Panconnectivity and Edge-Pancyclicity of Faulty Recursive Circulant $G(2^m, 4)$*

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Abstract. In this paper, we investigate a problem on embedding paths into recursive circulant $G(2^m, 4)$ with faulty elements (vertices and/or edges) and show that each pair of vertices in recursive circulant $G(2^m, 4)$, $m \geq 3$, are joined by a fault-free path of every length from $m + 1$ to $|V(G(2^m, 4)\backslash F)| - 1$ inclusive for any fault set F with $|F| \leq m - 3$. The bound $m-3$ on the number of acceptable faulty elements is the maximum possible. Moreover, recursive circulant $G(2^m, 4)$ has a fault-free cycle of every length from four to $|V(G(2^m, 4)\backslash F)|$ inclusive excluding five passing through an arbitrary fault-free edge for any fault set F with $|F| \leq m - 3$.

Key Words: Panconnected, edge-pancyclic, embedding, linear arrays, rings, recursive circulants, fault tolerance, interconnection networks.

1 Introduction

Linear arrays and rings are two of the most important computational structures in interconnection networks. So, embedding of linear arrays and rings into a faulty interconnection network is one of the important issues in parallel processing[5, 11, 19, 21–24]. An interconnection network is often modelled as a graph, in which vertices and edges correspond to nodes and communication links, respectively. Thus, the embedding problem can be modelled as finding fault-free paths and cycles in the graph with some faulty vertices and/or edges.

In the embedding problem, if the longest path or cycle is required the problem is closely related to well-known hamiltonian problems in graph theory. A graph G is called *f-fault hamiltonian* (resp. *f-fault hamiltonian-connected*) if there exists a hamiltonian cycle (resp. if each pair of vertices are joined by a hamiltonian path) in $G\backslash F$ for any set F of faulty elements with $|F| \leq f$. On the other hand, if the paths joining each pair of vertices of every length shorter than or equal to a hamiltonian path are required the problem is concerned with panconnectivity of the graph. If the cycles of arbitrary size (up to a hamiltonian cycle) are required the problem is concerned with pancyclicity of the graph.

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Definition 1. *A graph G is called* f-fault l-panconnected *if each pair of fault-free vertices are joined by a path in $G\backslash F$ of every length from l to $|V(G\backslash F)|-1$ inclusive for any set F of faulty elements with $|F| \le f$.*

Definition 2. *A graph G is called* f-fault almost edge-pancyclic *(resp.* f-fault nearly edge-pancyclic*) if for any set F of faulty elements with $|F| \le f$, there exists a cycle of every length from four to $|V(G\backslash F)|$ inclusive (resp. from four to $|V(G\backslash F)|$ inclusive excluding five) that passes through an arbitrary fault-free edge.*

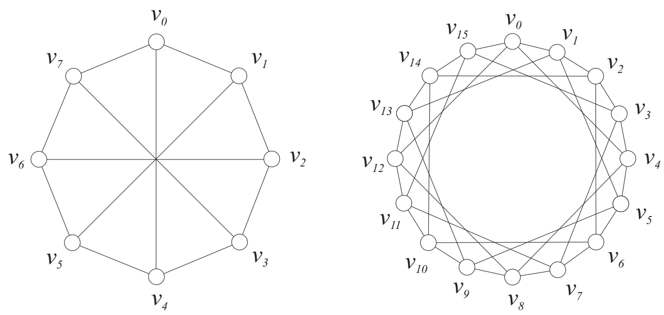
Panconnectivity of some interconnection networks without faulty elements was reported in the literature. A graph G is said to be *panconnected* (resp. *almost panconnected*) if each pair of vertices s and t in G are joined by an s-t path of every length from $d(s,t)$ to $V(G)-1$ (resp. from $d(s,t)+2$ to $V(G)-1$) inclusive. Here, $d(s,t)$ denotes the distance between s and t. Recursive circulant $G(2^m,2)$[16], alternating group graphs[5], and augmented cubes[13] are panconnected, and recursive circulant $G(2^m,4)$[16], locally twisted cubes[14], and twisted cubes[7] are almost panconnected. Recently, fault-panconnectivity of a family of hypercube-like interconnection networks called *restricted HL-graphs* was investigated in [20]. It was shown that every m-dimensional restricted HL-graph, $m \ge 3$, is $m-3$-fault $2m-3$-panconnected. The family includes many interconnection networks proposed in the literature such as twisted cubes, crossed cubes, multiply twisted cubes, Möbius cubes, Mcubes, and generalized twisted cubes.

Edge-pancyclicity of some fault-free interconnection networks such as recursive circulants, crossed cubes, twisted cubes was studied in [1], [9], and [8]. A graph G is called *f-fault l-edge-pancyclic* if for any fault set F with $|F| \le f$, there exists a cycle of every length from l to $|V(G\backslash F)|$ inclusive that passes through an arbitrary fault-free edge. An f-fault l-panconnected graph is obviously f-fault $l+1$-edge-pancyclic. In the presence of faulty elements, fault-panconnectivity result in [20] implies that every m-dimensional restricted HL-graph, $m \ge 3$, is $m-3$-fault $2m-2$-edge-pancyclic.

Pancyclicity and fault-pancyclicity of various interconnection networks were investigated. A graph G is called *f-fault pancyclic* (resp. *f-fault almost pancyclic*) if $G\backslash F$ contains a cycle of every length from three to $|V(G\backslash F)|$ inclusive (resp. four to $|V(G\backslash F)|$ inclusive) for any fault set F with $|F| \le f$. The works on fault-pancyclicity can be summarized as that many interconnection networks of degree δ are $\delta-2$-fault pancyclic or $\delta-2$-fault almost pancyclic depending on the existence of length three cycles in the network; for example, augmented cubes[13], recursive circulants[2, 17], Möbius cubes[11], crossed cubes[23], twisted cubes[24], and restricted HL-graphs[20].

Recursive circulant is an interconnection network proposed in [18]. Recursive circulant $G(N,d)$, $d \ge 2$, is defined as follows: the vertex set $V = \{v_0, v_1, v_2, \cdots, v_{N-1}\}$, and the edge set $E = \{(v_i, v_j) \mid$ there exists k, $0 \le k \le \lceil \log_d N \rceil - 1$, such that $i + d^k \equiv j \pmod{N}\}$. $G(N,d)$ is a circulant graph with N vertices and jumps of powers of d, $d^0, d^1, \cdots, d^{\lceil \log_d N \rceil - 1}$. Examples of $G(N,d)$ are shown in Figure 1.

In this work, our attention is restricted to $G(N, d)$ with $N = 2^m$ and $d = 4$. $G(2^m, 4)$, whose degree is m, compares favorably to the hypercube Q_m. While retaining attractive properties of hypercube Q_m such as node-symmetry, recursive structure, the maximum connectivity, etc., it achieves noticeable improvements in diameter[18] and possesses a complete binary tree with $2^m - 1$ vertices as a subgraph[12]. Recursive circulant has a cycle-based construction, and thus it is expected to have nice properties concerned with cycles. $G(N, d)$ with degree three or higher is hamiltonian-connected[6]. $G(N, d)$ with $N = cd^m$ and $1 \leq c < d$ is hamiltonian decomposable[3, 10, 15], that is, the set of edges can be partitioned into edge-disjoint hamiltonian cycles (and a 1-factor when the degree is odd). In [10], edge forwarding index and bisection width of recursive circulants were also analyzed.



(a) $G(8, 4)$ (b) $G(16, 4)$

Fig. 1. Examples of $G(N, d)$.

In this paper, we investigate panconnectivity and edge-pancyclicity of recursive circulant $G(2^m, 4)$ with faulty elements. It will be shown that $G(2^m, 4)$, $m \geq 3$, is $m-3$-fault $m+1$-panconnected and $m-3$-fault nearly edge-pancyclic. The bound $m - 3$ on the number of acceptable faulty elements for $G(2^m, 4)$ to be l-panconnected for any fixed l (less than the number of fault-free vertices) is the maximum possible in a sense that no graph of degree m is $m - 2$-fault l-panconnected as well as hamiltonian-connected.

In the rest of this paper, we will use standard terminology in graphs (see ref. [4]). This paper is organized as follows. In the next section, we will present some basic properties of recursive circulant $G(2^m, 4)$. Panconnectivity and edge-pancyclicity of faulty recursive circulant $G(2^m, 4)$ will be proved in Section 3 and 4, respectively. Finally in Section 5, concluding remarks of this paper will be given.

2 Recursive Circulant $G(2^m, 4)$

Recursive circulant $G(N, d)$ can also be defined as Cayley graph of the cyclic group $\mathbb{Z}_N$ with the generating set $\{d^0, d^1, \cdots, d^{\lceil \log_d N \rceil - 1}\}$. Every Cayley graph over a general group is vertex symmetric, and thus regular. Recursive circulant $G(N, d)$ has a recursive structure when $N = cd^m$, $1 \le c < d$ [18]. In other words, $G(cd^m, d)$ can be defined recursively by utilizing the following property.

Property 1. [18] Let V_i be a subset of vertices in $G(cd^m, d)$ such that $V_i = \{v_j \mid j \equiv i \pmod{d}\}$, $m \ge 1$. For $0 \le i \le d-1$, the subgraph of $G(cd^m, d)$ induced by V_i is isomorphic to $G(cd^{m-1}, d)$.

$G(cd^m, d)$, $m \ge 1$, can be constructed recursively on d copies of $G(cd^{m-1}, d)$ as follows. Let $G_i(V_i, E_i)$, $0 \le i \le d-1$, be a copy of $G(cd^{m-1}, d)$. We assume that $V_i = \{v_0^i, v_1^i, \cdots, v_{cd^{m-1}-1}^i\}$, and G_i is isomorphic to $G(cd^{m-1}, d)$ with the isomorphism mapping v_j^i to v_j. We relabel v_j^i by v_{jd+i}. The vertex set V of $G(cd^m, d)$ is $\bigcup_{0 \le i \le d-1} V_i$, and the edge set E is $\bigcup_{0 \le i \le d-1} E_i \cup X$, where $X = \{(v_j, v_{j'}) \mid j+1 \equiv j' \pmod{cd^m}\}$. The construction of $G(32, 4)$ on four copies of $G(8, 4)$ is illustrated in Figure 2. Note that recursive circulant $G(2^m, 4)$ has a recursive structure when $m \ge 2$. In the recursive structure, $G(2^m, 4)$ consists of four *components* G_0, G_1, G_2, and G_3; each of them is isomorphic to $G(2^{m-2}, 4)$. A vertex in G_i is represented by v_j^i, $0 \le j < 2^{m-2}$, $0 \le i \le 3$, as well as $v_{j'}$, $0 \le j' < 2^m$, without saying in which G_i the vertex is contained.

Hereafter in this paper, we denote by $G_i \oplus G_j$ and $G_i \oplus G_j \oplus G_k$ for some $0 \le i, j, k \le 3$ the subgraphs of $G(2^m, 4)$ induced by $V_i \cup V_j$ and $V_i \cup V_j \cup V_k$, respectively. Let F be the set of faulty elements in $G(2^m, 4)$. F_i denotes the set of faulty elements in G_i, $i = 0, 1, 2, 3$, and $F_{i,i+1 \bmod 4}$ denotes the set of faulty edges joining vertices in G_i and vertices in $G_{i+1 \bmod 4}$, so that $F = \bigcup_{0 \le i \le 3} (F_i \cup F_{i,i+1 \bmod 4})$. Let $f_i = |F_i|$ and $f_{i,i+1 \bmod 4} = |F_{i,i+1 \bmod 4}|$. We denote by f_v^i the number of faulty vertices in G_i, and by f_v the total number of faulty vertices, so that $f_v = \sum_{0 \le i \le 3} f_v^i$.

From now on, all arithmetic on the indices of vertices will be assumed to be done modulo 2^m. Some properties of recursive circulant $(2^m, 4)$ explored to establish our main results are listed below, where the *diameter* D_m of $G(2^m, 4)$ is defined as the maximum distance between any two vertices in the graph.

Lemma 1 (Shortest Path). *[18] Let G_0, G_1, G_2, and G_3 be the components of $(2^m, 4)$. (a) Every shortest path joining a pair of vertices v_0^i and v_j^i passes through only vertices in G_i. (b) There exists a shortest path between v_0^0 and v_j^i passing through v_j^0 when $i = 1$, and passing through v_{j+1}^0 when $i = 3$. In case $i = 2$, there exists a shortest path between v_0^0 and v_j^i passing through v_j^0 when $d(v_0^0, v_j^0) \le d(v_0^0, v_{j+1}^0)$, and passing through v_{j+1}^0 when $d(v_0^0, v_{j+1}^0) \le d(v_0^0, v_j^0)$.*

Lemma 2 (Diameter). *[18] (a) $D_{m-2} + 1 \le D_m \le D_{m-2} + 2$ for $m \ge 2$. (b) $D_m = \lceil \frac{3m-1}{4} \rceil$.*

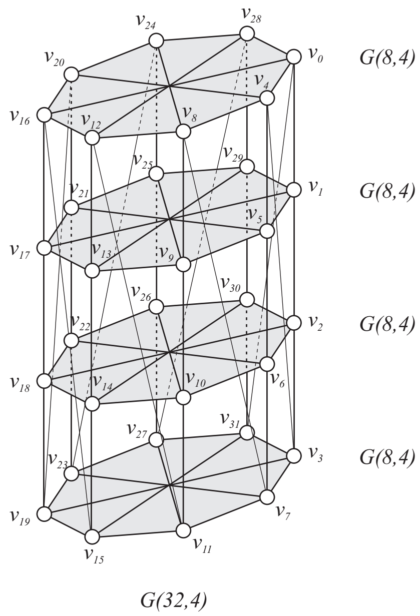


Fig. 2. Recursive structure of $G(32, 4)$.

Lemma 3 (Fault-Hamiltonicity). *[22, 19] (a) $G(2^m, 4)$, $m \geq 3$, is $m-3$-fault hamiltonian-connected and $m-2$-fault hamiltonian. (b) The product $G(2^m, 4) \times K_2$ of $G(2^m, 4)$ and K_2, $m \geq 3$, is $m-2$-fault hamiltonian-connected and $m-1$-fault hamiltonian.*

Lemma 3(a) implies that $G(2^m, 4)$, $m \geq 3$, with at most $m-1$ faulty elements has a hamiltonian path joining some pair of fault-free vertices.

3 Panconnectivity of Faulty $G(2^m, 4)$

In this section, we will show that $G(2^m, 4)$, $m \geq 3$, is $m-3$-fault $m+1$-panconnected. Throughout this paper, a path in a graph is represented as a sequence of vertices. A path joining a pair of vertices s and t is called an *s-t path*.

Panconnectivity of fault-free recursive circulants $G(2^m, 2^k)$ was investigated in [16]. It was shown that between any pair of vertices s and t, there exits a path of every length $d(s,t) + \Delta$ or longer for some Δ. One of the results is given in the following, which will be utilized for our purpose.

Lemma 4. *[16] $G(2^m, 4)$ is almost panconnected. That is, between any pair of vertices s and t in $G(2^m, 4)$, there exists a path of every length l, $d(s,t) + 2 \leq l \leq 2^m - 1$.*

A *concatenation* of two paths $(x_1, x_2, \ldots, x_p)$ and $(y_1, y_2 \ldots, y_q)$ is defined to be the path $(x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q)$.

Lemma 5. *(a) $G(2^m, 4)$, $m \geq 3$, is 0-fault $D_m + 1$-panconnected.*
(b) $G(2^m, 4)$, $m \geq 5$, is 0-fault D_m-panconnected.

Proof. We prove (a) by induction on m. Due to Lemma 4, it suffices to show that for any pair of vertices s and t with $d(s,t) = D_m$, there exists a path of length $D_m + 1$ between them. For $m = 3, 4$, the construction is by an immediate inspection. Let $m \geq 5$. We assume $s = v_1^0$ without loss of generality. There are two cases up to symmetry. If $t = v_j^1$ for some $j(\neq 1)$, we first find a v_1^1-t path P' in G_1 of length D_m. The path P' exists since $D_{m-2} + 1 \leq D_m$. Then, (s, P') is a desired path of length $D_m + 1$. Now, let $t = v_j^2$ for some $j(\neq 1)$. By Lemma 1, D_m is equal to $d(v_1^2, t) + 2$ or $d(v_0^2, t) + 2$. We assume w.l.o.g. $D_m = d(v_1^2, t) + 2$. Letting P'' be a shortest v_1^2-t path in G_2, we have a path (s, v_0^3, v_1^3, P'') of length $d(v_1^2, t) + 3 = D_m + 1$.

To prove (b), we assume that each G_i is $D_{m-2} + 1$-panconnected and furthermore, whenever $m - 2 \geq 5$, it is D_{m-2}-panconnected. It suffices to construct a path of length D_m joining every pair of vertices s and t with $d(s,t) = D_m - 1$. Let $s = v_1^0$. If $t = v_j^0$ for some $j(\neq 1)$, there exists an s-t path in G_0 of every length $D_{m-2} + 1$ or longer, and thus we are done. When $t = v_j^1$ for some $j(\neq 1)$, there exists a v_1^1-t path P' of length $D_{m-2} + 1$, and (s, P') is an s-t path of length $D_{m-2} + 2$. If $D_m = D_{m-2} + 2$, we are done. Suppose otherwise ($D_m = D_{m-2} + 1$), observe $m \geq 7$. Note that D_3, D_4, D_5, and D_6 are 2, 3, 4, 5, respectively. Employing the assumption that G_1 is D_{m-2}-panconnected, we have an s-t path (s, P'') of length D_m, where P'' is a v_1^1-t path in G_1 of length D_{m-2}. Finally when $t = v_j^2$ for some $j(\neq 1)$, assuming w.l.o.g. $d(s,t) = d(v_1^2, t) + 2$, a concatenation of (s, v_0^3, v_1^3) and a shortest v_1^2-t path in G_2 results in an s-t path of length $d(s,t) + 1 = D_m$. Thus, the proof is completed. □

Now, we are to investigate panconnectivity of faulty recursive circulants. We will show that $G(2^m, 4)$, $m \geq 3$, is $m-3$-fault $m+1$-panconnected. For $m = 3, 4$, we have the following lemma.

Lemma 6. *(a) $G(8, 4)$ is 0-fault 3-panconnected.*
(b) $G(16, 4)$ is 0-fault 4-panconnected and 1-fault 5-panconnected.

Proof. Lemma 5(a) says that $G(8, 4)$ is 0-fault 3-panconnected and $G(16, 4)$ is 0-fault 4-panconnected. To show $G(16, 4)$ is 1-fault 5-panconnected, it needs to construct an s-t path of every length 5 or longer for any pair of fault-free vertices s and t in $G(16, 4)$ with one faulty element. When the faulty element is a vertex v_f, the construction of an s-t path in $G(16, 4) \backslash v_f$ is by a case analysis and omitted here. Suppose there exists a faulty edge (x, y). If $\{x, y\} = \{s, t\}$, letting (x, y) be a *virtual* fault-free edge, Lemma 5(a) is applied. Otherwise, letting $x \notin \{s, t\}$ be a *virtual* faulty vertex, an s-t path of every length up to 14 is constructed. An s-t hamiltonian path of length 15 also exists due to Lemma 3(a). □

To prove the main result for $m \geq 5$, we exploit the recursive structure of $G(2^m, 4)$ and a technique so called "strong induction." In other words, assuming that each component G_i which is isomorphic to $G(2^{m-2}, 4)$ is not only $m-5$-fault $m-1$-panconnected but also $\frac{m-5}{2}$-fault $m-2$-panconnected and $\frac{m-5}{2^2}$-fault $m-3$-panconnected and so on, we show that $G(2^m, 4)$ is $m-3$-fault $m+1$-panconnected and $\frac{m-3}{2}$-fault m-panconnected and so on.

Theorem 1. *$G(2^m, 4)$, $m \geq 3$, is $\lfloor \frac{m-3}{2^k} \rfloor$-fault $m-k+1$-panconnected for any integer k, $0 \leq k \leq L(m-3)+1$, where $L(n) = \lfloor \log_2 n \rfloor$ for $n \geq 1$ and $L(0) = 0$.*

Proof. By Lemma 6, the theorem holds for $m = 3, 4$. Hereafter, we assume $m \geq 5$. Observe that $\lfloor \frac{m-3}{2^k} \rfloor = 0$ if $k = L(m-3)+1$, and that $\lfloor \frac{m-3}{2^k} \rfloor = 1$ if $k = L(m-3)$. When $k = L(m-3)+1$, due to Lemma 5(b), the theorem holds. We claim that $D_m = \lceil \frac{3m-1}{4} \rceil \leq m - L(m-3)$ for any $m \geq 5$. The inequality can be checked for small m in the following table. For $m \geq 19$, it suffices to show that $\frac{3m-1}{4}+1 \leq m - \lfloor \log_2(m-3) \rfloor$ or equivalently $4\lfloor \log_2(m-3) \rfloor + 3 \leq m$. Let $k \geq 4$ be an integer such that $2^k \leq m-3 < 2^{k+1}$. Then, we have $4\lfloor \log_2(m-3) \rfloor + 3 = 4k+3$ and $2^k + 3 \leq m$. Obviously $4k+3 \leq 2^k + 3$ for any $k \geq 4$, and thus the claim is proved.

m	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\lceil \frac{3m-1}{4} \rceil$	4	5	5	6	7	8	8	9	10	11	11	12	13	14
$m - L(m-3)$	4	5	5	6	7	8	8	9	10	11	12	13	14	15

From now on, we assume $0 \leq k \leq L(m-3)$. It suffices to consider the case $|F| = \lfloor \frac{m-3}{2^k} \rfloor$ since otherwise, we can choose $\lfloor \frac{m-3}{2^k} \rfloor - |F|$ fault-free edges and regard them as *virtual* faults when we construct an s-t path of every length $m-k+1$ or more. Observe that $|F| - 1 = \lfloor \frac{(m-2^k)-3}{2^k} \rfloor \leq \lfloor \frac{(m-2)-3}{2^k} \rfloor$ for any $k \geq 1$. Thus, assuming $f_0 \geq f_j$ for any $j = 1, 2, 3$, there are two cases.

Case 1: $f_i \leq \lfloor \frac{(m-2)-3}{2^k} \rfloor$ for every $i = 0, 1, 2, 3$.

It is straightforward to see that $k \leq L((m-2)-3)+1$. Thus, each $G_i \backslash F_i$ is $m-k-1$-panconnected. Furthermore when $m = 5$, G_i is fault-free and, by Lemma 6(a), it is 3-panconnected. We first consider panconnectivity of $G_0 \oplus G_1$.

Claim 1. Each pair of vertices x and y in $G_0 \oplus G_1$ are joined by an x-y path of every length l, $m-k+1 \leq l \leq 2^{m-1} - f_v^0 - f_v^1 - 1$.

To prove the claim, let x and y be vertices in G_0 first. There exists an x-y path P_0 in G_0 of every length l_0, $m-k-1 \leq l_0 \leq 2^{m-2} - f_v^0 - 1$. To construct a longer path P_1 that passes through vertices in G_1 as well as vertices in G_0, let P' be an x-y path in G_0 of every length l', $2m-5 \leq l' \leq 2^{m-2} - f_v^0 - 1$. Then, there is an edge (v_i^0, v_j^0) on P' such that all of v_i^1, (v_i^0, v_i^1), v_j^1, and (v_j^0, v_j^1) are fault-free since each faulty element can "block" at most two such candidate edges and the number of faulty elements is at most $m-3$. The path P_1 can be obtained from merging P' and a v_i^1-v_j^1 path P'' in G_1 with the edges (v_i^0, v_i^1) and (v_j^0, v_j^1). When $m \geq 6$, the length l'' of P'' is any integer in the range $m-k-1 \leq l'' \leq 2^{m-2} - f_v^1 - 1$ and thus the length l_1 of P_1 is in the range

$(2m-5)+(m-k-1)+1 \le l_1 \le 2^{m-1} - f_v^0 - f_v^1 - 1$. It is straightforward to see that $(2m-5)+(m-k-1)+1 \le (2^{m-2}-f_v^0-1)+1$ since $3m-5 \le 2^{m-2}-(m-5)$ for every $m \ge 6$. When $m = 5$, observing $f_i = 0$ for each i, we have $5 \le l' \le 7$. Furthermore, by Lemma 6(a), we have $3 \le l'' \le 7$. Thus, $9 \le l_1 \le 15$. It remains to construct an x-y path of length eight. Let v_p^0 (resp. v_q^0) be a vertex in G_0 which is either x (resp. y) or at least adjacent to it such that (i) $v_p^0 \ne y$ and $v_q^0 \ne x$, (ii) (v_p^0, v_p^1) and (v_q^0, v_q^1) are fault-free, and (iii) $v_p^0 \ne v_q^0$. Since there exists a v_p^1-v_q^1 path P'' of every length l'', $3 \le l'' \le 7$, we have an x-y path $P_1 = (s, v_p^0, v_p^1, P'', v_q^1, v_q^0, y)$ of length eight. Therefore, we have an x-y path of every length l, $m-k-1 \le l \le 2^{m-1} - f_v^0 - f_v^1 - 1$.

Now, let x be a vertex in G_0 and y be a vertex in G_1. Let v_p^1 be a vertex in G_1 which is either y or at least adjacent to it such that (i) $v_p^0 \ne x$ and (ii) v_p^0 ,v_p^1, and (v_p^0, v_p^1) are fault free. The existence of such a vertex v_p^1 is due to that there are $m-1$ candidates and at most $m-2$ blocking elements (the source x and at most $m-3$ faulty elements). Letting P' be an x-v_p^0 path in G_0 of every length l', $m-k-1 \le l' \le 2^{m-2} - f_v^0 - 1$, we have an x-y path $P_0 = (P', v_p^1, y)$ of every length l_0, $m-k+1 \le l_0 \le 2^{m-2} - f_v^0$. To construct a longer path, we let (v_q^0, v_q^1) be an edge such that (i) $v_q^0 \ne x$ and $v_q^1 \ne y$, and (ii) v_q^0, v_q^1, and (v_q^0, v_q^1) are fault-free. Letting P' be an x-v_q^0 path in G_0 of every length l', $m-k-1 \le l' \le 2^{m-2} - f_v^0 - 1$, and letting P'' be a v_q^1-y path in G_1 of every length l'', $m-k-1 \le l'' \le 2^{m-2} - f_v^1 - 1$, we have an x-y path $P_1 = (P', P'')$ of every length l_1, $2m-2k-1 \le l_1 \le 2^{m-1} - f_v^0 - f_v^1 - 1$. We have $2m-2k-1 \le 2^{m-2} - f_v^0 + 1$ since $2m-1 \le 2^{m-2} - (m-5) + 1$ for every $m \ge 5$. Therefore, we have an x-y path of every length $m-k+1$ or more. This completes the proof of Claim 1.

Note that for each of $G_1 \oplus G_2$, $G_2 \oplus G_3$, and $G_3 \oplus G_0$, we can establish the same statement as Claim 1 since we do not use the assumption of $f_0 \ge f_1, f_2, f_3$ in the proof. From now on, we will construct an s-t path of every length l, $m-k+1 \le l \le 2^m - f_v - 1$. We assume w.l.o.g. s is contained in G_0.

Subcase 1.1: t is a vertex in G_0, G_1, or G_3.
We assume w.l.o.g. t is contained in $G_0 \oplus G_1$. By Claim 1, there exists an s-t path P_0 in $G_0 \oplus G_1$ of every length l_0, $m-k+1 \le l_0 \le 2^{m-1} - f_v^0 - f_v^1 - 1$. Let P' be an s-t path in $G_0 \oplus G_1$ of every length l', $2m-5 \le l' \le 2^{m-1} - f_v^0 - f_v^1 - 1$. There is an edge (x, y) on P' such that $\bar{x}$, $(x, \bar{x})$, $\bar{y}$, and $(y, \bar{y})$ are fault-free, where $\bar{x}$ and $\bar{y}$ are the vertices in $G_2 \oplus G_3$ adjacent to x and y, respectively. Letting P'' be an $\bar{x}$-$\bar{y}$ path in $G_2 \oplus G_3$ of every length l'', $m-k+1 \le l'' \le 2^{m-1} - f_v^2 - f_v^3 - 1$, an s-t path P_1 can be obtained from merging P' and P'' with edges $(x, \bar{x})$ and $(y, \bar{y})$. The length l_1 of P_1 is any integer in the range $3m-k-3 \le l_1 \le 2^m - f_v - 1$. It holds true $3m-k-3 \le 2^{m-1} - f_v^0 - f_v^1 - 1 + 1$ since $3m-3 \le 2^{m-1} - (m-3)$ for any $m \ge 5$. Thus, we have an s-t path of every length $m-k+1$ or more.

Subcase 1.2: t is a vertex in G_2.
We let $s = v_1^0$ and $t = v_j^2$ for some j. First, we will construct an s-t path P_0 of every length l_0, $m-k+1 \le l_0 \le 2^{m-2} - \lfloor \frac{m-5}{2} \rfloor$. Let us consider the subcase when $|F| - 1 \le f_0 + f_2 \le |F|$. In this subcase, we assume w.l.o.g. $j \ne 1$. (Suppose otherwise, we can construct an s-t path P_0 with the roles of

G_1 and G_3 being interchanged in a symmetric way.) If all of v_1^1, (s, v_1^1), v_j^1, and (t, v_j^1) are fault-free, letting P' be a v_1^1-v_j^1 path in G_1 of every length l', $m-k-1 \leq l' \leq 2^{m-2} - f_v^1 - 1$, we have an s-t path $P_0 = (s, P', t)$ of every length l_0, $m-k+1 \leq l_0 \leq 2^{m-2} - f_v^1 + 1$. Obviously, $2^{m-2} - \lfloor \frac{m-5}{2} \rfloor \leq 2^{m-2} - f_v^1 + 1$. Suppose otherwise, exactly one among the four elements v_1^1, (s, v_1^1), v_j^1, and (t, v_j^1) is faulty. If $j \neq 0$, an s-t path P_0 passing through vertices in G_3 can be constructed symmetrically. Let $j = 0$ and let v_i^2 be a vertex in G_2 adjacent to t such that v_i^2 and (t, v_i^2) are fault-free. There is a v_0^3-v_i^3 path P' in G_3 of every length l', $3 \leq l' \leq 2^{m-2} - 1$, by Lemma 4. Note that G_3 is fault-free and v_0^3 is adjacent to v_i^3. Thus, the length l_0 of $P_0 = (s, P', v_i^2, t)$ is any integer in the range $6 \leq l_0 \leq 2^{m-2} + 2$. Observe that $6 \leq m - k + 1$ for any m and k with $m \geq 5$ and $0 \leq k \leq L(m-3)$ except only when $m = 5$ and $k = 1$ ($|F| = 1$). For the exceptional case, regarding the faulty element as a *virtual* fault-free one, we will construct two vertex-disjoint s-t paths of length five. Letting P' be an s-v_0^0 path of length three in G_0 and P'' be a v_1^2-t path of length three in G_2, we have two paths (P', v_0^1, t) and (s, v_1^1, P''). At least one of the two are fault-free paths since $|F| = 1$.

Now we will construct an s-t path P_0 of every length l_0, $m - k + 1 \leq l_0 \leq 2^{m-2} - \lfloor \frac{m-5}{2} \rfloor$, when $f_0 + f_2 \leq |F| - 2$ ($|F| \geq 2$). Remember $f_2 \leq f_0$. Then, in the following claim, we can obtain a stronger result than that $G_2 \backslash F_2$ is $m-k-1$-panconnected.

Claim 2. $G_2 \backslash F_2$ is $m - k - 2$-panconnected.

To prove the claim, it suffices to show that $f_2 \leq \lfloor \frac{(m-2)-3}{2^{k+1}} \rfloor$ and $k+1 \leq L((m-2)-3)+1$. Suppose $f_2 \geq \lfloor \frac{(m-2)-3}{2^{k+1}} \rfloor + 1$, we have $f_0 + f_2 \geq 2\lfloor \frac{(m-2)-3}{2^{k+1}} \rfloor + 2 \geq \lfloor \frac{(m-2)-3}{2^k} \rfloor + 1 \geq \lfloor \frac{m-3}{2^k} + \frac{2 \cdot 2^k - 2}{2^k} \rfloor - 1 \geq \lfloor \frac{m-3}{2^k} \rfloor - 1 = |F| - 1$, which is a contradiction. Suppose $k \geq L((m-2)-3)+1$, we have $|F| = \lfloor \frac{m-3}{2^k} \rfloor \leq \lfloor \frac{m-3}{2^{L((m-2)-3)+1}} \rfloor \leq 1$ since $m - 3 < 2 \cdot 2^{L((m-2)-3)+1}$ for any $m \geq 5$. This is a contradiction to $|F| \geq 2$. Thus, we have the claim.

In the subcase of $f_0 + f_2 \leq |F| - 2$, we assume w.l.o.g. that $p \neq j$ for each vertex v_p^0 adjacent to s. (Suppose otherwise, we can construct an s-t path P_0 passing through a vertex in G_3 instead of a vertex in G_1 in a symmetric way. Note that for any pair of vertices v_i and v_{i+1}, there exists no vertex adjacent to both v_i and v_{i+1} since $G(2^m, 4)$ does not have a cycle of length three.) There exists a vertex v_p^0 adjacent to s such that (s, v_p^0, v_p^1, v_p^2) is a fault-free path (and $v_p^2 \neq t$). Letting P' be a v_p^2-t path in G_2 of every length l', $m-k-2 \leq l' \leq 2^{m-2} - f_v^2 - 1$, we have an s-t path $P_0 = (s, v_p^0, v_p^1, P')$ of every length l_0, $m - k + 1 \leq l_0 \leq 2^{m-2} - f_v^2 + 2$. Obviously, $2^{m-2} - \lfloor \frac{m-5}{2} \rfloor \leq 2^{m-2} - f_v^2 + 2$.

We are to construct a longer path P_1 that passes through vertices in G_0, G_1, and G_2. There exists a fault-free vertex v_i^2 in G_2 adjacent to t such that all of v_i^1, (v_i^2, t), and (v_i^2, v_i^1) are fault-free. Letting P' be an s-v_i^1 path in $G_0 \oplus G_1$ of every length l', $m - k + 1 \leq l' \leq 2^{m-1} - f_v^0 - f_v^1 - 1$, we have an s-t path $P_1 = (P', v_i^2, t)$ of every length l_1, $m-k+3 \leq l_1 \leq 2^{m-1} - f_v^0 - f_v^1 + 1$. Observe $m-k+3 \leq 2^{m-2} - \lfloor \frac{m-5}{2} \rfloor + 1$ since $m+3 \leq 2^{m-2} - \lfloor \frac{m-5}{2} \rfloor + 1$ for any $m \geq 5$. Finally, it remains to construct a path P_2 longer than P_1. P_2 is constructed from

P_1 by replacing the edge (v_i^2, t) with a v_i^2-t path in $G_2 \oplus G_3$ of every length l'', $m-k+1 \le l'' \le 2^{m-1} - f_v^2 - f_v^3 - 1$. Then, the length l_2 of P_2 is any integer in the rance $2m-2k+3 \le l_2 \le 2^m - f_v - 1$. Observe $2m-2k+3 \le 2^{m-1} - f_v^0 - f_v^1 + 2$ since $2m+3 \le 2^{m-1} - (m-3) + 2 \le 2^{m-1} - f_v^0 - f_v^1 + 2$ for any $m \ge 5$.

Case 2: either $k \ge 1$ and $F_0 = F$ or $k = 0$ and $|F_0| \ge |F| - 1$.
In this case, we have $f_0 \ge 1$. Let us consider panconnectivity of G_1, $G_1 \oplus G_2$, $G_2 \oplus G_3$, and $G_1 \oplus G_2 \oplus G_3$ first in the following Claims 3 through 5.

Claim 3. $G_i \backslash F_i$ is $m-k-1$-panconnected for every $i = 1, 2, 3$, except only when $m = 5$, $k = 0$, $f_0 = 1$, and $f_j = 1$ for some $j = 1, 2, 3$.

Recall that G_i is $\lfloor \frac{(m-2)-3}{2^k} \rfloor$-fault $m-k-1$-panconnected. The claim holds for $k \ge 1$ or $m \ge 6$ and $k = 0$ since $|F_i| = 0$ for $k \ge 1$ and $\lfloor \frac{(m-2)-3}{2^k} \rfloor = m-5 \ge |F_i|$ for $m \ge 6$ and $k = 0$. If $m = 5$ and $k = 0$, we have $|F| = 2$, and thus the claim holds only when $f_1 = f_2 = f_3 = 0$. This completes proof of the claim.

For the exceptional case of Claim 3, it will be proved later in Lemma 7 that $G(2^5, 4) \backslash F$ with $|F| = 2$ and $f_0 = f_j = 1$ for some $j = 1, 2, 3$, is 6-panconnected. Hereafter in this proof, we will exclude the exceptional case. Then, we have $|F_i| \le \lfloor \frac{(m-2)-3}{2^k} \rfloor$ for every $i = 1, 2, 3$. By virtue of Claim 1, we have Claim 4.

Claim 4. $G_1 \oplus G_2 \backslash F$ and $G_2 \oplus G_3 \backslash F$ are $m-k+1$-panconnected.

Claim 5. $G_1 \oplus G_2 \oplus G_3 \backslash F$ is $m-k+1$-panconnected with an exception of $m = 5$ and $k = 1$.

To prove the claim, between any pair of vertices x and y, an x-y path of every length $m-k+1$ or more will be constructed. First, we consider the case that x and y are contained in $G_1 \oplus G_2$. There exists an x-y path P_0 in $G_0 \oplus G_1$ of every length l_0, $m-k+1 \le l_0 \le 2^{m-1} - f_v^1 - f_v^2 - 1$, by Claim 4. To construct a longer path, we assume w.l.o.g. $F_3 = F_{2,3} = \emptyset$ if both x and y are contained in G_2. Let P' be an x-y path in $G_1 \oplus G_2$ of every length $l' \ge 2^{m-2} + 4$. Then, there exists an edge (v_p^2, v_q^2) on P' such that v_p^3, (v_p^2, v_p^3), v_q^3, and (v_q^2, v_q^3) are fault-free. Letting P'' be a v_p^3-v_q^3 path in G_3 of every length l'', $l'' \ge m-k-1$ for $m \ge 6$ and $l'' \ge 3$ for $m = 5$. The length l_1 of an x-y path P_1 obtained from merging P' and P'' is any integer in the range $2^{m-2} + 4 + m - k \le l_1 \le 3 \cdot 2^{m-2} - f_v^1 - f_v^2 - f_v^3 - 1$ for $m \ge 6$ and in the range $2^{m-2} + 4 + 4 \le l_1 \le 3 \cdot 2^{m-2} - f_v^1 - f_v^2 - f_v^3 - 1$ for $m = 5$. It is straightforward to check that $2^{m-2} + 4 + m - k \le 2^{m-1} - f_v^1 - f_v^2$ for $m \ge 6$ and $2^{m-2} + 4 + 4 \le 2^{m-1} - f_v^1 - f_v^2 = 2^{m-1}$ for $m = 5$.

Let x and y be vertices in G_1 and G_3, respectively, and let $x = v_1^1$ and $y = v_j^3$. When $j \ne 1$, we assume w.l.o.g. path (x, v_1^2, v_1^3) and G_3 are fault-free. Letting P' be a v_1^3-y path in G_3 of every length $l' \ge m-k-1$, we have an x-y path $P_0 = (x, v_1^2, P')$ of every length l_0, $m-k+1 \le l_0 \le 2^{m-2} + 1$. When $j = 1$ ($y = v_1^3$), there exists a vertex v_i^1 adjacent to x such that path $(x, v_i^1, v_i^2, v_i^3, y)$ is fault-free. Assuming w.l.o.g. G_3 is fault-free, we find a v_i^3-y path P' in G_3 of every length l', $3 \le l' \le 2^{m-2} - 1$, by Lemma 4. Then, we have an x-y path $P_0 = (x, v_i^1, v_i^2, P')$ of every length l_0, $6 \le l_0 \le 2^{m-2} + 2$. Note that $6 \le m-k+1$ unless $m = 5$ and $k = 1$. To construct a longer path P_1, let (v_p^2, v_p^3) be a fault-free edge with v_p^2 and v_p^3 being fault-free. Letting P' be an x-v_p^2 path in $G_1 \oplus G_2$ of every length l', $m-k+1 \le l' \le 2^{m-1} - f_v^1 - f_v^2 - 1$, and P'' be a v_p^3-y path in G_3 of every length l'', $D_{m-2}+1 \le l'' \le 2^{m-2}-1$, we have an x-y path $P_1=(P',P'')$ of every length l_1, $m-k+D_{m-2}+3 \le l_1 \le 3\cdot 2^{m-2}-f_v^1-f_v^2-1$. Since $m+D_{m-2}+3 \le 2^{m-2}+2$ for any $m \ge 5$, an x-y path of every length $m-k+1$ or more is constructed. Thus, proof of the claim is completed.

The exceptional case of Claim 5 is considered later in Lemma 8. It will be proved that $G(2^5,4)\backslash F$ with $|F|=f_0=1$ is 5-panconnected. We also exclude the exceptional case in our discussion. Now, we will construct an s-t path of every length $m-k+1$ or more. An s-t path of every length between $m-k+1$ and $3\cdot 2^{m-2}-2$ is constructed in Subcases 2.1 through 2.4, and a path of every length $3\cdot 2^{m-2}-1$ or more is constructed in Subcases 2.5 through 2.7.

Subcase 2.1: both s and t are contained in G_0.
Assume w.l.o.g. $F_{0,1}\cup F_1=\emptyset$. Let $s=v_1^0$ and $t=v_j^0$. Letting P' be a v_1^1-v_j^1 path in G_1 of every length l', $m-k-1 \le l' \le 2^{m-2}-1$, we have an s-t path $P_0=(s,P',t)$ of every length l_0, $m-k+1 \le l_0 \le 2^{m-2}+1$. Letting P'' be a v_1^1-v_j^1 path in $G_1\oplus G_2\oplus G_3$ of every length $l'' \ge m-k+1$, we have a longer path $P_1=(s,P'',t)$ of every length l_1, $m-k+3 \le l_1 \le 3\cdot 2^{m-2}-f_v^1-f_v^2-f_v^3+1$.

Subcase 2.2: s and t are contained in G_0 and G_1, respectively.
Let $s=v_1^0$ and $t=v_j^1$. Let v_i^0 is a vertex in G_0 which is either s or at least adjacent to it such that $i\ne j$ and path (s,v_i^0,v_i^1) is fault-free. Letting P' be a v_i^1-t path in G_1 of every length $l' \ge m-k-1$, we have an s-t path $P_0=(s,v_i^0,P')$ of every length l_0, $m-k+1 \le l_0 \le 2^{m-2}-f_v^1$. Letting P'' be a v_i^1-t path in $G_1\oplus G_2\oplus G_3$ of every length $l'' \ge m-k+1$, we have an s-t path $P_1=(s,v_i^0,P'')$ of every length l_1, $m-k+3 \le l_1 \le 3\cdot 2^{m-2}-f_v^1-f_v^2-f_v^3$.

Subcase 2.3: s and t are contained in G_0 and G_2, respectively.
Let $s=v_1^0$ and $t=v_j^2$, and assume w.l.o.g. $j\ne 1$. When $F_{0,1}\cup F_1\cup F_{1,2}=\emptyset$, letting P' be a v_1^1-v_j^1 path in G_1 of every length $l' \ge m-k-1$, we have an s-t path P_0 of every length l_0, $m-k+1 \le l_0 \le 2^{m-2}+1$. If $F_{0,1}\cup F_1\cup F_{1,2}\ne\emptyset$, then $F_{0,3}\cup F_3\cup F_{3,2}=\emptyset$. When $j\ne 0$, an s-t path P_0 passing through vertices in G_3 can be constructed symmetrically. When $j=0$, letting P' be a v_0^3-v_1^3 path in G_3 of length three or more, we have an s-t path $P_0=(s,P',v_1^2,t)$ of every length l_0, $6(\le m-k+1) \le l_0 \le 2^{m-2}+2$. To construct a longer path, let v_i^0 be a vertex in G_0 adjacent to s such that (s,v_i^0,v_i^1) is a fault-free path. Letting P'' be a v_i^1-t path in $G_1\oplus G_2\oplus G_3$ of every length $m-k+1$ or more, we have an s-t path $P_1=(s,v_i^0,P'')$ of every length l_1, $m-k+3 \le l_1 \le 3\cdot 2^{m-2}-f_v^1-f_v^2-f_v^3+1$.

Subcase 2.4: both s and t are contained in $G_1\oplus G_2\oplus G_3$.
By Claim 5, we have an s-t path P_0 in $G_1\oplus G_2\oplus G_3$ of every length l_0, $m-k+1 \le l_0 \le 3\cdot 2^{m-2}-f_v^1-f_v^2-f_v^3-1$.

Subcase 2.5: both s and t are contained in $G_0\oplus G_1$.
For a vertex x in $G_0\oplus G_1$, we denote by $\bar{x}$ the vertex in $G_2\oplus G_3$ adjacent to x. If $f_0+f_{0,1}+f_1 \le m-4$, then there exists an s-t hamiltonian path P' in $G_0\oplus G_1$ by Lemma 3(b). Let (x,y) be an edge on P' such that $\bar{x}$, $(x,\bar{x})$, $\bar{y}$, and $(y,\bar{y})$ are fault-free, so that $P'=(s,Q_1,x,y,Q_2,t)$. Letting P'' be an $\bar{x}$-$\bar{y}$ path in $G_2\oplus G_3$ of every length $m-k+1$ or more, we have an s-t path $P_2=(s,Q_1,x,P'',y,Q_2,t)$ of every length l_2, $2^{m-1}-f_v^0-f_v^1+m-k+1 \le l_2 \le 2^m-f_v-1$. If $f_0+f_{0,1}+f_1=m-3$, there exists a hamiltonian cycle

(s, Q_1, x, t, Q_2, y) in $G_0 \oplus G_1$. Then, letting P'' be an $\bar{x}$-$\bar{y}$ path in $G_2 \oplus G_3$ of every length $m-k+1$ or more, we have an s-t path $P_2 = (s, Q_1, x, P'', y, Q_2^R, t)$ of every length $2^{m-1} - f_v^0 - f_v^1 + m - k + 1$ or more. Here, Q_2^R denotes the reverse of path Q_2, that is, $Q_2^R = (z_l, z_{l-1}, \ldots, z_1)$ for $Q_2 = (z_1, z_2, \ldots, z_l)$. Obviously, $2^{m-1} - f_v^0 - f_v^1 + m - k + 1 \leq 3 \cdot 2^{m-2} - 1$ for any $m \geq 5$.

Subcase 2.6: s is contained in $G_0 \oplus G_1$ and t is contained in $G_2 \oplus G_3$.
If $f_0 + f_{0,1} + f_1 \leq m-4$, we let x be a vertex in $G_0 \oplus G_1$ such that $x \neq s$, $\bar{x} \neq t$, and all of x, $\bar{x}$, $(x, \bar{x})$ are fault-free. Then, letting P' be an s-x hamiltonian path in $G_0 \oplus G_1$ and P'' be an $\bar{x}$-t path in $G_2 \oplus G_3$ of every length $m-k+1$ or more, we have an s-t path $P_2 = (P', P'')$ of every length l_2, $2^{m-1} - f_v^0 - f_v^1 + m - k + 1 \leq l_2 \leq 2^m - f_v - 1$. If $f_0 + f_{0,1} + f_1 = m-3$, there exists a hamiltonian cycle (s, x, Q, y) in $G_0 \oplus G_1$. Assume w.l.o.g. $\bar{y} \neq t$. Letting P'' be a $\bar{y}$-t path of every length $m-k+1$ or more, we have an s-t path $P_2 = (s, x, Q, y, P'')$ of every length $2^{m-1} - f_v^0 - f_v^1 + m - k + 1$ or more.

Subcase 2.7: both s and t are contained in G_2.
There exists a hamiltonian path in $G_0 \backslash F_0$ by Lemma 3(a) and let the hamiltonian path be (v_a^0, Q, v_b^0). Assume w.l.o.g. that v_a^1, (v_a^0, v_a^1), v_{b-1}^3, and (v_b^0, v_{b-1}^3) are all fault-free. We first find an s-t path P' in G_2 of every length $m-k-1$ or more by Claim 3. Since $m-k-1 \geq 4$, there exists an edge (v_x^2, v_y^2) on P' such that $x \neq a$, $y \neq b-1$, and all of v_x^1, (v_x^2, v_x^1), v_y^3, (v_y^2, v_y^3) are fault-free. Let $P' = (s, Q_1, v_x^2, v_y^2, Q_2, t)$. Then, we have an s-t path $P_2 = (s, Q_1, v_x^2, P'', v_a^0, Q, v_b^0, P''', v_y^2, Q_2, t)$, where P'' is a v_x^1-v_a^1 path in G_1 of length $m-k-1$ or more and P''' is a v_{b-1}^3-v_y^3 path in G_3 of length $m-k-1$ or more. The length l_2 of P_2 is any integer in the range $2^{m-2} - f_v^0 + 3(m-k) - 1 \leq l_2 \leq 2^m - f_v - 1$. It is straightforward to check that $2^{m-2} - f_v^0 + 3(m-k) - 1 \leq 3 \cdot 2^{m-2} - 1$ for any $m \geq 5$. □

Lemma 7. *$G(2^5, 4) \backslash F$ with $|F| = 2$ and $f_0 = f_j = 1$ for some $j = 1, 2, 3$, is 6-panconnected.*

Proof. We can see that $G(8, 4) \times K_2$ is 1-fault 5-panconnected since $G(8, 4) \times K_2$ is a 4-dimensional restricted HL-graph and every 4-dimensional restricted HL-graph was shown to be 1-fault 5-panconnected in [20]. Due to vertex symmetry, we assume $f_1 = 0$ (either $f_2 = 1$ or $f_3 = 1$). When s and t are contained in $G_0 \oplus G_1$, there exists an s-t path P_0 of every length l_0, $5 \leq l_0 \leq 2^4 - f_v^0 - 1$. For some edge (x, y) on P_0 such that the vertices $\bar{x}$ and $\bar{y}$ in $G_2 \oplus G_3$ adjacent to x and y, respectively, are fault-free, letting P' be an $\bar{x}$-$\bar{y}$ path in $G_2 \oplus G_3$ of every length $l' \geq 5$, we can obtain an s-t path P_1 from merging P_0 and P'. The length l_1 of P_1 is any integer in the range $11 \leq l_1 \leq 2^5 - f_v - 1$.

When s is contained in $G_0 \oplus G_1$ and t is contained in $G_2 \oplus G_3$, we first construct an s-t path of every length seven or more. There exists an edge $(x, \bar{x})$ joining a vertex x in $G_0 \oplus G_1$ and a vertex $\bar{x}$ in $G_2 \oplus G_3$ such that (i) $\bar{x}$ is adjacent to t, (ii) $x \neq s$, and (iii) path $(t, \bar{x}, x)$ is fault-free. Letting P' be an s-x path in $G_0 \oplus G_1$ of every length $l' \geq 5$, we have an s-t path $P_0 = (P', \bar{x}, t)$ of every length l_0, $7 \leq l_0 \leq 2^4 - f_v^0 + 1$. Replacing the edge $(\bar{x}, t)$ on P_0 with an $\bar{x}$-t path P'' in $G_2 \oplus G_3$ of every length $l'' \geq 5$ results in an s-t path $P_1 = (P', P'')$ of every length l_1, $11 \leq l_1 \leq 2^5 - f_v - 1$. It remains to construct an s-t path of length six.

If the vertex $\bar{t}$ in $G_0 \oplus G_1$ adjacent to t is fault-free and different from s, then the above construction with $\bar{t} = x$ and $t = \bar{x}$ will be sufficient. Symmetrically, if $\bar{s} \neq t$ and $\bar{s}$ is fault-free, we are done. Thus, we assume that s is adjacent to t, or both $\bar{s}$ and $\bar{t}$ are the faulty elements.

For the subcase that s is adjacent to t, let $s = v_1^0$ and $t = v_0^3$. If $f_3 = 0$, we are done since $G_3 \oplus G_0$ is 1-fault 5-panconnected. Otherwise ($f_1 = f_2 = 0$), letting P' be a v_1^1-v_0^1 path in G_1 of length three by Lemma 6(a), we have an s-t path (s, P', v_0^2, t) of length six. Finally, let us consider the subcase that both $\bar{s}$ and $\bar{t}$ are faulty vertices. Since $\bar{t}$ is faulty and $f_1 = 0$, t is contained in G_3. Let $t = v_i^3$. If s is contained in G_0, regarding $\bar{t}$ as a *virtual* fault-free vertex, we find an s-$\bar{t}$ path in G_0 of length five. Letting the path found be $(s, Q, v_j^0, \bar{t})$, we have an s-t path $(s, Q, v_j^0, v_{j-1}^3, t)$ of length six. If s is contained in G_1, let v_j^3 be a vertex adjacent to t such that $v_j^1 \neq s$. Observe that path (v_j^1, v_j^2, v_j^3, t) is fault-free. Letting P' be an s-v_j^1 path in G_1 of length three by Lemma 6(a), there exists an s-t path (P', v_j^2, v_j^3, t) of length six. □

Lemma 8. *$G(2^5, 4) \backslash F$ with $|F| = f_i = 1$ for some $i = 0, 1, 2, 3$, is 5-panconnected.*

Proof. By Lemma 7, it suffices to construct an s-t path of length five. If s and t are contained in $G_i \oplus G_{i+1 \bmod 4}$ for some $i = 0, 1, 2, 3$, we are done since $G(8, 4) \times K_2$ is 1-fault 5-panconnected. It is assumed w.l.o.g. that $s = v_1^0$ and $t = v_j^2$ for some $j \neq 1$. We can see that (i) $f_2 = 0$ and (s, v_1^1, v_1^2) is a fault-free path, or (ii) $f_0 = 0$ and (t, v_j^1, v_j^0) is a fault-free path. If condition (i) is satisfied, we have an s-t path (s, v_1^1, P'), where P' is a v_1^2-t path in G_2 of length three; otherwise, an s-t path can be constructed symmetrically. □

Remark 1. Let l_m^* be the minimum l_m such that $G(2^m, 4)$ is $m-3$-fault l_m-panconnected. Theorem 1 suggests an upper bound $m+1$ on l_m^*. Of course, l_m^* cannot be smaller than D_m, and thus we have $\lceil \frac{3m-1}{4} \rceil \leq l_m^* \leq m+1$.

4 Edge-Pancyclicity of Faulty $G(2^m, 4)$

In this section, we will show that $G(2^m, 4)$, $m \geq 3$, is $m-3$-fault nearly edge-pancyclic. Since an f-fault l-panconnected graph is always f-fault $l+1$-edge-pancyclic, by Theorem 1, we have the following lemma.

Lemma 9. *$G(2^m, 4)$, $m \geq 3$, is $m-3$-fault $m+2$-edge-pancyclic.*

We are to show that $G(2^m, 4)$, $m \geq 3$, with at most $m-3$ faulty elements has a cycle of every length l, $l = 4, 6, 7, 8, \ldots, m+1$, passing through an arbitrary fault-free edge.

Lemma 10. *(a) $G(2^3, 4)$ is 0-fault almost edge-pancyclic.*
(b) $G(2^4, 4)$ is 1-fault nearly edge-pancyclic.

Proof. The statement (a) is obvious from Lemma 6(a). To prove (b), it suffices to construct a cycle of length four that passes through an arbitrary edge e by Lemma 9. There are two cases up to symmetry. If $e = (v_0^0, v_1^0)$, then at least one of the two cycles $(v_0^0, v_1^0, v_2^0, v_3^0)$ and $(v_0^0, v_1^0, v_1^1, v_0^1)$ are fault-free. If $e = (v_0^0, v_0^1)$, then cycles $(v_0^0, v_0^1, v_1^1, v_1^0)$ or $(v_0^0, v_0^1, v_3^1, v_3^0)$ are fault-free. Thus, we have the lemma. □

Theorem 2. *$G(2^m, 4)$, $m \geq 3$, is $m-3$-fault nearly edge-pancyclic.*

Proof. For $m = 3, 4$, the theorem holds by Lemma 10. Assume $m \geq 5$. Let e be an arbitrary fault-free edge whose two endvertices are also fault-free. By Lemma 9, it suffices to construct a cycle of every length l, $l = 4, 6, 7, 8, \ldots, m+1$, that passes through e. There are two cases up to symmetry.

Case 1: $e = (v_i^0, v_j^0)$.
If $f_0 \leq (m-2)-3$, we have a cycle of every length l, $l = 4, 6, 7, 8, \ldots, 2^{m-2} - f_v^0$, and we are done since $m+1 \leq 2^{m-2} - (m-5) \leq 2^{m-2} - f_v^0$ for any $m \geq 5$. Let $f_0 \geq m-4$. Then, there exists at most one faulty element outside G_0, and thus $F_{0,1} \cup F_1 = \emptyset$ or $F_{0,3} \cup F_3 = \emptyset$. Assume w.l.o.g. $F_{0,1} \cup F_1 = \emptyset$. There exists a v_i^1-v_j^1 path P' in G_1 of length l', $l' = 1, 3, 4, 5, \ldots, 2^{m-1} - 1$, by Lemma 4. Thus, we have a cycle (v_i^0, P', v_j^0) of every length l, $l = 4, 6, 7, 8, \ldots, 2^{m-2} + 2$.

Case 2: $e = (v_i^0, v_i^1)$.
We first construct a cycle of every even length l, $4 \leq l \leq m+1$. Let $F' = \{v_a^0 | v_a^1 \in F \text{ or } (v_a^0, v_a^1) \in F\} \cup \{(v_a^0, v_b^0) | (v_a^1, v_b^1) \in F\}$. Obviously, $|F' \cup F_0| \leq m-3$. By Lemma 3(a), there exists a hamiltonian path in $G_0 \backslash F_0 \cup F'$ of length at least $2^{m-2} - (m-3) - 1$. The hamiltonian path passes through v_i^0, and thus we can construct a fault-free v_i^0-path of every length k, $1 \leq k \leq \lfloor \frac{2^{m-2}-(m-3)}{2} \rfloor$. Let the v_i^0-path in $G_0 \backslash F_0 \cup F'$ be $(v_i^0, v_{i_1}^0, v_{i_2}^0, \ldots, v_{i_k}^0)$. Then, by the construction, v_i^1-path $(v_i^1, v_{i_1}^1, v_{i_2}^1, \ldots, v_{i_k}^1)$ is also fault-free. Furthermore, the edge $(v_{i_k}^0, v_{i_k}^1)$ is fault-free. Thus, we have a cycle $(v_i^0, v_{i_1}^0, \ldots, v_{i_k}^0, v_{i_k}^1, \ldots, v_{i_1}^1, v_i^1)$ of length $2k+2$ for every k, $1 \leq k \leq \lfloor \frac{2^{m-2}-(m-3)}{2} \rfloor$. The construction of every even cycle passing through e is completed since $2\lfloor \frac{2^{m-2}-(m-3)}{2} \rfloor + 2 \geq 2^{m-2} - (m-3) + 1 \geq m+1$ for any $m \geq 5$.

Now, it remains to construct a cycle of every odd length l, $7 \leq l \leq m+1$. We first claim that for some vertex v_z^0 in G_0 adjacent to v_i^0, a cycle $C_z = (v_i^0, v_z^0, v_{z-1}^3, v_z^3, v_z^2, v_z^1, v_i^1)$ associated with v_z^0 is fault-free. There are totally $m-2$ cycles associated with vertices in G_0 adjacent to v_i^0, and any two cycles among them are disjoint excluding v_i^0, v_i^1, and (v_i^0, v_i^1). Note that it is impossible for both v_z^0 and v_{z-1}^0 to be adjacent to v_i^0 since $G(2^m, 4)$ has no cycle of length three. Since there are at most $m-3$ faulty elements, at least one of the cycles are fault-free. Thus, the claim is proved. Observe that C_z has a single edge in G_0, in G_1, and in G_3, respectively. It is straightforward to see that at least one of G_0, G_1, and G_3 have at most $m-5$ faulty elements. Assume w.l.o.g. $f_0 \leq m-5$. Remember the cycle C_z is of length seven. Since G_0 is $m-5$-fault nearly edge-pancyclic, $G_0 \backslash F_0$ has a cycle C passing through (v_i^0, v_z^0) of every even length l', $4 \leq l' \leq 2^{m-2} - f_v^0$. Thus, there exists a v_i^0-v_z^0 path $P = C \backslash (v_i^0, v_z^0)$ in G_0 of every odd length l'', $3 \leq l'' \leq 2^{m-2} - f_v^0 - 1$. If we replace the edge (v_i^0, v_z^0) of C_z with v_i^0-v_z^0 path P, we have a cycle of every odd length l, $9 \leq l \leq 2^{m-2} - f_v^0 + 5$. Obviously, $m+1 \leq 2^{m-2} - (m-5) + 5 \leq 2^{m-2} - f_v^0 + 5$ for any $m \geq 5$. This completes the proof. □

Remark 2. $G(2^4, 4)$ has a unique cycle $(v_0^0, v_0^1, v_0^2, v_0^3, v_1^0)$ of length five passing through edge (v_0^0, v_1^0). Thus, we cannot say that every $G(2^m, 4)$, $m \geq 3$, is $m-3$-fault almost edge-pancyclic.

5 Concluding Remarks

In this paper, we have proven that every recursive circulant $G(2^m, 4)$ with $m \geq 3$ is $m-3$-fault $m+1$-panconnected. Here, the upper bound $m-3$ on the number of faulty elements is the maximum possible in a sense that, for any f with $f \geq m-2$, there exists a fault set F with $|F| = f$ such that $G(2^m, 4)\backslash F$ is not l_m-panconnected for any l_m, $l_m \leq |V(G\backslash F)| - 1$. We have also shown that the result on fault-panconnectivity of $G(2^m, 4)$ leads to the fact that $G(2^m, 4)$, $m \geq 3$, is $m-3$-fault nearly edge-pancyclic. There remains a number of interesting issues for future research. Finding the minimum l_m^* such that $G(2^m, 4)$ is $m-3$-fault l_m^*-panconnected will be one of them.

References

1. T. Araki, "Edge-pancyclicity of recursive circulants," *Inform. Proc. Lett.* **88**, pp. 287-292, 2003.
2. T. Araki and Y. Shibata, "Pancyclicity of recursive circulant graphs," *Inform. Proc. Lett.* **81**, pp. 187-190, 2002.
3. D.K. Biss, "Hamiltonian decomposition of recursive circulant graphs," *Discrete Mathematics* **214**, pp. 89-99, 2000.
4. J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, 5th printing, American Elsevier Publishing Co., Inc., 1976.
5. J.-M. Chang, J.-S. Yang, Y.-L. Wang, and Y. Cheng, "Panconnectivity, fault-tolerant hamiltonicity and hamiltonian-connectivity in alternating group graphs," *Networks* **44**, pp. 302-310, 2004.
6. C. C. Chen and N. F. Quimpo, "On strongly hamiltonian abelian group graphs," *Lecture Notes in Mathematics* **884** (Springer, Berlin), Australian Conference on Combinatorial Mathematics, pp. 23-34, 1980.
7. J. Fan, X. Jia, and X. Lin, "Optimal embeddings of paths with various lengths in twisted cubes," *IEEE Trans. Parallel and Distributed Systems* **18**, pp. 511-521, 2007.
8. J. Fan, X. Lin, X. Jia, R.W.H. Lau, "Edge-pancyclicity of twisted cubes," in *Proc. of International Symposium on Algorithms and Computation ISAAC 2005*, pp. 1090-1099, Dec. 2005.
9. J. Fan, X. Lin, X. Jia, "Node-pancyclicity and edge-pancyclicity of crossed cubes," *Inform. Proc. Lett.* **93**, pp. 133-138, 2005.

10. G. Gauyacq, C. Micheneau, and A. Raspaud, "Routing in recursive circulant graphs: edge forwarding index and hamiltonian decomposition," in *Proc. of the 24th International Workshop on Graph-Theoretic Concepts in Computer Science (WG98)*, pp. 227-241, 1998.
11. S.-Y. Hsieh and N.-W. Chang, "Cycle embedding on the Möbius cube with both faulty nodes and faulty edges," in *Proc. of 11th International Conference on Parallel and Distributed Systems ICPADS 2005*, 2005.
12. H.-S. Lim, J.-H. Park, and K.-Y. Chwa, "Embedding trees into recursive circulants," *Discrete Applied Mathematics* **69**, pp. 83-99, 1996.
13. M. Ma, G. Liu, and J.-M. Xu, "Panconnectivity and edge-fault-tolerant pancyclicity of augmented cubes," *Parallel Computing* **33**, pp. 36-42, 2007.
14. M. Ma and J.-M. Xu, "Panconnectivity of locally twisted cubes," *Applied Mathematics Letters* **19**, pp. 673-677, 2006.
15. J.-H. Park, "Hamiltonian decomposition of recursive circulants," in *Proc. International Symposium on Algorithms and Computation ISAAC'98*, Taejon, Korea, pp. 297-306, 1998.
16. J.-H. Park, "Strong hamiltonicity of recursive circulants," *Journal of KISS* **28(8)**, pp. 399-405, Aug. 2001 (in Korean).
17. J.-H. Park, "Cycle embedding of faulty recursive circulants," *Journal of KISS* **31(2)**, pp. 86-94, 2004 (in Korean).
18. J.-H. Park and K.Y. Chwa, "Recursive circulants and their embeddings among hypercubes," *Theoretical Computer Science* **244**, pp. 35-62, 2000.
19. J.-H. Park, H.-C. Kim, and H.-S. Lim, "Fault-hamiltonicity of hypercube-like interconnection networks," in *Proc. IEEE International Parallel and Distributed Processing Symposium IPDPS 2005*, Denver, Apr. 2005.
20. J.-H. Park, H.-S. Lim, and H.-C. Kim, "Panconnectivity and pancyclicity of hypercube-like interconnection networks with faulty elements," *Theoretical Computer Science* **377**, pp. 170-180, 2007.
21. A. Sengupta, "On ring embedding in hypercubes with faulty nodes and links", *Inform. Proc. Lett.* **68**, pp. 207-214, 1998.
22. C.-H. Tsai, J.J.M. Tan, Y.-C. Chuang, and L.-H. Hsu, "Fault-free cycles and links in faulty recursive circulant graphs," in *Proc. of the Workshop on Algorithms and Theory of Computation ICS 2000*, pp. 74-77, 2000.
23. M.-C. Yang, T.-K. Li, J.J.M. Tan, and L.-H. Hsu, "Fault-tolerant cycle-embedding of crossed cubes," *Inform. Proc. Lett.* **88**, pp. 149-154, 2003.
24. M.-C. Yang, T.-K. Li, J.J.M. Tan, and L.-H. Hsu, "On embedding cycles into faulty twisted cubes," *Information Sciences* **176**, pp. 676-690, 2006.