltonian-laceable even if a single vertex or edge is deleted from \( G \).
Lemma 1 (Chen and Quimpo [21]). Let $G$ be an $m \times n$ rectangular grid with $m, n \geq 2$. (a) If $m$ is even, then $G$ has a Hamiltonian path from a corner vertex, i.e., a vertex of degree two, to any other vertex in the different color class. (b) If $m$ is odd, then $G$ has a Hamiltonian path from a corner vertex to any other vertex in the same color class.

Lemma 2 (Tsai, Tan, Chuang, and Hsu [22]). An $m \times n$ cylindrical grid with $m \geq 2$ and even $n \geq 4$ is 1-fault Hamiltonian-laceable.

A necessary and sufficient condition was established by Park and Ihm [18] for an $m \times n$ bipartite cylindrical grid to have a paired 2-DPC joining disjoint terminal sets $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$; furthermore, inadmissible configurations of the four terminals which would not permit a paired 2-DPC in the cylindrical grid were classified in four cases: (i) $m \geq 4$ and even $n \geq 6$, (ii) $n = 4$, (iii) $m = 2$ and even $n \geq 6$, and (iv) $m = 3$ and even $n \geq 6$, as shown in Lemmas 3 through 6.

Lemma 3. For $m \geq 4$ and even $n \geq 6$, an $m \times n$ cylindrical grid $G$ has a paired 2-DPC joining $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$ if and only if $S \cup T$ is balanced, and the four terminals in $S \cup T$ do not form an inadmissible configuration equivalent to A0, B0, or C0:

A0: $s_1 = v_0^i, s_2 = v_0^p, t_1 = v_0^j, t_2 = v_0^q$ for some $i, j, p, q$ such that $i < p < j < q$;

B0: $s_1 = v_0^i, t_1 = v_0^i+1, s_2 = v_0^i+1, t_2 = v_0^i+1$ for some $i$ and $r$;

C0: $s_1 = v_0^i, t_1 = v_0^i+1, t_2 = v_0^i+2, s_2 = v_0^i+3$ for some $i$.

Lemma 4. For $m \geq 2$, an $m \times 4$ cylindrical grid $G$ has a paired 2-DPC joining $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$ if and only if $S \cup T$ is balanced, and the four terminals in $S \cup T$ do not form an inadmissible configuration equivalent to A1, B0, or C1:

A1: $s_1, t_1 \in R_i, s_2, t_2 \in R_p, s_1 = c(t_1) \neq c(s_2) = c(t_2)$ for some $r_1$ and $r_2$;

C1: $s_1 = v_0^i, t_1 = v_0^i+1, t_2 = v_0^i+2, s_2 = v_0^i+3$ for some $i$ and $r$.

Lemma 5. For even $n \geq 6$, a $2 \times n$ cylindrical grid $G$ has a paired 2-DPC joining $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$ if and only if $S \cup T$ is balanced, and the four terminals in $S \cup T$ do not form an inadmissible configuration equivalent to A0, B2, C2, or D2:

B2: $S \cup T = \{v_0^i, v_1^i, v_0^j, v_1^j\}$ and $c(s_1) = c(t_1) \neq c(s_2) = c(t_2)$ for some $i$ and $j$ with $i \neq j$;

C2: $s_1 = v_0^i, t_1 = v_0^i, s_2 = v_0^j, t_2 = v_0^j, s_1 = c(t_1) \neq c(s_2) = c(t_2)$ for some $i, j, p, q$ and $q$ such that $\min\{i, j\} < \min\{p, q\}$;

D2: $s_1 = v_0^i, s_2 = v_0^j, t_1 = v_0^i, t_2 = v_0^j, c(s_1) = c(s_2) \neq c(t_1) = c(t_2)$ for some $i, j, p, q$ and $q$ such that $i < p < j < q$.

Lemma 6. For even $n \geq 6$, a $3 \times n$ cylindrical grid $G$ has a paired 2-DPC joining $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$ if and only if $S \cup T$ is balanced, and the four terminals in $S \cup T$ do not form an inadmissible configuration equivalent to A0, B0, C3, D3, E3, or F3:

C3: $s_1 = v_0^i, t_1 = v_1^i, t_2 = v_2^i, s_2 = v_0^p, c(s_1) = c(t_1) \neq c(s_2) = c(t_2)$ for some $i, j, p, q$ and $q$ such that $i < j < p < q < j+1$, and $(n - 1 - p) + i \geq 2$;

D3: $s_1 = v_1^i, s_2 = v_1^j, t_1 = v_1^j, t_2 = v_1^q, c(s_1) = c(t_2) \neq c(t_1) = c(s_2)$ for some $i, j, p, q$ and $q$ such that $i < p < j < q$, $p = i + 1$, and $q = j + 1$;

E3: $s_1 = v_0^i, s_2 = v_0^p, t_1 = v_1^j, t_2 = v_2^j, c(s_1) = c(s_2) \neq c(t_1) = c(t_2)$ for some $i, j, p, q$ and $q$ such that $i < p < j < q$, $p = i - 1$, and $(n - 1 - j) + i \geq 2$;

F3: $s_1 = v_0^p, t_2 = v_1^q, s_2 = v_0^p, t_1 = v_2^q, c(s_1) = c(t_2) \neq c(s_2) = c(t_1)$ for some $i, j, p, q$ and $q$ such that $q' < j' < i' - q' - 1 \geq 2$, and $(n - 1 - p') + i' \geq 2$, where $i' = \min\{i, q\}, q' = \max\{i, q\}, j' = \min\{i, j, p\}$, and $p' = \max\{i, j, p\}$.

Remark 1. The four terminals in $S \cup T$ form an inadmissible configuration in a bipartite cylindrical grid only if each row contains an even number of terminals.

III. DISJOINT PATH COVERS IN BIPARTITE CYLINDRICAL GRIDS

Suppose that disjoint source and sink sets $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ are given in an $m \times n$ bipartite toroidal grid. If we divide the toroidal grid into two cylindrical grids, $m_1 \times n$ and $m_2 \times n$ cylindrical grids for some $m_1, m_2 \geq 2$ with $m_1 + m_2 = m$, then each cylindrical grid may have an “incomplete” terminal set in a sense that $s_1$ is contained in its terminal set but $t_1$ is not for some $i \in \{1, 2, 3\}$, and vice versa. In this section, we derive some useful properties of a disjoint path cover in a bipartite cylindrical grid with an incomplete terminal set, where the notion of a disjoint path cover is “generalized” in a way that a one-vertex path is allowed. (Note that a disjoint path cover joining disjoint terminal sets $S$ and $T$ contains no one-vertex path.) A boundary row in an $m \times n$ cylindrical grid hereafter refers to the row $0$ or row $m - 1$.

Theorem 4. Let $G$ be an $m \times n$ cylindrical grid with $m \geq 2$ and even $n \geq 4$, in which three distinct terminals $s_1, s_2 \in S$ and $t_1 \in T$ are given such that not all the three are of the same color. Then, there exist two disjoint paths, $s_1-t_1$ and $s_2-x$ paths, possibly $x = s_2$, that altogether cover all the vertices of $G$.

- for every vertex $x$ in one boundary row and for at least one vertex $x$ in the other boundary row
such that \( \{s_1, t_1, s_2, x \} \) is balanced, or

- for every vertex \( x \) except one in one boundary row and for at least two vertices \( x \) in the other boundary row such that \( \{s_1, t_1, s_2, x \} \) is balanced.

Proof. Suppose we are given three distinct terminals \( s_1, t_1, \) and \( s_2 \) in \( G \) such that the three are not of the same color. Then, there is a terminal with a different color from the other two, so \( \{s_1, t_1, s_2, x \} \) is balanced if and only if \( x \) has the same color as the terminal. In addition, inspection of the inadmissible configurations in each of the four cases, where (i) \( m \geq 4 \) & even \( n \geq 6 \), (ii) \( n = 4 \), (iii) \( m = 2 \) & even \( n \geq 6 \), and (iv) \( m = 3 \) & even \( n \geq 6 \), can reveal that there exists an inadmissible configuration \( Z \) such that for every vertex \( x \in V(G) \setminus \{s_1, t_1, s_2 \} \), the four terminals in \( \{s_1, t_1, s_2, x \} \) do not form an inadmissible configuration or form an inadmissible configuration equivalent to \( Z \) only, i.e., the four terminals do not form an inadmissible configuration not equivalent to \( Z \).

Firstly, suppose \( m \geq 4 \) & even \( n \geq 6 \). From Lemma 3, there exists a pair 2-DPC, made of \( s_1–t_1 \) and \( s_2–x \) paths, in \( G \) for every vertex \( x \in (R_0 \cup R_{m-1}) \setminus \{s_1, t_1, s_2 \} \) such that \( \{s_1, t_1, s_2, x \} \) is balanced and the four terminals in \( \{s_1, t_1, s_2, x \} \) do not form an inadmissible configuration equivalent to \( A_0, B_0, \) or \( C_0 \). Also, if \( c(s_1) = c(t_1) \) and \( s_2 \in R_0 \cup R_{m-1}, \) then there exist two disjoint \( s_1–t_1 \) and \( s_2–x \) paths that cover all the vertices of \( G \) for \( x = s_2 \), because \( G \) is 1-fault Hamiltonian-laceable by Lemma 2. An inspection of the three inadmissible configurations each leads to that two disjoint \( s_1–t_1 \) and \( s_2–x \) paths exist, provided \( \{s_1, t_1, s_2, x \} \) is balanced, for every vertex \( x \) in one boundary row and at least one vertex \( x \) in the other boundary row, as required. Analogously, we can prove the theorem in each of the remaining three cases from Lemmas 4 through 6, and Lemma 2. Note that if the inadmissible configuration \( Z \) is not equal to \( F_3 \) (where \( m = 3 \) & even \( n \geq 6 \)), there exist required disjoint paths, \( s_1–t_1 \) and \( s_2–x \) paths, for every vertex \( x \) in one boundary row and at least one vertex \( x \) in the other boundary row such that \( \{s_1, t_1, s_2, x \} \) is balanced; otherwise, the required disjoint paths exist for every vertex \( x \) except one in one boundary row and at least two vertices \( x \) in the other boundary row such that \( \{s_1, t_1, s_2, x \} \) is balanced. This completes the proof. \( \square \)

Remark 2. The number of such vertices \( x \) in Theorem 4 is at least \( \frac{m}{2} + 1 \).

Theorem 5. For distinct terminals \( s_1, s_2, s_3 \in S \) and \( t_1 \in T \) in an \( m \times n \) cylindrical grid \( G \) with \( m \geq 2 \) and even \( n \geq 4 \) such that not all the four are of the same color, there exist vertices \( x \) and \( y \) in the boundary rows, possibly \( x = s_2 \) and/or \( y = s_3 \), such that \( G \) has three disjoint paths, \( s_1–t_1, s_2–x \), and \( s_3–y \) paths, that altogether cover all the vertices of \( G \).

Proof. The proof will proceed by induction on \( m \). Let \( m = 2 \) for the base step, where the two rows of \( G \) are both boundary ones. If \( c(s_2) \neq c(s_3) \), then a Hamiltonian \( s_2–s_3 \) path exists in \( G \) since \( G \) is 1-fault Hamiltonian-laceable by Lemma 2. It suffices to divide the Hamiltonian path, represented as \( \{s_2, \ldots, x, s_1', t_1', y, \ldots, s_3 \} \), where \( \{s_1', t_1' \ldots, y, \ldots, s_3 \} \) is the predecessor of \( s_1' \), and \( y \) is the successor of \( t_1' \), into three subpaths: \( \{s_2, \ldots, x \}, \{s_1', t_1' \}, \{y, \ldots, s_3 \} \). If \( c(s_2) = c(s_3) \), then \( c(s_3) \neq c(s_2) \) or \( c(t_1') \neq c(s_1') \), so we assume w.l.o.g. \( c(s_3) \neq c(s_2) \). Then, there exists a Hamiltonian \( s_2–s_3 \) path in \( G \) by Lemma 2. For a neighbor \( v \) of \( s_1 \) other than \( s_2 \) and \( s_3 \), the Hamiltonian path can be represented as \( \{s_2, \ldots, x, v', \ldots, t_1', y, \ldots, s_3 \} \), where \( \{v', t_1' \ldots, v, \ldots, t_1 \} \). It suffices to divide the Hamiltonian path into three subpaths, \( \{s_2, \ldots, x \}, \{v', \ldots, t_1', y, \ldots, s_3 \} \), and combine the one-vertex path \( \{s_3 \} \) with the second subpath through the edge \( (s_1, v) \).

Let \( m \geq 3 \) for the inductive step. We assume w.l.o.g. that \( R_0 \) contains no fewer terminals than \( R_{m-1}, \) i.e., \( |R_0 \cap (S^c)T| \geq |R_{m-1} \cap (S^c)T| \). There are several cases depending on the distribution of terminals.

Case 1: There is a boundary row that contains no terminal, i.e., \( R_{m-1} \cap (S^c)T = \emptyset \). By the induction hypothesis, there are two vertices \( x, y \in R_0 \cup R_{m-2} \) that admit three disjoint \( s_1–t_1, s_2–x \), and \( s_3–y \) paths that cover all the vertices of the subgraph \( G[R_0 \cup R_{m-2}] \) induced by \( R_0 \cup R_{m-2} \). If exactly one of \( x \) and \( y \) is contained in \( R_{m-2} \), say \( x \in R_0 \) and \( y \in R_{m-2} \), it suffices to extend the \( s_3–y \) path to cover the vertices of \( R_{m-1} \), i.e., concatenate the \( s_3–y \) path and a Hamiltonian \( w–y' \) path of the subgraph \( G[R_{m-1}] \) induced by \( R_{m-1} \) for the neighbor \( w \in R_{m-1} \) of \( y \) and a neighbor \( y' \in R_{m-1} \) of \( w \). If \( x, y \in R_{m-2} \), then it suffices to extend the \( s_2–x \) and \( s_3–y \) paths to cover the vertices of \( R_{m-1} \). That is, for the neighbor \( u \in R_{m-1} \) of \( x \) and \( w \in R_{m-1} \) of \( y \), we extract two disjoint \( u–x' \) and \( w–y' \) paths from a Hamiltonian cycle of \( G[R_{m-1}] \), and then concatenate the \( s_2–x \) and \( u–x' \) paths and concatenate again the \( s_3–y \) and \( w–y' \) paths.

Finally, suppose \( x, y \notin R_{m-2} \), i.e., \( x, y \in R_0 \). If there is a nonterminal vertex \( v \) in \( R_{m-2} \), i.e., \( v \notin \{s_1, t_1, s_2, s_3 \} \), then one of the three disjoint paths, \( s_1–t_1, s_2–x \), and \( s_3–y \) paths, of \( G[R_0 \cup R_{m-2}] \) passes through \( v \), hence passes through an edge \( (v, w) \) of \( G[R_{m-2}] \). It suffices to reroute the path, instead of passing through the edge \( (v, w) \), to traverse a Hamiltonian \( v'–w' \) path of \( G[R_{m-1}] \) for the neighbors \( v', w' \in R_{m-1} \) of \( v \) and \( w \), respectively. Now, let every vertex in \( R_{m-2} \) be a terminal, i.e., \( R_{m-2} = \{s_1, t_1, s_2, s_3 \} \) and \( n = 4 \). For the neighbors \( s_1' \ldots, t_1' \), \( s_2 \in R_{m-3} \), respectively, of \( s_1, t_1, \) and \( s_2 \), there are two disjoint \( s_1'–t_1' \) and \( s_2–x \) paths for some \( x \in R_0 \) that cover \( G[R_0 \cup R_{m-3}] \). (The existence is by Theorem 4 if \( m \geq 4 \); the existence is obvious if
m = 3.) It suffices to concatenate the one-vertex path \( \langle s_1 \rangle \), the \( s_1-t_1 \) path, and \( (t_1) \) into an \( s_1-t_1 \) path, concatenate again the one-vertex path \( \langle s_2 \rangle \) and the \( s_2-x \) path, and extend \( \langle s_3 \rangle \) to cover \( R_{m-1} \).

**Case 2:** There is a boundary row, say \( R_{m-1} \), that contains a single terminal in \( \{ s_2, s_3 \} \), say \( s_3 \), whose color is the same as at least one of the other terminals. That is, \( R_{m-1} \cap (S \cup T) = \{ s_3 \} \) and the three terminals \( s_1, t_1, s_2 \in R_{m-2} \) are not of the same color. Then, for some \( x \in R_0 \), there exist disjoint \( s_1-t_1 \) and \( s_2-x \) paths that cover \( G[R_{m-2}] \) by Theorem 4. It suffices to build a Hamiltonian \( s_3-y \) path of \( G[R_{m-1}] \) for some \( y \).

**Case 3:** \( R_0 \cap (S \cup T) = \{ s_1, t_1, s_2 \} \). Assume w.l.o.g. that the three terminals in \( \{ s_1, t_1, s_2 \} \) are not of the same color. It suffices to divide the Hamiltonian cycle \( \langle s_1, u, s_2, x, s_3, v, \ldots, v \rangle \) of \( G[R_0] \) into three paths \( \langle s_1, u, \ldots, u \rangle, \langle s_2, \ldots, x \rangle \), and \( \langle s_3, \ldots, v \rangle \), and then build two disjoint \( u-t_1 \) and \( v'-y \) paths that cover \( G[R_{m-1}] \) for some \( y \in R_{m-1} \). where \( u', v' \in R_1 \) are the neighbors of \( u \) and \( v \), respectively. Note that \( c(u') = c(s_2) \) and \( c(v') = c(s_1) \), meaning the three vertices of \( \{ u', v', t_1 \} \) are not of the same color.

**Case 4:** \( R_0 \cap (S \cup T) = \{ s_1, t_1, s_2 \} \). From the hypotheses of Cases 1 and 2, we can assume that \( s_3 \in R_{m-1} \) and \( c(s_1) = c(t_1) = c(s_2) \neq c(s_3) \). The proof is similar to that of Case 3. Dividing the Hamiltonian cycle \( \langle s_1, u, t_1, v, s_2, x, \ldots, v \rangle \) of \( G[R_0] \) into three paths \( \langle s_1, u, \ldots, u \rangle, \langle t_1, \ldots, v \rangle, \langle s_2, \ldots, x \rangle \) paths and building two disjoint \( u-t_1 \) and \( v-y \) paths that cover \( G[R_{m-1}] \) for some \( y \in R_{m-1} \) and then two disjoint \( u-t_1 \) and \( s_3-y \) paths that cover \( G[R_{m-1}] \) for some \( y \in R_{m-1} \). Finally, suppose \( i = j \). Let \( s_1, t_1, s_2, \) and \( s_3 \), respectively, be contained in columns \( C_{p}, C_{q}, \) and \( C_r \). Assume w.l.o.g. \( p \leq r \leq q = 0 \).

**Claim 1:** There exist three disjoint \( s_1-t_1, s_2-u, \) and \( v-x \) paths that cover \( G[R_{0i}], \) where \( u = v_1, v = v_{n-1}, \) and \( x = v_{p+1} \). Furthermore, each of the \( n \) edges \( \langle v_a, v_{a+1} \rangle \) for odd \( a \in \{1, 3, \ldots, n-3 \} \) is visited by one of the three paths.

**Proof of Claim 1:** It holds true that \( c(u) = c(v) = c(x) \neq c(s_1) = c(t_1) = c(s_2) \). If \( i \) is even, then \( R_{0i} \) has an odd number of rows, so possibly \( p \in \{0, \tau \} \); if \( i \) is odd, then \( R_{0i} \) has an even number of rows, so \( 0 < p < \tau \). (Refer to Fig. 2.) An \( s_1-t_1 \) path is obtained by concatenating a Hamiltonian \( s_1-v_{n-1}^0 \) path of \( G[R_{0i-1} \cap C_{p}], \) and the one-vertex path \( \langle t_1 \rangle \) set an \( s_2-u \) path be \( \langle v_1, v_{n-1}, \ldots, v_1 \rangle \); in addition, a \( v-x \) path is obtained from concatenating a Hamiltonian \( v_{n-1}^0-v_{n-2} \) path of \( G[R_{0i} \cap C_{p-2}], \ldots, a \) Hamiltonian \( v_{n-4}^0-v_{n-2} \) path of \( G[R_{0i} \cap C_{p+2}}, \) the one-vertex path \( \langle u_{r+1} \rangle \), and a Hamiltonian \( v_{n-1}^0-v_{p+1}^0 \) path of \( G[R_{0i} \cap C_{p+1}], \ldots, a \) Hamiltonian \( v_{n-4}^0-v_{p+1}^0 \) path of \( G[R_{0i} \cap C_{p+1} \rangle \). The existence of the Hamiltonian paths.
in the induced subgraphs that are isomorphic to rectangular grids is due to Lemma 1(a). Thus, the claim is proven.

Let \( u', v' \in R_{i+1} \) be the neighbors of \( u \) and \( v \), respectively. If \( i \leq m - 3 \), it suffices to build two disjoint \( u'-v' \) and \( s_3-y \) paths that cover \( G[R_{i+1,m-1}] \) for some \( y \in R_{m-1} \), which exist by Theorem 4, and combine them with the three disjoint paths of Claim 1. So, let \( i = m - 2 \) now, where \( u' = v^{i-1}_m \), \( v' = v^{i-1}_m \), and \( s_3 = v^{i-1}_b \) for some even \( b \in \{0, \ldots, n-2\} \) because \( c(s_3) \neq c(u') = c(v') \). If \( b = 0 \), it suffices to set \( s_3-y \) and \( u'-v' \) paths be \( \langle v^{i-1}_0 \rangle \) and \( \langle v^{i-1}_1, v^{i-1}_2, \ldots, v^{i-1}_{m-1} \rangle \), respectively, and combine the two with the three paths of Claim 1. If \( b \geq 2 \), we set an \( s_3-y \) path be \( \langle v^{i-1}_b, v^{i-1}_{b+1}, \ldots, v^{i-1}_{m-1} \rangle \) and set a \( u'-v' \) path be \( \langle u', v^{i-1}_{m-1}, v' \rangle \). To deal with the vertices \( v^{i-1}_{b+1}, \ldots, v^{i-1}_{m-2} \) not visited till now, we use the fact shown in Claim 1 that every edge \( \langle v^3, v^4 \rangle \) for odd \( a \in \{1, 3, \ldots, n-3\} \) is visited by one of the three disjoint paths of \( G[R_0] \). To cover each pair of unvisited vertices \( v^{i-1}_{a+1} \) and \( v^{i+1}_{a+1} \) for odd \( c \in \{b+1, \ldots, n-3\} \), it suffices to reroute the path that visits the edge \( \langle v^{i-1}_{a}, v^{i-1}_{a+1} \rangle \) to traverse \( \langle v^{i-1}_{a}, v^{i-1}_{a+1}, v^{i+1}_1, v^{i+1}_{a+1} \rangle \).

**Case 10:** \( R_0 \cap (S \cup T) = \{s_1\} \) and \( R_{m-1} \cap (S \cup T) = \{t_1\} \). Let \( s_2 \in R_i \) and \( s_3 \in R_j \) for some \( i, j \in \{1, \ldots, m-2\} \). Assume w.l.o.g. that the three terminals \( t_1, s_2 \), and \( s_3 \) are not of the same color. If \( i < j \), we first pick up an edge \( (u,v) \) with \( u \in R_i \) and \( v \in R_{i+1} \) such that \( c(u) \neq c(s_2) \) and \( v \neq s_3 \). Then, the three vertices of \( \{s_1, s_2, u\} \) are not of the same color; also, the three vertices of \( \{t_1, s_3, v\} \) are not of the same color because \( c(v) = c(s_3) \). It suffices to build two disjoint \( s_1-x \) and \( s_2-x \) paths that cover \( G[R_{i+1}] \) for some \( x \in R_0 \), and combine them with the two disjoint \( v-t_1 \) and \( s_3-y \) paths that cover \( G[R_{i+1,m-1}] \) for some \( y \in R_{m-1} \). The case where \( j < i \) is symmetric to the case where \( i < j \), so we consider the remaining case where \( i = j \) hereafter.

**Claim 2.** There exists an edge \((u,v)\) of \( G[R_1] \) with \((v,s_1) \notin E(G) \) such that for some \( y \in R_{m-1} \), the subgraph \( G[R_{i+1,m-1}] \) contains three disjoint paths, composed of

- either \( u-t_1, s_2-v, \) and \( s_3-y \) paths, or
- \( u-t_1, s_3-v, \) and \( s_2-y \) paths,

that cover all the vertices of \( G[R_{m-1}] \).

**Proof of Claim 2.** If \( i \leq m - 3 \), then \( G[R_{i+1,m-1}] \) contains three or more rows. For an edge \((u,v)\) of \( G[R_1] \) with \( u \notin \{s_2, s_3\} \) and \((v,s_1) \notin E(G) \), it suffices to decompose the Hamiltonian cycle of \( G[R_1] \), represented as \( \langle u, \ldots, w, s_2, \ldots, v, \rangle \), into three paths \( \langle u, \ldots, w, \rangle, \langle s_3, \ldots, z, \rangle, \) and \( \langle s_2, \ldots, v, \rangle \), and then build disjoint \( u'-v' \) and \( z'-y \) paths that cover \( G[R_{i+1,m-1}] \) for some \( y \in R_{m-1} \), where \( u', z' \in R_{i+1} \) are the neighbors of \( w \) and \( z \), respectively. Note that \( c(u') = c(s_3) \) and \( c(z') = c(s_2) \), so the vertices of \( \{t_1, u', z'\} \) are not of the same color. Now, suppose \( i = m-2 \), where \( G[R_{m-1}] \) contains exactly two rows. Let \( t_1 \in C_p, s_2 \in C_q, \) and \( s_3 \in C_r \) for some \( p, q, r \in \{0, \ldots, n-1\} \).

For the first case, suppose \( c(s_2) = c(s_3) \), so \( c(s_2) = c(s_3) \neq c(t_1) \) from our assumption. We further assume w.l.o.g. that \( q < p \leq r \) and \( r = n - 1 \). (See Fig. 3 (a) through (e).) If \( p \neq n - 1 \), it suffices to set an \( s_2-y \) path be a Hamiltonian \( s_2-v_{p-1}^m \) path of \( G[R_{m-2,m-1} \cap C_{0,1}] \), and then decompose the \( s_3-t_1 \) path, built by concatenating a Hamiltonian \( s_3-v_{p+1}^{m-1} \) path of \( G[R_{m-2,m-1} \cap C_{p+1,n-1}] \) and a Hamiltonian \( v_{p-1}^m-t_1 \) path of \( G[R_{m-2,m-1} \cap C_{p+1,n+1}] \), by deleting an edge \((u,v) = (v_{p-1}^m, v_{p+1}^m) \) or \((v_{p-1}^m, v_{p+1}^m) \) so that \((v,s_1) \notin E(G) \). If \( p = n - 1 \), the required three paths are obtained in one of the following two ways: (i) set an \( s_2-y \) path be a Hamiltonian \( s_2-v_{n-1}^m \) path of \( G[R_{m-2,m-1} \cap C_{0,1}] \), and then decompose the Hamiltonian \( s_3-t_1 \) path of \( G[R_{m-2,m-1} \cap C_{q+1,n-1}] \) through \((u,v) = (v_{n-1}^m, v_{n-1}^m) \) or (ii) concatenate \( (s_3) \), a Hamiltonian \( v_{n-1}^m-v_{n-1}^m \) path of \( G[R_{m-2,m-1} \cap C_{0,2}] \), and \( (v_{n-1}^m) \) into an \( s_3-y \) path, and then decompose the \( s_2-t_1 \) path, built by concatenating \( (s_2) \), a Hamiltonian \( v_{n-1}^m-v_{n-1}^m \) path of \( G[R_{m-2,m-1} \cap C_{q+1,n-2}] \), and \( (t_1) \), through \((u,v) = (v_{n-2}^m, v_{n-2}^m) \).

For the second case, suppose \( c(s_2) \neq c(s_3) \). Assume w.l.o.g. that \( c(t_1) = c(s_2) \neq c(s_3) \) and moreover, \( q < p \leq r = n - 1 \). (See Fig. 3 (d) through (j).) If \( p \neq n - 1 \), the required three paths are obtained in one of the following two ways: (i) set an \( s_2-y \) path be a Hamiltonian \( s_2-v_{p-1}^m \) path of \( G[R_{m-2,m-1} \cap C_{0,p}] \), and then decompose the Hamiltonian \( s_3-t_1 \) path of \( G[R_{m-2,m-1} \cap C_{p,n-1}] \) through \((u,v) = (v_{p-1}^m, v_{p-1}^m) \) or (ii) concatenate \( (s_3) \), a Hamiltonian \( s_3-c_{q-1} \) path of \( G[R_{m-2,m-1} \cap C_{q-1}] \) and \((v_{p+1}^m) \) into an \( s_3-y \) path, and then decompose the \( s_2-t_1 \) path, built by concatenating \( (s_2) \) and a Hamiltonian \( v_{q+1}^m-t_1 \).
path of $G[R_{m-2,m-1} \cap C_{q+1,p}]$, through $(u, v) = (v_{q+1}^{n-2}, v_q^{m-2})$. If $p = n - 1$, assuming w.l.o.g. $q \neq n - 2$, it suffices to set an $s_2$-$y$ path be a Hamiltonian $s_2$-$v_{0,q}^{m-1}$ path of $G[R_{m-2,m-1} \cap C_{0,q}]$ and then decompose the Hamiltonian $s_3$-$t_1$ path of $G[R_{m-2,m-1} \cap C_{q+1,n-1}]$ through an edge $(u, v) = (v_{n-1}^{n-2}, v_{n-2}^{m-2})$ or $(v_{n-2}^{n-2}, v_{n-2}^{m-2})$. Thus, the claim is proven. 

Let $u', v' \in R_{m-1}$ be the neighbors of $u$ and $v$, respectively. It remains to build two disjoint $s_1$-$u'$ and $v'$-$x$ paths that cover $G[R_{0,1}]$ for some $x \in R_0$. If $i \geq 2$, the two disjoint paths exist by Theorem 4; if $i = 1$, dividing the Hamiltonian cycle $(s_1, \ldots, u', v', \ldots, x)$, where $v' \neq s_1$, of $G[R_0]$ results in two paths $(s_1, \ldots, u')$ and $(v', \ldots, x)$, as required. If we combine the two paths of $G[R_{0,1}]$ with the three paths of Claim 2, we obtain the required three paths that cover $G$. This completes the entire proof. 

**Remark 3.** If distinct terminals $s_1, s_2 \in S$ and $t_1, t_2 \in T$ (instead of $s_1, s_2, s_3 \in S$ and $t_1, t_2 \in T$) are given in an $m \times n$ cylindrical grid with $m \geq 2$ and even $n \geq 4$, then there exist three disjoint paths, $s_1$-$t_1, s_2$-$x$, and $t_2$-$y$ paths (instead of $s_1$-$t_1, s_2$-$x$, and $s_3$-$y$ paths), that altogether cover all the vertices.

**IV. PAIRED 3-DPC IN BIPARTITE TOROIDAL GRIDS**

In this section, we will show that every $m \times n$ bipartite toroidal grid with $(m, n) \neq (4, 4)$ has a paired 3-DPC joining $S$ and $T$ for any disjoint source and sink sets $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ such that $S \cup T$ is balanced. The $6 \times 4$ and $6 \times 6$ toroidal grids admit a paired 3-DPC joining $S$ and $T$ for any such terminal sets $S$ and $T$, while the $4 \times 4$ toroidal grid does not, as shown in Fig. 4. Lemma 7 below was verified from a computer program that exhaustively searches for DPCs. The source code may be downloaded from [http://tcs.catholic.ac.kr/~jhpark/papers/toroidal_grid.zip](http://tcs.catholic.ac.kr/~jhpark/papers/toroidal_grid.zip). 

**Lemma 7.** Let $G$ be a $6 \times 4$ or $6 \times 6$ toroidal grid, in which disjoint source and sink sets $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ are given. Then, $G$ has a paired 3-DPC joining $S$ and $T$ if $S \cup T$ is balanced.

One of the natural approaches would be reduction of our problem to a problem on a smaller bipartite toroidal grid. This is possible if there are two consecutive rows that contain no terminal as follows:

**Lemma 8 (Row reduction).** An $m \times n$ bipartite toroidal grid $G$ with $m \geq 6$ has a paired 3-DPC joining $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ if (i) $S \cup T$ is balanced, (ii) there are two consecutive rows $R_p$ and $R_{p+1}$ that contain no terminal, and (iii) an $(m - 2) \times n$ toroidal grid has a paired 3-DPC joining $S'$ and $T'$ for any disjoint terminal sets $S'$ and $T'$ such that $S' \cup T'$ is balanced.

**Proof.** Let $H$ denote the $(m - 2) \times n$ toroidal grid, obtained from $G$ by deleting the vertices of $R_{p,p+1}$ and adding $n$ virtual edges $(v_j^{p-1}, v_j^{p+2})$ for $j \in \{0, \ldots, n - 1\}$, as shown in Fig. 5(a). Then, by the hypothesis (iii) of the lemma, $H$ has a paired 3-DPC joining $S$ and $T$. If none of the virtual edges is passed through by a path in the 3-DPC of $H$ (see Fig. 5(b)), then for an edge in row $p + 1$ or $p + 2$, say $(v_j^{p-1}, v_j^{p+1})$ w.l.o.g., that is covered by the 3-DPC of $H$, replacing the edge with a path obtained by concatenating $(v_j^{p-1})$, a Hamiltonian $v_j^{p-1}$-$v_j^{p+1}$ path of $G[R_{p,p+1}]$, and $(v_j^{p+1})$ results in a paired 3-DPC of $G$. Now, suppose that there is a virtual edge that is covered by the 3-DPC of $H$ (see Fig. 5(c)). Let $\{(v_j^{p-1}, v_j^{p+2}) : j \in \{j_1, \ldots, j_q\}\}$ be the set of such virtual edges, and assume $j_1 < \cdots < j_q$. A paired 3-DPC of $G$ can be built by replacing the virtual edge $(v_{j_i}^{p-1}, v_{j_i}^{p+2})$ with a path obtained by concatenating $(v_{j_i}^{p-1})$, a Hamiltonian $v_{j_i}^{p-1}$-$v_{j_i}^{p+1}$ path of $G[R_{p,p+1} \cap (C_{j_i,j_{i+1}-1}]$, and $(v_{j_i}^{p+2})$ if $i < q$; with a path obtained by concatenating $(v_{j_q}^{p-1})$, a Hamiltonian $v_{j_q}^{p-1}$-$v_{j_q}^{p+1}$ path of $G[R_{p,p+1} \cap (C_{j_q,j_{q+1}})]$, and $(v_{j_q}^{p+2})$ if $i = q$. Thus, the lemma is proven. 

![Fig. 4: A configuration that does not admit a paired 3-DPC. Every $s_i$-$t_i$ path that does not pass through a terminal as an intermediate vertex contains at least 6 vertices, whereas the toroidal grid has less than 18 vertices.](http://jcse.kiise.org)
An $m \times n$ bipartite toroidal grid with $m \geq 6$ is said to be row-reducible if there are two consecutive rows $R_p$ and $R_{p+1}$ that contain no terminal. Besides the row reduction of Lemma 8, we can try a partition of the $m \times n$ toroidal grid into two cylindrical grids each having at least two rows, so as to build a paired 3-DPC in the toroidal grid. Three types of such partitions are investigated in Lemmas 9, 10, and 11 below and illustrated in Fig. 6.

**Lemma 9 (Type-A partition).** An $m \times n$ bipartite toroidal grid $G$ has a paired 3-DPC joining $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ if $S \cup T$ is balanced and there are $r$, $2 \leq r \leq m - 2$, consecutive rows $R_p, \ldots, R_{p+r-1}$ that contain four terminals $s_a, t_a, s_b$, and $t_b$ for some $a, b \in \{1, 2, 3\}$ in total such that the subgraph $G[R_{p, p+r-1}]$ induced by $R_{p, p+r-1}$ has a paired 2-DPC composed of $s_a-t_a$ and $s_b-t_b$ paths.

**Proof.** The subgraph $G - R_{p, p+r-1}$ contains two terminals $s_c$ and $t_c$ with $c(s_c) \neq c(t_c)$, so there exists a Hamiltonian $s_c-t_c$ path in the subgraph by Lemma 2. A paired 2-DPC of $G[R_{p, p+r-1}]$ along with the Hamiltonian path form a paired 3-DPC of $G$. □

**Lemma 10 (Type-B partition).** An $m \times n$ bipartite toroidal grid $G$ has a paired 3-DPC joining $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ if $S \cup T$ is balanced and there are $r$, $2 \leq r \leq m - 2$, consecutive rows $R_p, \ldots, R_{p+r-1}$ that contain three terminals $s_a, t_a, s_b$ and $t_b$ for some $a, b \in \{1, 2, 3\}$ in total such that the three are not of the same color.

**Proof.** In the subgraph $G[R_{p, p+r-1}]$, there are two disjoint $s_a-t_a$ and $s_b-x$ paths for some $x \in R_p \cup R_{p+r-1}$ that cover all the vertices of the subgraph; moreover, the number of such vertices $x$ is at least $\frac{r}{2} + 1$ by Theorem 4. Consider the subgraph $H$ of $G$ induced by $R_{p-1} \cup R_{p+r-1}$ now (i.e., $H = G - R_{p, p+r-1}$), in which there are three terminals $s_c$, $t_c$, and $t_b$ for some $c \in \{1, 2, 3\}$ with $c \neq a, b$. Also, the three terminals of $H$ are not of the same color, so there exist two disjoint $s_c-t_a$ and $t_b-y$ paths that cover $H$ for at least $\frac{r}{2} + 1$ choices of $x \in R_{p-1} \cup R_{p+r-1}$ by Theorem 4 again. It follows that there is an edge $(x, y)$ of $G$, where $x \in R_p \cup R_{p+r-1}$ and $y \in R_{p-1} \cup R_{p+r-1}$, that admits not only a 2-DPC, made of $s_a-t_a$ and $s_b-x$ paths, of $G[R_{p, p+r-1}]$ but also a 2-DPC, made of $s_a-t_a$ and $t_b-y$ paths, of $H$, because $c(x) \neq c(y)$ and there are at least $\frac{r}{2} + 1$ choices of $x$ and $y$ each. It suffices to combine the $s_b-x$ path with the $t_b-y$ path into an $s_b-t_b$ path through the edge $(x, y)$, completing the proof. □

**Lemma 11 (Type-C partition).** An $m \times n$ bipartite toroidal grid $G$ has a paired 3-DPC joining $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ if $S \cup T$ is balanced, $G$ is not row-reducible, and there are $r$, $2 \leq r \leq m - 2$, consecutive rows $R_p, \ldots, R_{p+r-1}$ that contain two terminals $\alpha$ and $\beta$ in total such that

- $c(\alpha) = c(\beta)$ or $\alpha, \beta \notin R_p \cup R_{p+r-1}$ when $r \geq 4$,
- $c(\alpha) = c(\beta)$ or $(\alpha, \beta) \notin R_{p+1}$ or $c(\alpha) = c(\beta)$ or $(\alpha, \beta) \notin K$ and $(\alpha, \beta) \neq \{(\alpha, \beta) \cap R_p\} = 1$
- or $c(\alpha) = c(\beta)$ or $(\alpha, \beta) \notin K$ and $\alpha, \beta \in R_{p+1}$
- or $c(\alpha) \neq c(\beta)$ or $(\alpha, \beta) \notin K$ and $\alpha, \beta \in R_{p+1}$ and $(\alpha, \beta) \notin E(G)$ when $r = 3$,
• $c(\alpha) = c(\beta)$ & $(\alpha, \beta) \not\in K$ & $|\{\alpha, \beta\} \cap R_p| = 1$ when $r = 2,$

where $K = \{(s_1, t_1), (s_2, t_2), (s_3, t_3)\}.$

Proof. Let $H$ be the subgraph $G - R_{p+r-1}$ induced by $R_{0,p+1} \cup R_{p+r,n-1},$ in which there are four terminals, say $s_a, t_a, \alpha',$ and $\beta'$ for some $a \in \{1, 2, 3\},$ so that $S \cup T = \{s_a, t_a, \alpha', \beta', \alpha, \beta\},$ where $(\alpha, \alpha'), (\beta, \beta') \in K, \text{ or } (\alpha, \beta), (\alpha', \beta') \in K.$ or $(\alpha', \beta') \in K.$ The four terminals of $H$ are not of the same color since $S \cup T$ is balanced. So, from Theorem 5, there exist three disjoint $s_a-t_a, \alpha'-x,$ and $\beta'-y$ paths that cover $H$ for some $x, y \in R_{p+1} \cup R_{p+r}.$ Let $x', y' \in R_p \cup R_{p+r-1}$ be the neighbors of $x$ and $y,$ respectively.

CLAIM 3. For the two terminals $\alpha$ and $\beta$ of $G[R_{p+r-1}],$ satisfying the hypothesis of the lemma, (i) $\{x', y'\} \cap \{\alpha, \beta\} = \emptyset$; moreover, (ii) $G[R_{p+r-1}],$ has three kinds of a paired 2-DPC, a DPC made of $\alpha-x'$ and $\beta-y'$ paths, a DPC made of $\alpha-y'$ and $\beta-x'$ paths, and a DPC made of $\alpha-\beta$ and $x'-y'$ paths.

Proof of Claim 3. Within the scope of this proof, $x'$ and $y'$ as well as $\alpha$ and $\beta$ are said to be terminals. Observing that $\{\alpha, \beta, x', y'\}$ is balanced, we prove the assertion (i) first. If $c(\alpha) = c(\beta),$ then $c(x') = c(y') \neq c(\alpha) = c(\beta),$ so $\{x', y'\} \cap \{\alpha, \beta\} = \emptyset$; if $\alpha, \beta \not\in R_p \cup R_{p+r-1},$ then $\{x', y'\} \cap \{\alpha, \beta\} = \emptyset$ obviously. An inspection of the hypothesis of the lemma leads to $c(\alpha) = c(\beta)$ or $\alpha, \beta \not\in R_p \cup R_{p+r-1},$ proving (i). For the proof of the assertion (ii), let $\alpha \in R_i$ and $\beta \in R_j$ for some $i, j \in \{p, \ldots, p + r - 1\}.$ Firstly, let $r \geq 4.$ It follows that $i \neq j$ and $i, j \neq \{p, p + r - 1\};$ suppose otherwise, $G$ would be row-reducible. This leads to that there is a (non-boundary) row that contains a single terminal, meaning the required 2-DPCs exist by Lemmas 3 and 4 (also, by Remark 1). Secondly, let $r = 3.$ If $|\{\alpha, \beta\} \cap R_{p+1}| = 1,$ then $R_{p+1}$ contains a single terminal, so the required 2-DPCs exist. If $|\{\alpha, \beta\} \cap R_p| = |\{\alpha, \beta\} \cap R_{p+2}| = 1,$ $c(\alpha) = c(\beta),$ and $(\alpha, \beta) \in K,$ then the four terminals in $\{\alpha, \beta, x', y'\}$ cannot form an inadmissible configuration of Lemmas 4 and 6, so the required 2-DPCs exist. Analogously, we can see that the required 2-DPCs exist for the remaining two cases where $\alpha, \beta \in R_{p+1}.$ Finally, let $r = 2.$ If $c(\alpha) = c(\beta),$ $(\alpha, \beta) \not\in K,$ and $i \neq j$ (i.e., $|\{\alpha, \beta\} \cap R_p| = |\{\alpha, \beta\} \cap R_{p+1}| = 1,$ then the four terminals in $\{\alpha, \beta, x', y'\}$ cannot form an inadmissible configuration of Lemmas 4 and 5, so the required 2-DPCs exist. Thus, the claim is proven.

Combining the $\alpha'-x$ and $\beta'-y$ paths of $H$ with one of the three paired 2-DPCs of $G[R_{p+r-1}],$ through the edges $(x, x')$ and $(y, y')$ leads to a paired 3-DPC of $G,$ as required. This completes the proof.

Now, we are ready to prove our main theorem.

THEOREM 6. An $m \times n$ bipartite toroidal grid $G$ with $(m, n) \neq (4, 4)$ has a paired 3-DPC joining disjoint terminal sets $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ if and only if $S \cup T$ is balanced.

Proof. The necessity part is straightforward from the fact that the two color classes of $G$ are always the same in size. The sufficiency proof will proceed by induction on $m + n,$ where $m$ and $n$ are both even integers with $m, n \geq 4$ and $m + n \geq 10.$ Assume w.l.o.g. $m \geq n.$ The base step of $(m, n) = (6, 4)$ is due to Lemma 7. Moreover, the theorem holds true for the case of $(m, n) = (6, 6)$ by Lemma 7 again, so we assume $m \geq 8$ for the inductive step. Keep in mind that if $G$ is row-reducible, then $G$ has a paired 3-DPC joining $S$ and $T$ by Lemma 8 because by the induction hypothesis, an $(m - 2) \times n$ bipartite toroidal grid has a paired 3-DPC joining any disjoint terminal sets $S'$ and $T'$ of size 3 each such that $S' \cup T'$ is balanced. We assume w.l.o.g. that $R_0$ contains as many terminals as the other rows, i.e., $|R_0 \cap (S \cup T)| \geq |R_0 \cap (S \cap T)|$ for all $i \in \{1, \ldots, m - 1\}.$ There are three cases according to the size of $R_0 \cap (S \cup T)$.

Case 1: $|R_0 \cap (S \cup T)| \geq 3.$ The $m - 1 \geq 7$ rows other than $R_0$ contain 3 or less terminals in total, so (i) $G$ is row-reducible, or (ii) $m = 8$ and the three rows $R_2, R_4,$ and $R_6$ each contains a single terminal. For possibility (i), $G$ has a paired 3-DPC joining $S$ and $T$ by the induction hypothesis and Lemma 8; for possibility (ii), $G$ admits a type-C partition w.r.t. $R_{1,5},$ and hence $G$ has a paired 3-DPC joining $S$ and $T$ by Lemma 11.

Case 2: $|R_0 \cap (S \cup T)| = 2.$

Case 2.1: $|R_i \cap (S \cup T)| = 2$ for some $i \in \{1, \ldots, m - 1\}.$ In this case, there are at most three rows other than $R_0$ each of which contains a terminal. It follows that $G$ is row-reducible, or $m = 8$ and the three rows $R_2, R_4,$ and $R_6$ each contains a terminal. If $G$ is row-reducible, we are done by the induction hypothesis and Lemma 8. If $i = 2,$ i.e., $R_2$ contains two terminals, then $G$ has a paired 3-DPC joining $S$ and $T$ by Lemma 11 because $G$ admits a type-C partition w.r.t. $R_{3,7};$ symmetrically in case of $i = 6,$ $G$ is also type-C-partitionable. Let $i = 4$ now. There are two possibilities: (i) $R_0 \cap (S \cup T) = \{s_a, t_a\}$ for some $a,$ and (ii) $R_0 \cap (S \cup T) \neq \{s_a, t_a\}$ for all $a.$

For the first possibility, suppose $s_a, t_a \in R_0.$ If $c(s_a) \neq c(t_a),$ then $G$ admits a type-A partition w.r.t. $R_{2,7},$ hence $G$ has a required 3-DPC by Lemma 9. (Note that the four terminals in $(S \cup T) - \{s_a, t_a\}$ do not form an inadmissible configuration in the induced subgraph $G[R_{2,7}].$)

For the second possibility, suppose $s_a, t_a \in R_0$ for some $a, b \in \{1, 2, 3\}$ with $a \neq b$ (or symmetrically,
For the two terminals, denoted α and β, in $R_q$, if $\{\alpha, \beta\} = \{s_a, t_b\}$ for some $c \in \{1, 2, 3\}$ with $c \neq a, b$, then a paired 3-DPC can be constructed in a symmetric way to the first possibility where $s_a, t_b \in R_0$. So, we assume $\{\alpha, \beta\} \neq \{s_a, t_b\}$. If either $c(s_a) = c(s_b)$ or $c(s_a) \neq c(s_b) \notin E(G)$, then $G$ admits a type-C partition w.r.t. $R_7 \cup R_{0,1}$, hence $G$ has a required 3-DPC by Lemma 11. Similarly, if either $c(\alpha) = c(\beta)$ or $c(\alpha) \neq c(\beta)$ and $c(\alpha), c(\beta) \notin E(G)$, then $G$ is type-C-partitionable w.r.t. $R_{3,5}$ and has a required 3-DPC. So, we further assume $(s_a, s_b), (\alpha, \beta) \notin E(G)$ and $c(s_a) \neq c(s_b)$ and $c(\alpha) \neq c(\beta)$. If $t_b \in R_2$ or $t_b \in R_2$, then $G$ is type-B-partitionable w.r.t. $R_{0,2}$ and thus $G$ has a required 3-DPC by Lemma 10; also, $G$ is type-B-partitionable w.r.t. $R_{0,7} \cup R_0$ if $t_a \in R_0$ or $t_b \in R_0$.

Finally, there remains a case where $t_a, t_b \in R_4$ and $s_a, t_b \in R_5 \cup R_6$, say $s_a \in R_5$ and $t_b \in R_6$, and moreover $(s_a, s_b), (t_a, t_b) \notin E(G)$ and $c(s_a) \neq c(t_b)$. None of the three types of a partition can be applied in this case, so we will devise a direct construction of a paired 3-DPC joining $S$ and $T$. We assume w.l.o.g. that $c(s_a) = c(s_b)$, $s_a = v^{0}_{n-2}$, and $s_b = v^{0}_{n-1}$, and let $t_b = v^{4}_{j}$ for some $j$. The construction will be completed in five steps as follows (see Fig. 7(a)):

1. Find a Hamiltonian $s_a - v^{0}_{n-2}$ path, $(v^0_{n-2}, \ldots, v^0_{i})$, in $G[R_0] - s_b$.
2. Let $x = v_{j+1}^3$ if $t_a \neq v_{j+1}^3$; let $x = v_{j-1}^3$ otherwise. For $s'_a = v_{n-1}^3$ and $t'_a = v_{j}^3$, find a paired 2-DPC composed of $s'_a - t'_a$ and $s_a - x$ paths in $G[R_{1,3}]$.
3. Let $s'_a$ be the neighbor of $x$ in $R_1$. Divide the Hamiltonian $s'_a - t'_a$ path of $G[R_4] = t_b$ into $s'_a - y$ and $z - t_a$ paths, by deleting an arbitrary edge $(y, z)$ of the Hamiltonian path.
4. Let $y'$ and $z'$, respectively, be the neighbors of $y$ and $z$ in $R_3$. Find a paired 2-DPC composed of $y' - t_a$ and $v^3_{n-2}$-paths in $G[R_{0,7}]$.
5. Concatenate the $s_a - t'_a$, $v^3_{j-2}$-$z'$, and $z - t_a$ paths results in an $s_a - t_a$ path; concatenating the one-vertex path $(s'_a)$, the $s'_a - t'_a$ path, and $(t_b)$ leads to an $s_a - t_b$ path; finally, concatenating the $s_a - x$, $s'_a - y$, and $y' - t_a$ paths leads to an $s_a - t_a$ path.

The paired 2-DPCs in Steps 2 and 4 exist due to Lemmas 4 and 6 (also, due to Remark 1).

**Case 2.2:** $|R_i \cap (S \cup T)| \leq 1$ for all $i \in \{1, \ldots, m-1\}$. There are exactly four rows other than $R_0$ each of which contains a terminal, so $G$ is row-reducible (and we are done) or $m \leq 10$. If $m = 10$, then the four rows $R_2, R_4, R_6$, and $R_8$ each contains a single terminal, hence $G$ admits a type-C partition w.r.t. $R_{1,5}$ and has a required 3-DPC by Lemma 11. Suppose $m = 8$ from now on. Let $r$ be the maximum number of consecutive rows, including $R_0$, each of which contains a terminal; also, let $R_{p}, \ldots, R_q$ denote the remaining $8 - r$ consecutive rows. (Note that $R_{p,q}$ contains $5 - r$ terminals; but $R_p$ and $R_q$ contain no terminal.) It follows that $r \leq 3$ because $G$ is not row-reducible. If $r = 3$, then $R_{p+1}$ and $R_{p+3}$ each contains a single terminal, hence $G$ admits a type-C-partition w.r.t. $R_{p,p+4}$ and has a required 3-DPC. If $r = 2$, then $R_{p+1}$ and $R_{p+4}$ each contains a single terminal; also, either $R_{p+2}$ or $R_{p+3}$ contains a single terminal. This leads to that $G$ is type-C-partitionable (w.r.t. $R_{p,p+3}$ for the former case and w.r.t. $R_{p+2,p+5}$ for the latter case) and has a required 3-DPC. Finally, if $r = 1$, then $R_2$ and $R_6$ each contains a single terminal; also, two of the three $R_3, R_4$, and $R_5$ each contains a single terminal. If $R_3$ and $R_5$ each contains a single terminal (but $R_4$ does not), then $G$ is type-C-partitionable w.r.t. $R_{1,4}$. So, we assume w.l.o.g. $R_4$ and $R_6$ each contains a single terminal, i.e., $|R_i \cap S| = 1$ for $i \in \{2, 4, 5, 6\}$.

Let $\alpha$ and $\beta$ denote the two terminals in $R_0$. Firstly, suppose $c(\alpha) = c(\beta)$. If $\{\alpha, \beta\} = \{s_a, t_a\}$ for some $a \in \{1, 2, 3\}$, then assuming w.l.o.g. that the terminal in $R_2$ has a color different from $c(\alpha)$, $G$ is type-B-partitionable w.r.t. $R_{0,2}$. If $\{\alpha, \beta\} \neq \{s_a, t_a\}$ for all $a$, then $G$ is type-C-partitionable w.r.t. $R_{7} \cup R_{0,1}$. Secondly, suppose $c(\alpha) \neq c(\beta)$. If $\{\alpha, \beta\} = \{s_a, t_a\}$ for some $a$, then $G$ is type-A-partitionable w.r.t. $R_{2,7}$. If $\{\alpha, \beta\} \neq \{s_a, t_a\}$ for all
a, and moreover $(\alpha, \beta) \notin E(G)$, then $G$ is type-C-partitionable w.r.t. $R_7 \cup R_0.1$. So, we further assume $\{\alpha, \beta\} = \{s_\alpha, s_\beta\}$ for some $a, b \in \{1, 2, 3\}$ with $a \neq b$, and $(s_\alpha, s_\beta) \in E(G)$. If $R_2$ contains $t_a$ or $t_b$, then $G$ is type-B-partitionable w.r.t. $R_0.2$; if $R_6$ contains $t_a$ or $t_b$, then $G$ is also type-B-partitionable w.r.t. $R_6.7 \cup R_0$. There remains a case where $(R_2 \cup R_6) \cap (S \cup T) = \{s_c, t_c\}$ for some $c \in \{1, 2, 3\}$ with $c \neq a, b$. Assume w.l.o.g. $s_\alpha \in R_2$ and $t_c \in R_6$, and moreover $t_a \in R_4$ and $t_b \in R_5$. If $c(t_a) = c(t_b)$, then $G$ is type-C-partitionable w.r.t. $R_3.5$; also, if $c(t_b) = c(t_c)$, then $G$ is type-C-partitionable w.r.t. $R_5.7$. Under the condition $c(t_a) = c(t_c) \neq c(t_b) = c(s_c)$, we give a direct construction of a paired 3-DPC below for the remaining case (see Fig. 7(b)).

1: Find a Hamiltonian $s_a - s_b$ path in $G[R_0]$.

Let the Hamiltonian path be represented as $(s_a, \ldots, x, y, \ldots, s_b)$, possibly $x = s_a$, for some $x$ with $c(x) = c(s_c)$.

2: For the neighbor $s'_a \in R_1$ of $x$, the neighbor $t'_a \in R_3$ of $t_a$, and a neighbor $z \in R_3$ of $t'_a$, find a paired 2-DPC made of $s'_a - t'_a$ and $s_c - z$ paths in $G[R_1.3]$.

3: For the neighbor $z' \in R_3$ of $z$ and the neighbor $w \in R_4$ of $t_a$ other than $z'$, find a Hamiltonian $z' - w$ path in $G[R_4] - t_a$.

4: For the neighbor $s'_b \in R_7$ of $y$ and the neighbor $w' \in R_5$ of $w$, find a paired 2-DPC composed of $s'_b - t_b$ and $w' - t_c$ paths in $G[R_5.7]$.

5: Concatenating the $s_a - x$ path, the $s'_a - t'_a$ path, and $(t_a)$ results in an $s_a - t_a$ path; concatenating the $s_b - y$ and $s'_b - t_b$ paths leads to an $s_b - t_b$ path; finally, concatenating the $s_c - z$, $z' - w$, and $w' - t_c$ paths leads to an $s_c - t_c$ path.

**Case 3:** $|R_0 \cap (S \cup T)| = 1$. Let $r$ denote the maximum number of consecutive rows each of which contains a terminal; assume w.l.o.g. that $R_0, \ldots, R_{r-1}$ are such consecutive rows. Firstly, suppose $r = 1$. Then, $G$ is type-C-partitionable w.r.t. $R_{r-1} \cup R_{r,q+1}$ for some $q \geq 1$ such that $R_q$ contains a terminal but $R_j$ does not for all $j \in \{1, \ldots, q-1\}$. Secondly, suppose $r = 2$. Then, $G$ is also type-C-partitionable w.r.t. $R_{r-1} \cup R_0.2$. Thirdly, suppose $r = 3$. Then, $G$ is row-reducible or $m \leq 10$. If $m = 10$, then $R_4$, $R_6$, and $R_8$ each contains a single terminal, so $G$ is type-C-partitionable w.r.t. $R_{3.7}$.

Let $m = 8$ now. The rows $R_3$ and $R_7$ contain no terminal, so each of $R_4$, $R_5$, $R_6$ contains a terminal, i.e., $|R_j \cap (S \cup T)| = 1$ iff $j \in \{0, 1, 2, 4, 5, 6\}$. Let $\alpha_j$ denote the terminal in $R_j$. If $c(\alpha_0) = c(\alpha_1)$, then $G$ is type-C-partitionable; if $c(\alpha_1) = c(\alpha_2)$, then $G$ is also type-C-partitionable; so, $c(\alpha_0) = c(\alpha_2) \neq c(\alpha_1)$. A similar argument leads to $c(\alpha_4) = c(\alpha_6) \neq c(\alpha_5)$. It follows that $c(\alpha_0) = c(\alpha_2) = c(\alpha_3) \neq c(\alpha_1) = c(\alpha_4) = c(\alpha_6)$. Furthermore, if $c(\alpha_0, \alpha_1, \alpha_2)$ contains $s_a, t_a$ for some $a$, then $G$ is type-B-partitionable; if $\{\alpha_1, \alpha_2, \alpha_4\}$ contains $s_a, t_a$ for some $a$, then $G$ is also type-B-partitionable, and so on. Thus, we can assume w.l.o.g. that $s_1 \in R_0$, $s_2 \in R_1$, $s_3 \in R_2$, $t_1 \in R_4$, $t_2 \in R_5$, and $t_3 \in R_6$. A paired 3-DPC for the remaining case can be constructed as follows (see Fig. 7(c)).

1: For a vertex $x \in R_1$ with $c(x) = c(s_1)$, there exists a vertex $y \in R_0$ that admits a disjoint path cover composed of $s_1 - x$ and $s_2 - y$ paths in $R_{0.1}$.

2: For the neighbor $s'_1 \in R_2$ of $x$, there exists a vertex $z \in R_3$ that admits a disjoint path cover composed of $s'_1 - t_1$ and $s_3 - z$ paths in $R_{2.4}$.

3: For the neighbor $s'_4 \in R_5$ of $z$ and the neighbor $s'_2 \in R_7$ of $y$, there exists a paired 2-DPC composed of $s'_2 - t_2$ and $s'_1 - t_3$ paths in $R_{5.7}$.

4: Concatenating the $s_1 - x$ and $s'_1 - t_1$ paths results in an $s_1 - t_1$ path; concatenating the $s_2 - y$ and $s'_2 - t_2$ paths leads to an $s_2 - t_2$ path; finally, concatenating the $s_3 - z$ and $s'_1 - t_3$ paths leads to an $s_3 - t_3$ path.

The vertices $y$ in Step 1 and $z$ in Step 2 exist due to Theorem 4. The paired 2-DPC in Step 3 exists by Lemmas 4 and 6 (also, by Remark 1).

Finally, suppose $r \geq 4$. Then, $G$ is row-reducible, or $m = 8$ and $r \in \{4, 5\}$. Let $m = 8$. If $r = 4$, then $R_4$ and $R_7$ contain no terminal, but $R_0$ and $R_6$ each contains a single terminal, hence $G$ is type-C-partitionable w.r.t. $R_{4.7}$. If $r = 5$, then $R_0$ contains a terminal but $R_5$ and $R_7$ does not. Let $\alpha_j$ denote the terminal in $R_j$ again. If $c(\alpha_0) = c(\alpha_4)$, then $G$ is type-C-partitionable w.r.t. $R_{5.7}$; also, if $c(\alpha_0) = c(\alpha_6)$, then $G$ is type-C-partitionable w.r.t. $R_{4.7}$; in addition, if $c(\alpha_0) = c(\alpha_0)$, then $G$ is type-C-partitionable w.r.t. $R_{5.7} \cup R_0$; finally, if $c(\alpha_0) = c(\alpha_1)$, then $G$ is type-C-partitionable w.r.t. $R_7 \cup R_0.1$. It follows that $c(\alpha_3) \neq c(\alpha_0) \neq c(\alpha_0) \neq c(\alpha_0)$, and thus $c(\alpha_0) = c(\alpha_2) = c(\alpha_4) = c(\alpha_3) = c(\alpha_0)$. Furthermore, if $c(\alpha_0, \alpha_1, \alpha_2)$ contains $s_a, t_a$ for some $a$, then $G$ is type-B-partitionable; if $\{\alpha_1, \alpha_2, \alpha_3\}$ contains $s_a, t_a$ for some $a$, then $G$ is also type-B-partitionable, and so on. Thus, we can assume w.l.o.g. that $s_1 \in R_0$, $s_2 \in R_1$, $s_3 \in R_2$, $t_1 \in R_4$, $t_2 \in R_5$, and $t_3 \in R_6$. The construction, shown below, is almost the same as in the previous case where $r = 3$, $m = 8$, $s_1 \in R_0$, $s_2 \in R_1$, $s_3 \in R_2$, $t_1 \in R_4$, $t_2 \in R_5$, and $t_3 \in R_6$.

1: For a vertex $x \in R_1$ with $c(x) = c(s_1)$, there exists a vertex $y \in R_0$ that admits a disjoint path cover composed of $s_1 - x$ and $s_2 - y$ paths in $R_{0.1}$.

2: For the neighbor $s'_1 \in R_2$ of $x$, there exists a vertex $z \in R_3$ that admits a disjoint path cover composed of $s'_1 - t_1$ and $s_3 - z$ paths in $R_{2.4}$.

3: For the neighbor $s'_4 \in R_5$ of $z$ and the neighbor $s'_2 \in R_7$ of $y$, there exists a paired 2-DPC composed of $s'_2 - t_2$ and $s'_1 - t_3$ paths in $R_{5.7}$.

4: Concatenating the $s_1 - x$ and $s'_1 - t_1$ paths results in an $s_1 - t_1$ path; concatenating the $s_2 - y$ and $s'_2 - t_2$ paths leads to an $s_2 - t_2$ path; finally,
concatenating the $s_3^z$ and $s_3^t$ paths leads to an $s_3^t$ path.

This completes the entire proof.

REFERENCES


