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Paired 3-disjoint path covers in bipartite torus-like graphs with edge faults

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Given two disjoint vertex-sets, $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ in a graph, a *paired many-to-many k-disjoint path cover* joining S and T is a set of pairwise vertex-disjoint paths $\{P_1, \ldots, P_k\}$ that altogether cover every vertex of the graph, in which each path P_i runs from s_i to t_i. In this paper, we reveal that a bipartite torus-like graph, if built from lower dimensional torus-like graphs that have good disjoint-path-cover properties, retain such good property. As a result, an m-dimensional bipartite torus, $m \geq 3$, with at most $2m - 4$ edge faults has a paired many-to-many 3-disjoint path cover joining arbitrary disjoint sets S and T of size 3 each such that $S \cup T$ contains the equal numbers of vertices from different parts of the bipartition.

Keywords: Disjoint path; path cover; path partition; torus; toroidal grid.

1. Introduction

An interconnection network is frequently modeled as a graph, in which the vertices and edges represent nodes and links, respectively. One of the central issues in the study of interconnection networks is finding parallel paths, which is naturally related to routing among nodes and fault tolerance of the network [6, 15]. Parallel paths correspond to disjoint paths of the underlying graph. Disjoint path is, moreover, a fundamental notion from which many graph properties can be deduced [2, 15].

It is often important to find disjoint paths that collectively pass through all the vertices. A disjoint path cover of a graph is a set of vertex-disjoint paths that altogether cover every vertex of the graph. The disjoint path cover problems are applicable in many areas such as software testing, database design, and code optimization [1, 17]. In addition, the problem is concerned with applications where full utilization of network nodes is important [22]. For instance, basic communication problems for the dissemination of information, such as broadcasting and information gathering, require that every node in a network be visited at least once, but not more than once to avoid unnecessary overhead.

Let G be a finite, simple undirected graph. A *path* from v to w, referred to as a v–w path, is a sequence $\langle u_1, \ldots, u_l \rangle$ of distinct vertices of G such that $u_1 = v$,

$u_l = w$, and $(u_i, u_{i+1}) \in E(G)$ for all $i \in \{1, \dots, l-1\}$. If $l \geq 3$ and $(u_l, u_1) \in E(G)$, the sequence is called a *cycle*. A path that visits each vertex exactly once is a *Hamiltonian path*; a cycle that visits each vertex exactly once is a *Hamiltonian cycle*. A *bipartite* graph is a graph whose vertices can be colored in two colors, green and white, in such a way that every pair of adjacent vertices is colored differently. Throughout the paper, the vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively.

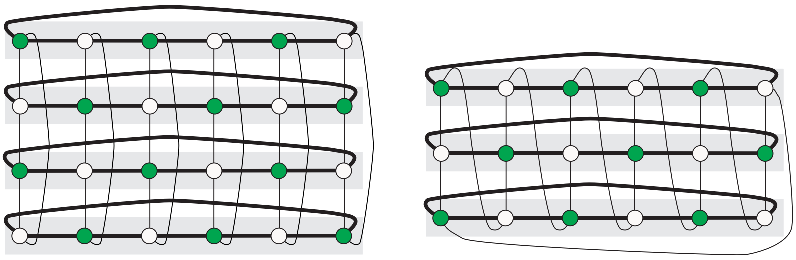
A *path cover* of a graph G is a set of paths in G such that every vertex of G is contained in at least one path. A *disjoint path cover* (DPC for short) of G is a path cover in which every vertex of G is covered by exactly one path. Given disjoint subsets $S = \{s_1, \dots, s_k\}$ and $T = \{t_1, \dots, t_k\}$ of $V(G)$ for a positive integer k, a *many-to-many k-disjoint path cover* is a DPC composed of k paths that collectively join S and T. If each source $s_i \in S$ must be joined to a specific sink $t_i \in T$, the many-to-many k-DPC is called *paired*, and it is *unpaired* if no such constraint is imposed. As intuitively clear, we will call the vertices in S and in T *sources* and *sinks*, respectively, which together form a set of *terminals*.

Definition 1. *(See [23].) A graph G is called* f-fault paired *(resp.* unpaired*)* k-disjoint path coverable *if $f + 2k \leq |V(G)|$ and G has a paired (resp. unpaired) k-DPC joining arbitrary disjoint set S of k sources and set T of k sinks in $G - F$ for any fault set $F \subseteq V(G) \cup E(G)$ with $|F| \leq f$.*

No bipartite graph is paired k-disjoint path coverable for any fixed $k \geq 1$, with the unique exception of the complete graph on two vertices for $k = 1$. This stems from the nature of bipartite graphs that vertices of different parts of the bipartition appear alternatively in a path. In particular, consider an *equitable* bipartite graph G, where the two parts of the bipartition have the same number of vertices. The graph G has a paired k-DPC joining terminal sets S and T only if $S \cup T$ is *balanced*, i.e., the number of terminals in $S \cup T$ that belong to each of the two parts is equal. This naturally leads to the notion of disjoint path bicoverability, as defined below.

Definition 2. *An equitable bipartite graph G is* paired *(resp.* unpaired*)* k-bicoverable *if G has a paired (resp. unpaired) k-DPC joining S and T for any disjoint terminal sets S and T of size k each such that $S \cup T$ is balanced. Moreover, the graph G is* f-edge-fault paired *(resp.* unpaired*)* k-bicoverable *if $G - F$ is paired (resp. unpaired) k-bicoverable for any edge fault set $F \subseteq E(G)$ with $|F| \leq f$.*

Among the interconnection networks proposed in the literature, torus is one of the widely recognized networks. An *m-dimensional torus* is defined as a Cartesian product of m cycles, $C_{d_1} \times \cdots \times C_{d_m}$, where each C_{d_j}, $j \in \{1, \dots, m\}$, is a cycle of length $d_j \geq 3$. Given two graphs, G_0 and G_1, of the same order and a bijection ϕ from $V(G_0)$ to $V(G_1)$, we denote by $G_0 \oplus_\phi G_1$ the graph whose vertex set is $V(G_0) \cup V(G_1)$ and edge set is $E(G_0) \cup E(G_1) \cup \{(v, \phi(v)) : v \in V(G_0)\}$. To simplify the notation, we often omit the bijection ϕ from $\oplus_\phi$. Given d graphs $G_0, \dots, G_{d-1}$



(a) 4×6 Torus (b) Double loop network

Fig. 1: Examples of 2-dimensional bipartite torus-like graphs, where an intra-component edge is indicated by a thick edge.

of the same order n, if we apply the graph constructor $\oplus$ to each pair G_i and $G_{(i+1) \bmod d}$ for $i \in \{0, \ldots, d-1\}$, then we obtain a graph with nd vertices. This graph is said to be obtained through the *cycle-based recursive construction.*

Definition 3. *(See [19].) An* m-dimensional torus-like graph*,* $m \geq 1$*, is a graph obtained through the cycle-based recursive construction from* $(m-1)$*-dimensional torus-like graphs* $G_0, \ldots, G_{d-1}$*,* $d \geq 3$*, of the same order, where the* 0*-dimensional torus-like graph is a one-vertex graph* K_1*.*

Here, the graphs $G_0, \ldots, G_{d-1}$ are called the *components* of the torus-like graph. Refer to Fig. 1 for examples of torus-like graphs. Each vertex v in component G_i has two neighbors outside G_i: one in $G_{(i+1) \bmod d}$, denoted by v^+, and the other in $G_{(i-1) \bmod d}$, denoted by v^-. Contracting the components of the torus-like graph into single vertices results in a cycle C_d of length d.

The disjoint path cover problems of a graph are closely related to the Hamiltonian properties (as well as the vertex connectivity) of the graph. The problems have been studied for various classes of graphs, including recent studies on dense graphs [11], cube of connected graphs [21], balanced hypercubes [12,13], hypercube-like networks [5,16], directed graphs [3], k-ary n-cubes [9], and torus networks [10]. In particular, the paired disjoint path cover problem for nonbipartite torus-like graphs was investigated in [19]. Some studies on bipartite torus networks can be summarized as follows:

Theorem 4. *(a) A* 2*-dimensional bipartite torus is paired* 2*-bicoverable [8, 14].*
(b) A 2*-dimensional bipartite torus is* 1*-edge-fault paired* 2*-bicoverable [20].*
(c) An m*-dimensional bipartite torus,* $m \geq 2$*, is* $(2m-3)$*-edge-fault paired* 2*-bicoverable [4].*
(d) With the unique exception of a 4×4 *torus, a* 2*-dimensional bipartite torus is paired* 3*-bicoverable [18].*

In this paper, we reveal that a bipartite torus-like graph has a good disjoint-path-bicover property if every component of the graph has such good property.

Specifically, we prove that an m-dimensional bipartite torus-like graph, $m \geq 3$, composed of d components $G_0, \ldots, G_{d-1}$ is $(2m-4)$-edge-fault paired 3-bicoverable, if each component G_i is $(2m-3-k)$-edge-fault paired k-bicoverable for all $k \in \{1, 2, 3\}$. As a result, we obtain that an m-dimensional bipartite torus, $m \geq 3$, is $(2m-4)$-edge-fault paired 3-bicoverable, settling down the following conjecture for the case $k = 3$:

Conjecture 1. *An m-dimensional bipartite torus, $m \geq 3$, is $(2m-1-k)$-edge-fault paired k-bicoverable.*

Note that Theorem 4(c) indicates that Conjecture 1 is true for the case $k = 2$. Also, the conjecture is true for the case $k = 1$ due to Theorem 5 below. An equitable bipartite graph is *Hamiltonian-laceable* if each pair of vertices contained in different parts is joined by a Hamiltonian path. By definition, an equitable bipartite graph is Hamiltonian-laceable if and only if it is paired 1-bicoverable.

Theorem 5. *(See [7].) An m-dimensional bipartite torus G, $m \geq 2$, is $(2m-2)$-edge-fault Hamiltonian-laceable.*

2. Preliminaries

We begin this section by discussing some basic properties of an m-dimensional bipartite torus-like graph G built from d components $G_0, \ldots, G_{d-1}$. Each component G_i is an $(m-1)$-dimensional torus-like graph and moreover, is a bipartite graph. The number d of components is, by definition, at least 3; the number is at least 4 for $m = 1$ because G is isomorphic to a cycle of length d. This leads to that the number of vertices in G is at least $4 \cdot 3^{m-1}$. In addition, the graph G is a regular graph of degree $2m$.

Lemma 6. *An m-dimensional bipartite torus-like graph, $m \geq 1$, is equitable.*

Proof. The proof is by induction on m. A 1-dimensional bipartite torus-like graph, which is isomorphic to an even cycle, is connected and equitable. Let $m \geq 2$. Supposing each component is a connected and equitable bipartite graph leads to that an m-dimensional bipartite torus-like graph built from the components is also connected and equitable, completing the proof. □

A graph G' is a *subgraph* of a graph G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. The subgraph G' is an *induced subgraph* of G if G' includes all the edges of G whose endvertices belong to $V(G')$. An induced subgraph is obtained from G by vertex deletions only. Hereafter in this paper, we denote the color of a vertex v in a bipartite graph by $c(v) \in \{\mathsf{green}, \mathsf{white}\}$. In addition, an edge (u, v) is said to be *free* if it is nonfaulty and both u and v are nonterminals. A paired k-DPC joining $\{(x_1, y_1), \ldots, (x_k, y_k)\}$ refers to a DPC composed of x_1–y_1, …, x_k–y_k paths.

3. Chain of torus-like graphs

Let G be an m-dimensional bipartite torus-like graph built from d components $G_0, \ldots, G_{d-1}$. The subgraph of G induced by $V(G_0) \cup \cdots \cup V(G_r)$, $0 \le r \le d-2$, forms a *chain of torus-like graphs* and will be denoted by $G_0 \oplus \cdots \oplus G_r$ or simply by $G_{0,r}$. In this section, we show that the chain $H = G_0 \oplus \cdots \oplus G_r$, $r \ge 1$, has a good disjoint-path-bicover property if every component G_i has. Throughout the paper, F will denote a set of edge faults. We let F_i denote the set of edge faults contained in G_i, i.e., $F_i = F \cap E(G_i)$. Also, let $F_{i,j}$ denote the fault set of $G_{i,j}$, so $F_{i,i} = F_i$. Then, the fault set F of H will be $F = F_{0,r-1} \cup F_r \cup F'_{r-1,r}$, where $F'_{i,i+1}$ denotes the set of edge faults bridging G_i and G_{i+1}.

Let $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ be the source and sink sets given in the chain $H = G_0 \oplus \cdots \oplus G_r$, $r \ge 1$. We denote by k_i and $k_{i,j}$ the numbers of source-sink pairs in G_i and in $G_i \oplus \cdots \oplus G_j$, respectively, so $k_{i,i} = k_i$. In addition, let $k'_{r-1,r}$ denote the number of source-sink pairs between H' and G_r, where $H' = G_0 \oplus \cdots \oplus G_{r-1}$, so that $k_{0,r-1} + k_r + k'_{r-1,r} = k = 3$. For an index set $I = \{1, 2, 3\}$ of the sources and sinks, we let $I_0 = \{i \in I : s_i, t_i \in V(H')\}$, $I_1 = \{j \in I : s_j, t_j \in V(G_r)\}$, and $I_2 = I \setminus (I_0 \cup I_1)$. A source s_i and its sink t_i are interchangeable, so we assume that for every $i \in I_2$, s_i is contained in H' and t_i is contained in G_r. Note that $|I_0| = k_{0,r-1}$, $|I_1| = k_r$, and $|I_2| = k'_{r-1,r}$.

Theorem 7. *Let G_i, $i \in \{0, \ldots, r\}$, be an $(m-1)$-dimensional bipartite torus-like graph of the same order, $m \ge 3$, such that G_i is $(2m-3-k)$-edge-fault paired k-bicoverable for all $k \in \{1, 2, 3\}$. Then, the chain $H = G_0 \oplus \cdots \oplus G_r$, $r \ge 1$, is $(2m-5)$-edge-fault paired 3-bicoverable if it is bipartite.*

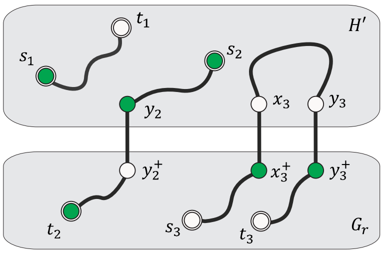
Proof. Given F, S, and T in the chain H such that $|F| \le 2m-5$, $|S| = |T| = 3$, and $S \cup T$ is balanced, we will build a paired 3-DPC joining S and T in $H - F$. We assume w.l.o.g. $|F| = 2m-5$ because for a set F' of arbitrary $(2m-5) - |F|$ fault-free edges, a paired k-DPC joining S and T in $H - (F \cup F')$ is a required k-DPC. As a result, we have $|F| \ge 1$. Also, we assume w.l.o.g. that (i) the number of terminals belonging to G_0 is greater than that of G_r, or (ii) G_0 and G_r have the same number of terminals and $|F_0| \ge |F_r|$.

First of all, a basic procedure for building a paired 3-DPC of H is introduced below. This procedure is expected to be applicable when $H' - F_{0,r-1}$ is paired 3-bicoverable, where $H' = G_0 \oplus \cdots \oplus G_{r-1}$. The number of terminals contained in G_r is $2k_r + k'_{r-1,r} \le 3$.

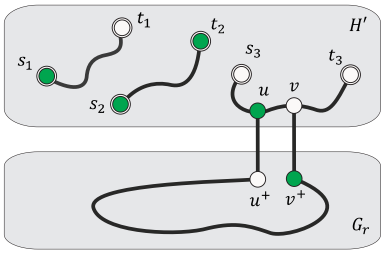
Procedure PAIRED-DPC(S, T, F, $G_0 \oplus \cdots \oplus G_r$) // See Fig. 2.

1: Pick up $2k_r + k'_{r-1,r}$ free edges as follows: For each source s_i contained in G_r, $i \in I_1$, pick up a free edge (x_i, x_i^+) with $x_i \in V(G_{r-1})$ such that $c(x_i) = c(s_i)$; for each sink t_j contained in G_r, $j \in I_1 \cup I_2$, pick up a free edge (y_j, y_j^+) with $y_j \in V(G_{r-1})$ such that $c(y_j) = c(t_j)$.

2: Build a paired 3-DPC in $H' - F_{0,r-1}$ joining $\{(s_i, t_i) : i \in I_0\} \cup \{(x_j, y_j) : j \in$



(a) Case when $2k_r + k'_{r-1,r} \geq 1$



(b) Case when $2k_r + k'_{r-1,r} = 0$

Fig. 2: Illustrations of Procedure Paired-DPC.

$I_1\} \cup \{(s_j, y_j) : j \in I_2\}$.

3: Case when $2k_r + k'_{r-1,r} \geq 1$:

- a: Build a paired $(2k_r + k'_{r-1,r})$-DPC in $G_r - F_r$ joining $\{(s_i, x_i^+) : i \in I_1\} \cup \{(y_j^+, t_j) : j \in I_1 \cup I_2\}$.
- b: Merge the two DPCs with the free edges selected in Step 1 into a paired 3-DPC joining S and T.

4: Case when $2k_r + k'_{r-1,r} = 0$:

- a: Pick up an edge $(u, v) \in E(G_{r-1})$ on a path P in the DPC such that (u, u^+) and (v, v^+) both are fault-free.
- b: Replace the edge (u, v) of P with a path (u, Q, v) for some Hamiltonian u^+–v^+ path Q of $G_r - F_r$.

The proof is by induction on r. Cases 1 and 2 deal with the base step of $r = 1$, and Case 3 deals with the inductive step of $r \geq 2$.

Case 1: *$r = 1$ and $|F_0| < |F|$.* We will prove a claim that Procedure Paired-DPC builds a paired 3-DPC in $G_0 \oplus G_1 - F$ when $r = 1$ and $|F_0| < |F|$. For the existence of the free edges in Step 1, we let p and q respectively denote the numbers of white and green terminals contained in G_1, where $p + q \leq 3$ by the assumption that G_1 has the same or fewer terminals than G_0. If $p \geq 1$, then there exist p or more free edges (u, u^+) with $u \in V(G_0)$ such that $c(u) = \mathsf{white}$. This is because there are $\frac{|V(G_0)|}{2}$ candidate edges whereas at most $|F| + (3 - p) + q$ of them could be blocked ($|F|$ edge faults, $3 - p$ white terminals in G_0, and q green terminals in G_1), for which $\frac{|V(G_0)|}{2} - (|F| + (3 - p) + q) \geq \frac{4 \cdot 3^{m-2}}{2} - (2m - 5) - (3 - p) + (p - 3) = \frac{4 \cdot 3^{m-2}}{2} - (2m - 5) - 6 + 2p \geq p$ for $m \geq 3$. Similarly, we can show that if $q \geq 1$, then there exist q or more free edges (u, u^+) with $u \in V(G_0)$ such that $c(u) = \mathsf{green}$ (because $\frac{|V(G_0)|}{2} - (|F| + (3 - q) + p) \geq q$ for $m \geq 3$). Thus, there exist $2k_r + k'_{r-1,r} (= p + q)$ free edges of Step 1. The paired 3-DPC of Step 2 exists by the hypothesis of this theorem because $|F_0| \leq |F| - 1 = 2m - 6$. Also, the paired $(2k_r + k'_{r-1,r})$-DPC of Step 3a exists because either $2k_r + k'_{r-1,r} = 3$

and $|F_1| \leq |F_0| \leq 2m-6$ or $2k_r + k'_{r-1,r} \leq 2$ and $|F_1| \leq |F| = 2m-5$ holds. The Hamiltonian path Q of Step 4b exists because $|F_1| \leq |F| = 2m-5$. Therefore, the procedure builds a required DPC, proving the claim.

Case 2: $r = 1$ *and* $|F_0| = |F|$. There are four subcases depending on the number of terminals contained in G_0. Firstly, suppose G_0 has all the six terminals. For an edge fault $e_f \in F_0$, let $F_0' = F_0 \setminus e_f$. Then, $|F_0'| = 2m-6$ and thus $G_0 - F_0'$ has a paired 3-DPC joining S and T. Let (u,v) be e_f if a path in the DPC passes through e_f; otherwise, let (u,v) be an arbitrary edge on a path in the DPC. It suffices to replace (u,v) with (u,Q,v), where Q is a Hamiltonian u^+–v^+ path of G_1.

Secondly, suppose G_0 has five terminals and G_1 has a single terminal. Assume that t_3 is contained in G_1 and is colored green. If there exists an edge e between a green terminal in G_0 and the neighbor t_3^- of t_3, let $F_0' = F_0 \cup \{e\}$ and α be the green terminal not incident with e; otherwise, let $F_0' = F_0$ and α be an arbitrary green terminal. Since $|F_0'| \leq 2m-4$, there exists a Hamiltonian path in $G_0 - F_0'$ joining the green terminal α and a white terminal β with $\beta^+ \neq t_3$. Let the Hamiltonian path be $(\alpha, \ldots, u, \gamma, \ldots, v, \sigma, \ldots, x, \tau, \ldots, y, \beta)$, where γ, σ, τ are terminals in G_0. Observe that the predecessor w of a white terminal is colored green, meaning $w^+ \neq t_3$; the predecessor w of a green terminal other than α is a white vertex other than t_3^-, also meaning $w^+ \neq t_3$. It follows that the Hamiltonian path can be decomposed into five path segments $(\alpha, \ldots, u)$, $(\gamma, \ldots, v)$, $(\sigma, \ldots, x)$, $(\tau, \ldots, y)$, and (β), each of which runs from a terminal to a vertex whose neighbor in G_1 is a nonterminal. A required 3-DPC can be constructed by combining the five path segments extended to G_1, namely $(\alpha, \ldots, u, u^+)$, $(\gamma, \ldots, v, v^+)$, $(\sigma, \ldots, x, x^+)$, $(\tau, \ldots, y, y^+)$, (β, β^+), with one-vertex path (t_3), utilizing a paired 3-DPC of G_1 joining proper pairs of vertices belonging to $\{u^+, v^+, x^+, y^+, \beta^+, t_3\}$.

Thirdly, suppose G_0 has four terminals and G_1 has two terminals. Assume that t_3 is contained in G_1 and is colored green. Let τ be the terminal in G_1 other than t_3. If $c(\tau) = \mathsf{green}$, then we first build a Hamiltonian path, $(\alpha, \ldots, u, \gamma, \ldots, v, \sigma, \ldots, x, \beta)$, in G_0 joining a green terminal α and a white terminal β such that β^+ is a nonterminal, where γ and σ are white terminals in G_0. Similar to the previous subcase, four path segments, $(\alpha, \ldots, u)$, $(\gamma, \ldots, v)$, $(\sigma, \ldots, x)$, and (β) are extracted from the Hamiltonian path, and then extended by one to nonterminals in G_1. It suffices to combine the extended path segments with two one-vertex paths (t_3) and (τ) into a required 3-DPC through a paired 3-DPC of G_1 joining proper pairs of vertices.

Now, let $c(\tau) = \mathsf{white}$. Assume that α and β are green terminals while γ and σ are white terminals in G_0. There exists a paired 2-DPC in $G_0 - F_0$ composed of α–β path P_1 and γ–σ path P_2, from which four path segments will be extracted as before. If P_i in the DPC has 5 or more vertices, then there is an edge (u,v) on P_i such that both u^+ and v^+ are nonterminals, hence deleting (u,v) from P_i results in desired path segments. So, we assume $P_1 = (\alpha, x, \beta)$ and P_2 is of order $|V(G_0)| - 3 \geq 9$. If x^+ is a nonterminal, then P_1 can be decomposed into desired path segments (α, x) and (β) or (α) and (x, β) according to whether β^+ is a nonterminal.

Now let x^+ be a terminal, i.e., $x^+ = t_3$. The vertex x has $2m-2$ neighbors in G_0, among them there exists a neighbor y on P_2 such that $(x,y) \notin F_0$ because $|F_0| = 2m-5$ and only two neighbors, α and β, are not on P_2. Assume w.l.o.g. β^+ is a nonterminal and $P_2 = (\gamma, \dots, u, y, v, w, \dots, \sigma)$ with $v \neq \sigma$, possibly $\gamma = u$. The path segments (α, x, y), (β), $(\gamma, \dots, u)$, $(v, w, \dots, \sigma)$ are desired one if y^+ is a nonterminal; otherwise, (α, x, y, v), (β), $(\gamma, \dots, u)$, $(w, \dots, \sigma)$ are desired path segments. It suffices to extend the four path segments to nonterminals in G_1 and then combine the four with (t_3) and (τ) into a required 3-DPC through a paired 3-DPC of G_1 joining proper pairs of vertices.

Finally, suppose G_0 and G_1 each has three terminals. Assume t_3 is contained in G_1. Let α, β, γ be the terminals in G_0, and σ, τ be the terminals in G_1 other than t_3. Suppose $c(\alpha) = c(\beta) = c(\gamma) = \mathsf{white}$ first. There exists a Hamiltonian path in $G_0 - F_0$ joining α and a green vertex z. The Hamiltonian path, say $(\alpha, \dots, u, \beta, \dots, v, \gamma, \dots, z)$, can be decomposed into three path segments $(\alpha, \dots, u)$, $(\beta, \dots, v)$, $(\gamma, \dots, z)$, each of which runs from a terminal to a (green) vertex whose neighbor in G_1 is a nonterminal. Using the three path segments, a required 3-DPC can be built as in the previous subcase. Now, suppose $c(\alpha) = c(\beta) = \mathsf{white}$ and $c(\gamma) = \mathsf{green}$. There exists a paired 2-DPC in $G_0 - F_0$ composed of α–β path P_1 and γ–z path P_2 for some green vertex z such that z^+ is a nonterminal. If there exists an edge (u,v) on P_1 such that u^+ and v^+ both are nonterminals, $P_1 - (u,v)$ and P_2 form desired path segments, from which we can build a required 3-DPC as before. So, we assume no such edge (u,v) exists in P_1, leading to that P_1 has 5 or less vertices.

Suppose P_1 has 5 vertices, say $P_1 = (\alpha, u, x, v, \beta)$. Then, α^+, x^+, v^+ are terminals up to symmetry (possibly, u^+, x^+, β^+ are terminals). If there is a neighbor $y \in V(P_2)$ of x such that $(x,y) \notin F_0$ and $y \neq \gamma$, then representing P_2 as $(\gamma, \dots, w, y, \dots, z)$, we can build three path segments (α, u), $(\beta, v, x, y, \dots, z)$, $(\gamma, \dots, w)$, from which a required 3-DPC can be constructed as before. If there is no such vertex y on P_2, meaning all edge faults are incident with x, then there exists a neighbor $y' \in V(P_2)$ of α with $(\alpha, y') \notin F_0$ and $y' \neq \gamma$, representing $P_2 = (\gamma, \dots, w', y', \dots, z)$. Hence, we have three path segments $(\alpha, y', \dots, z)$, (β, v, x, u), $(\gamma, \dots, w')$, from which a required 3-DPC can be built.

Suppose P_1 has 3 vertices now, say $P_1 = (\alpha, x, \beta)$. Let x^+ be a nonterminal first. Then, α^+ and β^+ are (green) terminals, and there exists a neighbor $y \in V(P_2)$ of β with $(\beta, y) \notin F_0$ and $y \neq \gamma$. Representing P_2 as $(\gamma, \dots, w, y, \dots, z)$, we have three path segments (α, x), $(\beta, y, \dots, z)$, $(\gamma, \dots, w)$, from which a required 3-DPC can be constructed as before. Let x^+ be a (white) terminal and let α^+ or β^+, say β^+, be a nonterminal now. Then, there exists a neighbor $y \in V(P_2)$ of x with $(x,y) \notin F_0$. Representing P_2 as $(\gamma, \dots, w, y, \dots, z)$, we have three path segments (β), $(\alpha, x, y, \dots, z)$, $(\gamma, \dots, w)$, from which we can build a required 3-DPC. Let x^+, α^+, and β^+ be the terminals of G_1 finally. There exists a neighbor $y \in V(P_2)$ of x with $(x,y) \notin F_0$; also, there exists a neighbor $y' \in V(P_2)$ of β with $(\beta, y') \notin F_0$

and $y' \neq \gamma$. Note that $c(y) \neq c(y')$, so $y \neq y'$. If y precedes y' on P_2 that runs from γ to z, representing P_2 as $(\gamma, \ldots, w, y, \ldots, w', y', \ldots, z)$, then we have three path segments $(\alpha, x, y, \ldots, w')$, $(\beta, y', \ldots, z)$, $(\gamma, \ldots, w)$; similarly, we can build desired path segments if y' precedes y on P_2. We can construct a required 3-DPC from the path segments as before.

Case 3: $r \geq 2$. We will prove a claim that Procedure Paired-DPC builds a paired 3-DPC in $H - F$ when $r \geq 2$. The free edges of Step 1 exist for the same reason as in Case 1. The subgraph H' is $(2m-5)$-edge-fault paired 3-bicoverable by the induction hypothesis, so the paired 3-DPC of Step 2 exists. The paired DPC of Step 3 and the Hamilton path of Step 4 also exist for the same reason as in Case 1, proving the claim. This completes the entire proof. □

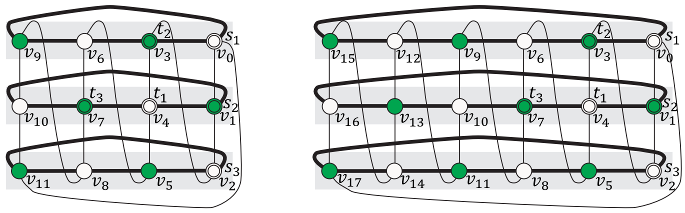
4. Disjoint path covers in torus-like graphs

Utilizing the disjoint-path-bicover property of the chain of torus-like graphs revealed in Theorem 7, we prove in this section that a bipartite torus-like graph has good disjoint-path-bicover property if each component of the graph has such good property, as shown below.

Theorem 8. *Let G be an m-dimensional bipartite torus-like graph, $m \geq 3$, composed of d components $G_0, \ldots, G_{d-1}$ each of which is $(2m-3-k)$-edge-fault paired k-bicoverable for all $k \in \{1, 2, 3\}$. Then, G is $(2m-4)$-edge-fault paired 3-bicoverable.*

Proof. For the proof, assume we are given terminal sets $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ in G, along with an edge fault set F with $|F| = 2m - 4$. If F contains an *inter-component edge*, an edge bridging two components G_i and G_{i+1} for some i, then it suffices to build a paired 3-DPC in the chain $G_{i+1} \oplus \cdots \oplus G_i$ by Theorem 7. Hereafter in this proof, we assume F contains no inter-component edges. Let G_q be a component with the minimum number of terminals among the components with edge faults. Let H be the chain $G - G_q$, i.e., $H = G_{q+1} \oplus \cdots \oplus G_{q-1}$. Then, $H - (F \setminus F_q)$ is paired 3-bicoverable by Theorem 7 because $|F \setminus F_q| \leq 2m - 5$. In addition, $G_q - F_q$ is paired 1-bicoverable because $|F_q| \leq |F| = 2m - 4$. Thus, Procedure Paired-DPC can be recycled in case when G_q has one or less terminals.

Suppose G_q has two terminals, say α, β, now. If $c(\alpha) \neq c(\beta)$, then there exists a Hamiltonian path $(\alpha, \ldots, u, v, \ldots, \beta)$ in $G_q - F_q$ with both u^+ and v^+ being nonterminals. It suffices to extract two path segments $(\alpha, \ldots, u)$ and $(v, \ldots, \beta)$ from the Hamiltonian path, and then extend the path segments by one to nonterminals in G_{q+1} and combine them with a paired 3-DPC of H joining proper pairs of vertices in $\{u^+, v^+\} \cup ((S \cup T) \setminus \{\alpha, \beta\})$. If $c(\alpha) = c(\beta) = \mathsf{green}$, then there exists a Hamiltonian path $(\alpha, \ldots, x, \beta, \ldots, z)$ in $G_q - F_q$ for some white vertex z such that z^+ is a nonterminal. The vertices x^+ and x^- are green, so one of them, say x^+, is a nonterminal. It suffices to combine two path segments $(\alpha, \ldots, x, x^+)$, $(\beta, \ldots, z, z^+)$ with a paired 3-DPC of H joining proper pairs of vertices.



(a) 3×4 Double loop network (b) 3×6 Double loop network

Fig. 3: There exists no paired 3-DPC joining S and T with $(s_1, s_2, s_3) = (v_0, v_1, v_2)$ and $(t_1, t_2, t_3) = (v_4, v_3, v_7)$ in a $3 \times n$ double loop network with even $n \geq 4$.

Suppose G_q has three terminals, say α, β, γ. There is a Hamiltonian cycle $(\alpha, \ldots, u, \beta, \ldots, v, \gamma, \ldots, w)$ in $G_q - F_q$ because $G_q - F_q$ is paired 1-bicoverable. If $c(\alpha) = c(\beta) = c(\gamma) = \mathsf{green}$, then u^+, v^+, w^+ are all green and nonterminals, hence using three path segments $(\alpha, \ldots, u)$, $(\beta, \ldots, v)$, $(\gamma, \ldots, w)$, a required 3-DPC can be built as in the previous case. If $c(\alpha) = c(\beta) = \mathsf{green}$ and $c(\gamma) = \mathsf{white}$, then u^+ or u^- is a nonterminal; also, w^+ or w^- is a nonterminal. If v^+ or v^- is a nonterminal, then desired path segments can be built; if both v^+ and v^- are nonterminals, traversing the Hamiltonian cycle in reverse direction produces $(\alpha, \ldots, v_1, \gamma, \ldots, u_1, \beta, \ldots, w_1)$, from which desired path segments can be extracted, because v_1^+ and v_1^- are both nonterminals. It suffices to build a required 3-DPC from the path segments as before.

Finally, suppose G_q has p terminals for $p \geq 4$, meaning H has $6-p \leq 2$ terminals. Let α, β be terminals in G_q with $c(\alpha) \neq c(\beta)$. In addition, we assume that (i) if H has two terminals of the same color, then β has a different color from the two, and (ii) β^+ or β^- is a nonterminal. The graph $G_q - F_q$ has a Hamiltonian α–β path $(\alpha, \ldots, u, \gamma_1, \ldots, v_1, \ldots, \gamma_{p-2}, \ldots, v_{p-2}, \beta)$, where $\gamma_1, \ldots, \gamma_{p-2}$ are terminals in G_q other than α, β. It suffices to extract p path segments $(\alpha, \ldots, u)$, $(\gamma_1, \ldots, v_1)$, …, $(\gamma_{p-2}, \ldots, v_{p-2})$, (β) from the Hamiltonian path and then extend the path segments by one to nonterminals in H and combine them with a paired 3-DPC of H joining proper pairs of vertices as before. This completes the proof. □

Remark 9. *The precondition of Theorem 8 that each component G_i is $(2m-3-k)$-edge-fault paired k-bicoverable for all $k \in \{1, 2, 3\}$ is not always satisfied. An example is that, as Theorem 4(d) indicates, a 4×4 torus is not paired 3-bicoverable. As another example, it can be proven that no bipartite $3 \times n$ double loop network, $n \geq 4$, is paired 3-bicoverable, as shown in Fig 3.*

5. Disjoint path covers in bipartite torus networks

A 4×4 torus, defined as a Cartesian product $C_4 \times C_4$, is not paired 3-bicoverable by Theorem 4(d). Accordingly, Theorem 8 does not lead to that a 3-dimensional bipartite torus built using the 4×4 torus as a building block is 2-edge-fault paired

3-bicoverable. Fortunately, every 3-dimensional bipartite torus excluding a $4 \times 4 \times 4$ torus can be constructed from components each of which is different from the 4×4 torus. For example, a $6 \times 4 \times 4$ torus can be constructed from four 6×4 tori (instead of six 4×4 tori).

To deal with the exceptional $4 \times 4 \times 4$ torus, we decompose the graph into two subgraphs H_0 and H_1 isomorphic to $C_4 \times C_4 \times K_2$, where K_2 is a complete graph of order 2. Note that the subgraph H_i is also isomorphic to a 5-dimensional hypercube Q_5. Moreover, the $4 \times 4 \times 4$ torus is isomorphic to a Cartesian product of the subgraph H_i and K_2, and isomorphic to a 6-dimensional hypercube Q_6. We are concerned with the disjoint-path-bicover properties of the subgraph H_i shown below. Lemma 10 was verified from a computer program that exhaustively searches for DPCs. The source code may be downloaded from http://tcs.catholic.ac.kr/~jhpark/papers/Lemma_10.zip.

Lemma 10. *(a) The graph $C_4 \times C_4 \times K_2$ is 1-edge-fault paired 3-bicoverable.*
(b) The graph $C_4 \times C_4 \times K_2$ is paired 4-bicoverable.

Now, we are ready to state and prove the paired 3-bicoverability of an m-dimensional bipartite torus, $m \geq 3$, with edge faults.

Theorem 11. *An m-dimensional bipartite torus G, $m \geq 3$, is $(2m-4)$-edge-fault paired 3-bicoverable.*

Proof. The proof proceeds by induction on m. Let $m = 3$ for the base step. If G is not isomorphic to a $4 \times 4 \times 4$ torus, then G can be built from components that are 2-dimensional bipartite tori not isomorphic to the 4×4 tori. Each component is paired 3-bicoverable, 1-edge-fault paired 2-bicoverable, and 2-edge-fault paired 1-bicoverable by Theorems 4 and 5. Thus, G is 2-edge-fault paired 3-bicoverable by Theorem 8. Suppose G is isomorphic to a $4 \times 4 \times 4$ torus now. Then, G is isomorphic to a Cartesian product $(C_4 \times C_4 \times K_2) \times K_2$, which can be represented $H_0 \oplus H_1$ for two graphs H_0 and H_1 isomorphic to $C_4 \times C_4 \times K_2$. Note that H_0 and H_1 are 1-edge-fault paired 3-bicoverable by Lemma 10(a). Also, we can easily derive that H_0 and H_1 are 3-edge-fault paired 1-bicoverable. Furthermore, similar to the proof of Theorem 7 for the base step of $r = 1$, we can prove that H_0 and H_1 are 2-edge-fault paired 2-bicoverable. Utilizing the disjoint-path-bicover properties of H_0 and H_1 just mentioned above, we can prove that G is 2-edge-fault paired 3-bicoverable in the same way as the proof of Theorem 7 for the base step of $r = 1$. The details of the proof are omitted here. Now, let $m \geq 4$ for the inductive step. Suppose G is built from components $G_0, \ldots, G_{d-1}$, which are $(m-1)$-dimensional bipartite tori. Each component G_i is $(2m-6)$-edge-fault paired 3-bicoverable by the induction hypothesis; also, it is $(2m-5)$-edge-fault paired 2-bicoverable by Theorems 4(c), and is $(2m-4)$-edge-fault paired 1-bicoverable by Theorem 5. Therefore, G is $(2m-4)$-edge-fault paired 3-bicoverable by Theorem 8, as required. This completes the proof. □

6. Concluding remarks

In this paper, we have studied the disjoint-path-bicover property of a bipartite torus-like graph made of components with good disjoint-path-bicover property. Specifically, we have proved that an m-dimensional bipartite torus-like graph, $m \geq 3$, composed of d components $G_0, \ldots, G_{d-1}$ is $(2m-4)$-edge-fault paired 3-bicoverable, if each component G_i is $(2m-3-k)$-edge-fault paired k-bicoverable for all $k \in \{1, 2, 3\}$. As a result, we know that an m-dimensional bipartite torus, $m \geq 3$, is $(2m-4)$-edge-fault paired 3-bicoverable. It is open to settle down Conjecture 1 for $k \geq 4$.

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References

[1] K. Asdre and S. D. Nikolopoulos, The 1-fixed-endpoint path cover problem is polynomial on interval graphs, *Algorithmica* **58**(3) (2010) 679–710.
[2] J. A. Bondy and U. S. R. Murty, Graph theory, 2nd printing (2008).
[3] H. Cao, B. Zhang and Z. Zhou, One-to-one disjoint path covers in digraphs, *Theoretical Computer Science* **714** (2018) 27–35.
[4] X.-B. Chen, Paired 2-disjoint path covers of multidimensional torus networks with faulty edges, *Information Processing Letters* **116**(2) (2016) 107–110.
[5] S.-Y. Hsieh, C.-W. Lee and C.-H. Huang, Conditional edge-fault hamiltonian-connectivity of restricted hypercube-like networks, *Information and Computation* **251** (2016) 314–334.
[6] L.-H. Hsu and C.-K. Lin, *Graph Theory and Interconnection Networks* (CRC press, 2008).
[7] H.-C. Kim and J.-H. Park, Paths and cycles in d-dimensional tori with faults, *Workshop on Algorithms and Computation WAAC2001*, (2001), pp. 67–74.
[8] B. G. Kronenthal and W. H. T. Wong, Paired many-to-many disjoint path covers of hypertori, *Discrete Applied Mathematics* **218** (2017) 14–20.
[9] J. Li, C. Melekian, S. Zuo and E. Cheng, Unpaired many-to-many disjoint path covers on bipartite k-ary n-cube networks with faulty elements, *International Journal of Foundations of Computer Science* **31**(03) (2020) 371–383.
[10] J. Li, G. Wang and L. Chen, Paired 2-disjoint path covers of multi-dimensional torus networks with $2n-3$ faulty edges, *Theoretical Computer Science* **677** (2017) 1–11.
[11] H.-S. Lim, H.-C. Kim and J.-H. Park, Ore-type degree conditions for disjoint path covers in simple graphs, *Discrete Mathematics* **339**(2) (2016) 770–779.
[12] H. Lü, Paired many-to-many two-disjoint path cover of balanced hypercubes with faulty edges, *The Journal of Supercomputing* **75**(1) (2019) 400–424.
[13] H. Lü and T. Wu, Unpaired many-to-many disjoint path cover of balanced hypercubes, *International Journal of Foundations of Computer Science* **32**(08) (2021) 943–956.
[14] K. Makino, 2-disjoint path covers in mesh-torus, *Publ. Res. Inst. Math. Sci, Kyoto Univ.* **1489** (2006) 181–187.
[15] J. A. McHugh, *Algorithmic Graph Theory* (Prentice-Hall, New Jersey, 1990).

[16] R. Niu and M. Xu, The unpaired many-to-many k-disjoint paths in bipartite hypercube-like networks, *Theoretical Computer Science* **911** (2022) 26–40.

[17] S. C. Ntafos and S. L. Hakimi, On path cover problems in digraphs and applications to program testing, *IEEE Transactions on Software Engineering* **5**(5) (1979) 520–529.

[18] J.-H. Park, Paired many-to-many 3-disjoint path covers in bipartite toroidal grids, *Journal of Computing Science and Engineering* **12**(3) (2018) 115–126.

[19] J.-H. Park, Torus-like graphs and their paired many-to-many disjoint path covers, *Discrete Applied Mathematics* **289** (2021) 64–77.

[20] J.-H. Park and I. Ihm, Many-to-many two-disjoint path covers in cylindrical and toroidal grids, *Discrete Applied Mathematics* **185** (2015) 168–191.

[21] J.-H. Park and I. Ihm, A linear-time algorithm for finding a one-to-many 3-disjoint path cover in the cube of a connected graph, *Information Processing Letters* **142** (2019) 57–63.

[22] J.-H. Park, H.-C. Kim and H.-S. Lim, Many-to-many disjoint path covers in hypercube-like interconnection networks with faulty elements, *IEEE Transactions on Parallel and Distributed Systems* **17**(3) (2006) 227–240.

[23] J.-H. Park, H.-C. Kim and H.-S. Lim, Many-to-many disjoint path covers in the presence of faulty elements, *IEEE Transactions on Computers* **58**(4) (2009) 528–540.