# Panconnectivity and Pancyclicity of Hypercube-Like Interconnection Networks with Faulty Elements*

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**Abstract.** In this paper, we deal with the graph $G_0 \oplus G_1$ obtained from merging two graphs $G_0$ and $G_1$ with $n$ vertices each by $n$ pairwise nonadjacent edges joining vertices in $G_0$ and vertices in $G_1$. The main problems studied are how fault-panconnectivity and fault-pancyclicity of $G_0$ and $G_1$ are translated into fault-panconnectivity and fault-pancyclicity of $G_0 \oplus G_1$, respectively. Many interconnection networks such as hypercube-like interconnection networks can be represented in the form of $G_0 \oplus G_1$ connecting two lower dimensional networks $G_0$ and $G_1$. Applying our results to a class of hypercube-like interconnection networks called *restricted HL-graphs*, we show that in a restricted HL-graph $G$ of degree $m(\geq 3)$, each pair of vertices are joined by a path in $G \backslash F$ of every length from $2m-3$ to $|V(G \backslash F)|-1$ for any set $F$ of faulty elements (vertices and/or edges) with $|F| \leq m-3$, and there exists a cycle of every length from 4 to $|V(G \backslash F)|$ for any fault set $F$ with $|F| \leq m-2$.

**Key Words:** Embedding, panconnected, pancyclic, edge-pancyclic, fault-hamiltonicity, fault tolerance, restricted HL-graphs, interconnection networks.

## 1 Introduction

Linear arrays and rings are two of the most important computational structures in interconnection networks. So, embedding of linear arrays and rings into a faulty interconnection network is one of the important issues in parallel processing[15, 22, 24]. An interconnection network is often modeled as a graph, in which

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vertices and edges correspond to nodes and communication links, respectively. Thus, the embedding problem can be modeled as finding fault-free paths and cycles in the graph with some faulty vertices and/or edges. In the embedding problem, if the longest path or cycle is required the problem is closely related to well-known hamiltonian problems in graph theory. In the rest of this paper, we will use standard terminology in graphs (see ref. [3]).

**Definition 1.** *A graph $G$ is called* $f$-fault hamiltonian *(resp.* $f$-fault hamiltonian-connected*) if there exists a hamiltonian cycle (resp. if each pair of vertices are joined by a hamiltonian path) in $G \backslash F$ for any set $F$ of faulty elements with $|F| \leq f$.*

For a graph $G$ to be $f$-fault hamiltonian (resp. $f$-fault hamiltonian-connected), it is necessary that $f \leq \delta(G)-2$ (resp. $f \leq \delta(G)-3$), where $\delta(G)$ is the minimum degree of $G$. On the other hand, if the paths joining each pair of vertices of every length shorter than or equal to a hamiltonian path are required the problem is concerned with panconnectivity of the graph. If the cycles of arbitrary size (up to a hamiltonian cycle) are required the problem is concerned with pancyclicity of the graph.

**Definition 2.** *A graph $G$ is called* $f$-fault $q$-panconnected *if each pair of fault-free vertices are joined by a path in $G \backslash F$ of every length from $q$ to $|V(G \backslash F)|-1$ inclusive for any set $F$ of faulty elements with $|F| \leq f$.*

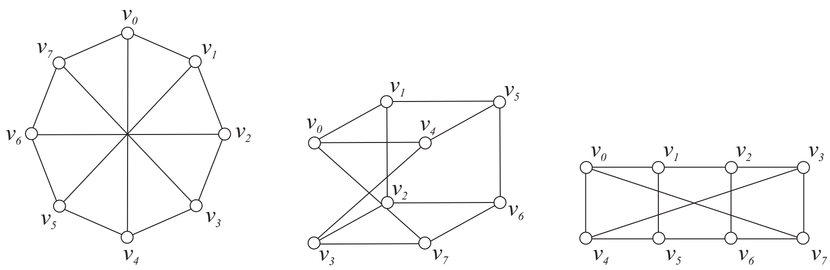
**Definition 3.** *A graph $G$ is called* $f$-fault pancyclic *(resp.* $f$-fault almost pancyclic*) if $G \backslash F$ contains a cycle of every length from 3 to $|V(G \backslash F)|$ (resp. 4 to $|V(G \backslash F)|$) inclusive for any set $F$ of faulty elements with $|F| \leq f$.*

Pancyclicity of various interconnection networks was investigated in the literature. It was shown in [16] that star graph of degree $m-1$ with at most $m-3$ edge faults has every cycle of even length 6 or more. Recursive circulant $G(2^m, 4)$ of degree $m$ was shown to be 0-fault almost pancyclic in [2] and then $m-2$-fault almost pancyclic in [20]. Möbius cube of degree $m$ is 0-fault almost pancyclic[10] and $m-2$-fault almost pancyclic[14]. Crossed cube and twisted cube of degree $m$ were also shown to be $m-2$-fault almost pancyclic in [28] and in [29]. Edge-pancyclicity of some fault-free interconnection networks such as recursive circulants, crossed cubes, twisted cubes was studied in [1], [12], and [11]. The work on panconnectivity of interconnection networks has a relative paucity and some results can be found in [4, 17]. As the authors know, no results on fault-panconnectivity were reported in the literature.

Many interconnection networks can be expanded into higher dimensional networks by connecting two lower dimensional networks. As a graph modeling of the expansion, we consider the graph obtained by connecting two graphs $G_0$ and $G_1$ with $n$ vertices. We denote by $V_i$ and $E_i$ the vertex set and edge set of $G_i$, $i = 0, 1$, respectively. We let $V_0 = \{v_1, v_2, \ldots, v_n\}$ and $V_1 = \{w_1, w_2, \ldots, w_n\}$. With respect to a permutation $M = (i_1, i_2, \ldots, i_n)$ of $\{1, 2, \ldots, n\}$, we can "merge" the two graphs into a graph $G_0 \oplus_M G_1$ with $2n$ vertices in such a way that the vertex set $V = V_0 \cup V_1$ and the edge set $E = E_0 \cup E_1 \cup E_2$, where $E_2 = \{(v_j, w_{i_j}) | 1 \leq j \leq n\}$. We denote by $G_0 \oplus G_1$ a graph obtained by merging $G_0$ and $G_1$ w.r.t. an arbitrary permutation $M$. Here, $G_0$ and $G_1$ are called *components* of $G_0 \oplus G_1$.

Fault-hamiltonicity of $G_0 \oplus G_1$ was investigated in [22]. One of the results is that if each $G_i$ is $f$-fault hamiltonian-connected and $f+1$-fault hamiltonian, then for any $f \geq 2$, $G_0 \oplus G_1$ is $f+1$-fault hamiltonian-connected and for any $f \geq 1$, it is $f+2$-fault hamiltonian.

Vaidya *et al.*[26] introduced a class of hypercube-like interconnection networks, called *HL-graphs*, which can be defined by applying the $\oplus$ operation repeatedly as follows: $HL_0 = \{K_1\}$; for $m \geq 1$, $HL_m = \{G_0 \oplus G_1 | G_0, G_1 \in HL_{m-1}\}$. Then, $HL_1 = \{K_2\}$; $HL_2 = \{C_4\}$; $HL_3 = \{Q_3,\ G(8,4)\}$. Here, $C_4$ is a cycle graph with 4 vertices, $Q_3$ is a 3-dimensional hypercube, and $G(8,4)$ is a recursive circulant[21] which is isomorphic to twisted cube $TQ_3$[13] and Möbius ladder[18] with 4 spokes as shown in Figure 1. An arbitrary graph which belongs to $HL_m$ is called an *$m$-dimensional HL-graph.* It was shown by Park and Chwa in [19] that every nonbipartite HL-graph is hamiltonian-connected, and that every bipartite HL-graph is hamiltonian-laceable, that is, every bipartite HL-graph has a hamiltonian path between any two vertices that belong to different partite sets. Obviously, some $m$-dimensional HL-graphs such as an $m$-dimensional hypercube are bipartite. They are not $f$-fault almost pancyclic for any $f \geq 0$, and thus they are not $f$-fault $q$-panconnected for any $f \geq 0$ and $q \geq 1$.




(a) $G(8,4)$ (b) $TQ_3$ (c) Möbius ladder

**Fig. 1.** Isomorphic graphs.

In [22], a subclass of nonbipartite HL-graphs, called *restricted HL-graphs* was introduced, which is defined recursively as follows: $RHL_m = HL_m$ for $0 \leq m \leq 2$; $RHL_3 = HL_3 \backslash Q_3 = \{G(8,4)\}$; $RHL_m = \{G_0 \oplus G_1 | G_0, G_1 \in RHL_{m-1}\}$ for $m \geq 4$. A graph which belongs to $RHL_m$ is called an *$m$-dimensional restricted HL-graph.* Many of the nonbipartite hypercube-like interconnection networks such as crossed cube[8], Möbius cube[6], twisted cube[13], multiply twisted cube[7], Mcube[25], generalized twisted cube[5], locally twisted cube[27], etc. proposed in the literature are restricted HL-graphs with the exception of recursive circulant $G(2^m, 4)$[21] and "near" bipartite interconnection networks such as twisted $m$-cube[9]. It was shown in [22] that every $m$-dimensional restricted HL-graph, $m \geq 3$, is $m-3$-fault hamiltonian-connected and $m-2$-fault hamiltonian. In [23], it was shown that every $m$-dimensional restricted HL-graph with $f$ or less faulty elements has $k$ disjoint paths, covering all the fault-free vertices, joining any $k$ distinct source-sink pairs for any $f \geq 0$ and $k \geq 1$ with $f+2k \leq m-1$. In this paper, we are concerned with panconnectivity and pancyclicity of restricted HL-graphs with faulty elements.

We first investigate panconnectivity and pancyclicity of $G_0 \oplus G_1$ with faulty elements. It will be shown that if each $G_i$, $i = 0, 1$, is $f$-fault $q$-panconnected and $f + 1$-fault hamiltonian (with additional conditions $n \geq f + 2q + 1$ and $q \geq 2f + 3$), then $G_0 \oplus G_1$ is $f + 1$-fault $q + 2$-panconnected for any $f \geq 2$. To study pancyclicity of $G_0 \oplus G_1$, the notion of *hypohamiltonian-connectivity* is introduced. A graph $G$ is called *$f$-fault hypohamiltonian-connected* if each pair of vertices can be joined by a path of length $|V(G \backslash F)| - 2$, that is one less than the longest possible length, in $G \backslash F$ for any fault set $F$ with $|F| \leq f$. We will show that if each $G_i$, $i = 0, 1$, is $f$-fault hamiltonian-connected, $f$-fault hypohamiltonian-connected, and $f + 1$-fault almost pancyclic, then $G_0 \oplus G_1$ is $f + 2$-fault almost pancyclic for any $f \geq 1$.

Our main results are applied to restricted HL-graphs. We will show that every $m$-dimensional restricted HL-graph with $m \geq 3$ is $m - 3$-fault $2m - 3$-panconnected and $m - 2$-fault almost pancyclic. Both bounds $m - 3$ and $m - 2$ on the number of acceptable faulty elements are the maximum possible. Notice that $f$-fault $q$-panconnected graph is $f$-fault hamiltonian-connected, and that $f$-fault almost pancyclic graph is $f$-fault hamiltonian. Our results are not only the extension of some works of [14, 28, 29] on fault-pancyclicity of restricted HL-graphs, but also a new investigation on fault-panconnectivity of restricted HL-graphs.

The organization of this paper is as follows. In the next section, panconnectivity and pancyclicity of $G_0 \oplus G_1$ with faulty elements will be investigated. In Section 3, fault-panconnectivity and fault-pancyclicity of restricted HL-graphs will be studied. Finally in Section 4, concluding remarks of this paper will be given.

## 2 Panconnectivity and Pancyclicity of $G_0 \oplus G_1$

For a vertex $v$ in $G_0 \oplus G_1$, we denote by $\bar{v}$ the vertex adjacent to $v$ which is in a component different from the component in which $v$ is contained. We denote by $F$ the set of faulty elements. When we are to construct a path from $s$ to $t$, $s$ and $t$ are called a *source* and a *sink*, respectively, and both of them are called *terminals*. Throughout this paper, a path in a graph is represented as a sequence of vertices.

**Definition 4.** *A vertex $v$ in $G_0 \oplus G_1$ is called* free *if $v$ is fault-free and not a terminal, that is, $v \notin F$ and $v$ is neither a source nor a sink. An edge $(v, w)$ is called* free *if $v$ and $w$ are free and $(v, w) \notin F$.*

We denote by $V_i$ and $E_i$ the sets of vertices and edges in $G_i$, $i = 0, 1$, and by $E_2$ the set of edges joining vertices in $G_0$ and vertices in $G_1$. We let $n = |V_0| = |V_1|$. $F_0$ and $F_1$ denote the sets of faulty elements in $G_0$ and $G_1$, respectively, and $F_2$ denotes the set of faulty edges in $E_2$, so that $F = F_0 \cup F_1 \cup F_2$. Let $f_0 = |F_0|$, $f_1 = |F_1|$, and $f_2 = |F_2|$.

When we find a path/cycle, sometimes we regard some fault-free vertices and/or edges as faulty elements. They are called *virtual* faults. If $G_i$ is $f$-fault hamiltonian-connected and $f+1$-fault hamiltonian, $i = 0, 1$, then

$$f \le \delta(G_i) - 3, \text{ and thus } f + 4 \le n,$$

where $\delta(G_i)$ is the minimum degree of $G_i$.

### 2.1 Panconnectivity of $G_0 \oplus G_1$

Hamiltonian-connectivity of $G_0 \oplus G_1$ with faulty elements was considered in [22]. In this subsection, we study panconnectivity of $G_0 \oplus G_1$ in the presence of faulty elements. We denote by $f_v^0$ and $f_v^1$ the numbers of faulty vertices in $G_0$ and $G_1$, respectively, and by $f_v$ the number of faulty vertices in $G_0 \oplus G_1$, so that $f_v = f_v^0 + f_v^1$. Note that the length of a hamiltonian path in $G_0 \oplus G_1 \backslash F$ is $2n - f_v - 1$.

**Theorem 1.** *Let $G_0$ and $G_1$ be graphs with $n$ vertices each. Let $f$ and $q$ be nonnegative integers satisfying $n \ge f + 2q + 1$ and $q \ge 2f + 3$. If each $G_i$ is $f$-fault $q$-panconnected and $f+1$-fault hamiltonian, then*
*(a) for any $f \ge 2$, $G_0 \oplus G_1$ is $f+1$-fault $q+2$-panconnected,*
*(b) for $f = 1$, $G_0 \oplus G_1$ with $2(= f+1)$ faulty elements has a path of every length $q+2$ or more joining $s$ and $t$ unless $s$ and $t$ are contained in the same component and $\bar{s}$ and $\bar{t}$ are the faulty elements(vertices), and*
*(c) for $f = 0$, $G_0 \oplus G_1$ with $1(= f+1)$ faulty element has a path of every length $q+2$ or more joining $s$ and $t$ unless $s$ and $t$ are contained in the same component and the faulty element is contained in the other component.*

*Proof.* To prove (a), assuming the number of faulty elements $|F| \le f+1$, we will construct a path of every length $l$, $q + 2 \le l \le 2n - f_v - 1$, in $G_0 \oplus G_1 \backslash F$ joining any pair of vertices $s$ and $t$.

*Case 1:* $f_0, f_1 \le f$.
When both $s$ and $t$ are contained in $G_0$, there exists a path $P_0$ of length $l_0$ in $G_0$ joining $s$ and $t$ for every $q \le l_0 \le n - f_v^0 - 1$. We are to construct a longer path $P_1$ that passes through vertices in $G_1$ as well as vertices in $G_0$. We first claim that there exists an edge $(x, y)$ on $P_0$ such that all of $\bar{x}$, $(x, \bar{x})$, $\bar{y}$, and $(y, \bar{y})$ are fault-free. There are $l_0$ candidate edges on $P_0$ and at most $f+1$ faulty elements can "block" the candidates, at most two candidates per one faulty element. By assumption $l_0 \ge q \ge 2f + 3$, and the claim is proved. The path $P_1$ can be obtained by merging $P_0$ and a path $P'$ in $G_1$ between $\bar{x}$ and $\bar{y}$ with the edges $(x, \bar{x})$ and $(y, \bar{y})$. Here, of course the edge $(x, y)$ is discarded.

Letting $l'$ be the length of $P'$, the length $l_1$ of $P_1$ can be anything in the range $2q+1 \leq l_1 = l_0+l'+1 \leq 2n-f_v-1$. Since $n \geq f+2q+1$, we have $2q+1 \leq n-f_v^0$ and we are done.

When $s$ is in $G_0$ and $t$ is in $G_1$, we first find a free edge $(x, \bar{x})$ in $E_2$ such that $(\bar{x}, t)$ is an edge and fault-free. The existence of such a free edge $(x, \bar{x})$ is due to the fact that there are $\delta(G_1)$ candidates and that at most $f+1$ faulty elements and the source $s$ can block the candidates. Remember $f \leq \delta(G_1) - 3$. Assuming $x \in V_0$, a path joining $s$ and $x$ in $G_0$ and an edge $(\bar{x}, t)$ are merged with $(x, \bar{x})$ into a path $P_0$. The length $l_0$ of $P_0$ is any integer in the range $q+2 \leq l_0 \leq n-f_v^0+1$. A longer path $P_1$ is obtained by replacing the edge $(\bar{x}, t)$ with a path in $G_1$ between $\bar{x}$ and $t$ of length $l''$, $q \leq l'' \leq n - f_v^1 - 1$. The length $l_1$ of $P_1$ is in the range $2q+1 \leq l_1 \leq 2n - f_v - 1$. We are done since $2q+1 \leq n - f_v^0$ as shown in the previous subcase.

*Case 2:* $f_0 = f+1$ (or symmetrically, $f_1 = f+1$).
We have $f_1 = f_2 = 0$. First, we consider the subcase $s, t \in V_0$. Letting $P'$ be a path in $G_1$ joining $\bar{s}$ and $\bar{t}$, we have a path $P_0 = (s, P', t)$ between $s$ and $t$. The length $l_0$ of $P_0$ is any integer in the range $q+2 \leq l_0 \leq n+1$. To construct a longer path $P_1$, we select an arbitrary faulty element $\alpha$ in $G_0$. Regarding $\alpha$ as *a virtual fault-free element*, find a path $P''$ in $G_0$ between $s$ and $t$. If $\alpha$ is a faulty vertex on $P''$, let $x$ and $y$ be the two vertices on $P''$ next to $\alpha$; else if $P''$ passes through the faulty edge $\alpha$, let $x$ and $y$ be the endvertices of $\alpha$; else let $(x, y)$ be an arbitrary edge on $P''$. The path $P_1$ is obtained by merging $P'' \backslash \alpha$ and a path in $G_1$ joining $\bar{x}$ and $\bar{y}$ with edges $(x, \bar{x})$ and $(y, \bar{y})$. If $\alpha$ is faulty vertex on $P''$, the length $l_1$ of $P_1$ is in the range $2q \leq l_1 \leq 2n - f_v - 1$; otherwise, we have $2q+1 \leq l_1 \leq 2n - f_v - 1$. In any cases, we are done since $2q+1 \leq n+2$.

Secondly, we consider the subcase $s \in V_0$ and $t \in V_1$. We first find a hamiltonian cycle $C$ in $G_0 \backslash F_0$ and let $C = (s = z_0, z_1, z_2, ..., z_k)$, where $k = n - f_v^0 - 1$. Assuming $\bar{z}_l \neq t$ without loss of generality, we can construct a path $P_0$ by merging $(z_0, z_1, ..., z_l)$ and a path in $G_1$ between $\bar{z}_l$ and $t$ with the edge $(z_l, \bar{z}_l)$. The length $l_0$ of $P_0$ is any integer in the range $q+l+1 \leq l_0 \leq n-f_v^1+l$. Since $l$ itself is any integer in the range $1 \leq l \leq n - f_v^0 - 1$, we have $q+2 \leq l_0 \leq 2n - f_v - 1$.

Finally, we consider the subcase $s, t \in V_1$. We have a path $P_0$ in $G_1$ joining $s$ and $t$, and the length $l_0$ of $P_0$ is in the range $q \leq l_0 \leq n-1$. To construct a longer path $P_1$, we let $C = (z_0, z_1, z_2, ..., z_k)$ be a hamiltonian cycle in $G_0 \backslash F_0$, where $k = n - f_v^0 - 1$. If $\bar{s} \notin F$, we assume w.l.o.g. $\bar{s} = z_0$. Then, letting w.l.o.g. $\bar{z}_l \neq t$, $P_1$ is a concatenation of $(s, z_0, z_1, \ldots, z_l)$ and a path in $G_1 \backslash s$ between $\bar{z}_l$ and $t$. The length $l_1$ of $P_1$ is in the range $q+3 \leq l_1 \leq 2n - f_v - 1$. If $\bar{s} \in F$, we let $(x, \bar{x})$ be a free edge such that $\bar{x}$ is adjacent to $s$. Then, letting w.l.o.g. $x = z_0$ and $\bar{z}_l \neq t$, $P_1$ is a concatenation of $(s, \bar{x}, z_0, z_1, \ldots, z_l)$ and a path in $G_1 \backslash \{s, \bar{x}\}$ between $\bar{z}_l$ and $t$. Here, the length $l_1$ of $P_1$ is in the range $q+4 \leq l_1 \leq 2n-f_v-1$. By the condition of $n \geq f+2q+1$ and $q \geq 2f+3$, we can observe $q+4 \leq n$. Therefore, we are done. This completes the proof of (a).

It immediately follows from Case 1 and the first and second subcases of Case 2, where the assumption $f \geq 2$ is never used, that for $f = 0, 1$, $G_0 \oplus G_1$ with $f+1$ faulty elements has a path of every length $q+2$ or more joining $s$ and $t$

unless $s$ and $t$ are contained in the same component and all the faulty elements are contained in the other component. Thus, the proof of (c) is done. To prove (b), assuming w.l.o.g. $\bar{s} \notin F$, it suffices to employ the construction of the last subcase of Case 2. Note that in the construction, $G_1$ is 1-fault $q$-panconnected. This completes the proof. □

**Corollary 1.** *Let $G_0$ and $G_1$ be graphs with $n$ vertices each. Let $f$ and $q$ be nonnegative integers satisfying $n \geq f + 2q + 1$ and $q \geq 2f + 3$. If each $G_i$ is $f$-fault $q$-panconnected and $f + 1$-fault hamiltonian, then $G_0 \oplus G_1$ is $f$-fault $q + 2$-panconnected.*

*Proof.* It is sufficient to consider the case $f = 0, 1$ by Theorem 1(a). To obtain a path of length $q + 2$ or more in $G \backslash F$ joining $s$ and $t$, we can apply Theorem 1 (b) and (c) after we choose $f + 1 - |F|$ fault-free edges in $E_2$ and regard them as virtual faults. □

## 2.2 Pancyclicity of $G_0 \oplus G_1$

In the presence of faulty elements, the existence of hamiltonian cycle in $G_0 \oplus G_1$ was considered in [22] as in Theorem 2. In this subsection, we investigate almost pancyclicity of $G_0 \oplus G_1$ with faulty elements. We denote by $H[v, w|G, F]$ a hamiltonian path in $G \backslash F$ joining a pair of fault-free vertices $v$ and $w$ in a graph $G$ with a set $F$ of faulty elements. $HH[v, w|G, F]$ denotes a hypohamiltonian path in $G \backslash F$ between $v$ and $w$.

**Theorem 2.** *[22] Let a graph $G_i$ be $f$-fault hamiltonian-connected and $f + 1$-fault hamiltonian, $i = 0, 1$. Then,*
*(a) for any $f \geq 1$, $G_0 \oplus G_1$ is $f + 2$-fault hamiltonian, and*
*(b) for $f = 0$, $G_0 \oplus G_1$ with $2 (= f + 2)$ faulty elements has a hamiltonian cycle unless one faulty element is contained in $G_0$ and the other faulty element is contained in $G_1$.*

Before presenting our theorem on pancyclicity, we will give two lemmas. They imply that to show an $f$-fault hamiltonian graph is $f$-fault almost pancyclic, it is sufficient to consider only vertex faults and further the maximum number of vertex faults. We call a graph $G$ to be *$f$-vertex-fault almost pancyclic*, if $G \backslash F_v$ contains a cycle of every length from 4 to $|V(G \backslash F_v)|$ for any set of faulty vertices $F_v$ with $|F_v| \leq f$.

**Lemma 1.** *Let a graph $G$ be $f$-fault hamiltonian and $f$-vertex-fault almost pancyclic. Then, $G$ is $f$-fault almost pancyclic.*

*Proof.* We prove that for any faulty set $F$ with $|F| \leq f$, $G \backslash F$ is almost pancyclic by induction on the number of faulty edges $f_e$ in $F$. It holds true for $f_e = 0$. Assume $f_e \geq 1$. Let $f_v$ be the number of faulty vertices and let $n$ be the number of vertices in $G$. There is a cycle of every length from 4 to $n - f_v - 1$ if we regard a faulty edge $(x, y)$ as a vertex fault of $x$ when $x$ is fault-free, or $y$ when $y$ is fault-free, or an arbitrary fault-free vertex when both $x$ and $y$ are faulty. The cycle of length $n - f_v$ exists since $G$ is $f$-fault hamiltonian. □

**Lemma 2.** *Let a graph $G$ be $f$-fault hamiltonian and almost pancyclic when the number of faulty vertices $f_v = f$. Then, $G$ is $f$-vertex-fault almost pancyclic.*

*Proof.* We show that $G$ is almost pancyclic when $f_v < f$. There exists a cycle of every length from 4 to $n - f$ by the condition in lemma. The cycle of length $l$, $n - f < l \leq n - f_v$, can be found by constructing a hamiltonian cycle taking account of fault-free vertices as virtual faults one by one (starting from 0). □

**Theorem 3.** *Let a graph $G_i$ be $f$-fault hamiltonian-connected, $f$-fault hypohamiltonian-connected, and $f+1$-fault almost pancyclic, $i = 0, 1$. Then,*
*(a) for any $f \geq 1$, $G_0 \oplus G_1$ is $f+2$-fault almost pancyclic, and*
*(b) for $f = 0$, $G_0 \oplus G_1$ with $2(= f+2)$ faulty elements is almost pancyclic unless one faulty element is contained in $G_0$ and the other faulty element is contained in $G_1$.*

*Proof.* To prove (a), we let $|F| = f + 2$, and assume $F$ has only vertex faults by virtue of the above two lemmas. Note that, by Theorem 2(a), $G_0 \oplus G_1$ is $f+2$-fault hamiltonian. Assuming $f_0 \geq f_1$ without loss of generality, we will construct cycles in $G_0 \oplus G_1 \backslash F$. By the condition in the theorem, there exist cycles of length from 4 to $n - f_1$ in $G_1 \backslash F_1$. Also, the cycle of length $2n - f_0 - f_1$ exists. So, the construction of remaining cycles of length from $n - f_1 + 1$ to $2n - f_0 - f_1 - 1$ will be given.

*Case 1:* $f_0 \leq f$.

*Subcase 1.1:* $n > f_0 + 2f_1$.

There exists a hamiltonian cycle $C_0$ of length $n - f_0$ in $G_0 \backslash F_0$. On $C_0$, we have $n - f_0$ different paths $P_k$'s of length $k$ for every $1 \leq k \leq n - f_0 - 1$. Among them, there exists a $P_k$ joining $x_k$ and $y_k$ such that both $\bar{x_k}$ and $\bar{y_k}$ are fault-free, since we have $n - f_0$ candidates and each of $f_1$ faulty vertices in $G_1$ can block at most two candidates. Then, $C = (P_k, HH[\bar{y_k}, \bar{x_k}|G_1, F_1])$ is a cycle of length $n - f_1 + k$, $1 \leq k \leq n - f_0 - 1$.

*Subcase 1.2:* $n \leq f_0 + 2f_1$.

We find two free edges $(x, \bar{x})$ and $(y, \bar{y})$ in $E_2$. Such free edges exist since there are $n(\geq f+4)$ candidates and $f+2$ blocking elements. Note that there are no terminals. We will construct a cycle by merging $H[x, y|G_0, F']$ or $HH[x, y|G_0, F']$ with $H[\bar{x}, \bar{y}|G_1, F'']$ or $HH[\bar{x}, \bar{y}|G_1, F'']$. Here, $F'$ (resp. $F''$) is a set of faulty elements in $G_0$ (resp. $G_1$) regarding some fault-free vertices as virtual faults. By taking account of $f - f_0$ vertices in $G_0 \backslash F_0$ excluding $\{x, y\}$ as virtual faults one by one, we can construct paths of length from $n - f - 2$ to $n - f_0 - 1$ between $x$ and $y$. Also, by taking account of $f - f_1$ vertices in $G_1 \backslash F_1$ excluding $\{\bar{x}, \bar{y}\}$ as virtual faults one by one, we can construct paths of length from $n - f - 2$ to $n - f_1 - 1$ between $\bar{x}$ and $\bar{y}$. By merging two paths in $G_0$ and $G_1$, we can obtain cycles of length from $2n - 2f - 2$ to $2n - f_0 - f_1$. If $2n - 2f - 2 \leq n - f_1 + 1$, we will have all cycles of desired lengths. First, we have $2n - 2f - 2 \leq n - f_1 + 2$ since $(2n - 2f - 2) - (n - f_1 + 2) = n - 2f + f_1 - 4 \leq (f_0 + 2f_1) - 2f + f_1 - 4 = f_0 + 3f_1 - 2f - 4 = 2f_1 - f - 2 \leq 0$. Furthermore, careful observation on the above equation leads to $2n - 2f - 2 \leq n - f_1 + 1$ unless $n = f_0 + 2f_1$ and $f_0 = f_1$.

For the remaining case that $n = f_0 + 2f_1$ and $f_0 = f_1$, it is sufficient to construct a cycle of length $n - f_1 + 1$. To do this, we claim that there exists an edge $(x, y)$ in $G_0$ such that both $\bar{x}$ and $\bar{y}$ are fault-free. Let $W = \{w | w \in V_0 \backslash F_0, \bar{w} \notin F\}$, and let $B = V_0 \backslash (F_0 \cup W)$. It holds true that $|W| \geq |B|$ since $|W| \geq n - f_0 - f_1 = f_1$ and $|B| \leq f_1$. Let $C_0$ be a hamiltonian cycle in $G_0 \backslash F_0$. If there is an edge $(a, b)$ on $C_0$ such that $a, b \in W$, we are done. Suppose otherwise, we have $|W| = |B|$ and the vertices on $C_0$ should alternate in $W$ and $B$. Since $G_0 \backslash F_0$ is hamiltonian-connected, we always have such an edge $(x, y)$ joining vertices in $W$. Note that $|W|, |B| \geq 2$, and that if there are no edges between vertices in $W$, there can not exist a hamiltonian path joining vertices in $B$. Then, we have a desired cycle $(x, y, HH[\bar{y}, \bar{x}|G_1, F_1])$ of length $n - f_1 + 1$.

*Case 2:* $f_0 = f + 1$.
We find a hamiltonian cycle $C_0$ in $G_0 \backslash F_0$, and let $x_k$ and $y_k$ be two vertices in $C_0$ such that both $\bar{x_k}$ and $\bar{y_k}$ are fault-free and there is a path of length $k$ between $x_k$ and $y_k$ on $C_0$, $1 \leq k \leq n - f_0 - 1$. The existence of such $x_k$ and $y_k$ is due to the fact that the length of $C_0$ is at least three and $f_1 = 1$. Let $P_k$ be the path of length $k$ on $C_0$ whose endvertices are $x_k$ and $y_k$. We construct cycles $(P_k, HH[\bar{y_k}, \bar{x_k}|G_1, F_1])$, $1 \leq k \leq n - f_0 - 1$, of length from $n - f_1 + 1$ to $2n - f_0 - f_1 - 1$. The hypohamiltonian path in $G_1$ between $\bar{y_k}$ and $\bar{x_k}$ exists since $f_1 = 1 \leq f$.

*Case 3:* $f_0 = f + 2$.
We select an arbitrary faulty vertex $v_f$ in $G_0$, regarding it as *a virtual fault-free vertex*, find a hamiltonian cycle $C_0$ in $G_0 \backslash F'$, where $F' = F_0 \backslash v_f$. The existence of $C_0$ is due to $|F'| = f + 1$. Let $P_k$ be an arbitrary path of length $k$ on $C_0 \backslash v_f$ whose endvertices are $x_k$ and $y_k$, $1 \leq k \leq n - f_0 - 1$. Then, we have a cycle $(P_k, HH[\bar{y_k}, \bar{x_k}|G_1, \emptyset])$ of length $n - f_1 + k$ for every $1 \leq k \leq n - f_0 - 1$.

The proof of (b) follows immediately from the proof of (a), where the assumption $f \geq 1$ is used only when $f_1 = 1$ in Case 2. □

*Remark 1.* For $f = 0$, Theorem 3(a) does not hold true. We can construct a counter example using 3-dimensional hypercube $Q_3$. Let $W_4$ be a wheel graph which consists of length four cycle $C_4$ and a center vertex adjacent to all the vertices in $C_4$. It is easy to verify that $W_4$ is 0-fault hamiltonian-connected, 0-fault hypohamiltonian-connected, and 1-fault almost pancyclic. Let $G$ be $W_4 \times K_2$, that is, a graph obtained by joining two identical $W_4$ by an identity permutation. If we remove both center vertices in two component graphs, the resulting graph is isomorphic to $Q_3$ which is a bipartite graph and thus does not possess any odd length cycle. So, $G$ is not 2-fault almost pancyclic.

## 3 Restricted HL-graphs

In this section, we will show that every $m$-dimensional restricted HL-graph is $m - 3$-fault $2m - 3$-panconnected and $m - 2$-fault almost pancyclic. Fault-hamiltonicity of restricted HL-graphs was studied in [22] as follows. Of course, panconnectivity implies the existence of a hamiltonian path and pancyclicity implies the existence of a hamiltonian cycle. Thus, the result given in this section is a generalization of the work in [22].

**Theorem 4.** *[22] Every $m$-dimensional restricted HL-graph, $m \geq 3$, is $m-3$-fault hamiltonian-connected and $m-2$-fault hamiltonian.*

### 3.1 Panconnectivity of restricted HL-graphs

By induction on $m$, we will prove that every $m$-dimensional restricted HL-graph, $m \geq 3$, is $m-3$-fault $2m-3$-panconnected. Recursive circulant $G(8,4)$ shown in Figure 1 is a graph defined as follows: vertex set is $\{v_i | 0 \leq i \leq 7\}$ and the edge set is $\{(v_i, v_j) | i+1 \text{ or } i+4 \equiv j \pmod 8\}$.

**Lemma 3.** *The 3-dimensional restricted HL-graph $G(8,4)$ is 0-fault 3-panconnected.*

*Proof.* The proof is by an immediate inspection. □

To prove that every 4-dimensional restricted HL-graph $G(8,4) \oplus G(8,4)$ is 1-fault 5-panconnected and every 5-dimensional restricted HL-graph is 2-fault 7-panconnected, we employ useful properties on disjoint paths in $G(8,4)$ and in $G(8,4) \oplus G(8,4)$, as shown in Lemmas 4, 5, and 6. Two paths joining $\{s_1, s_2\}$ and $\{t_1, t_2\}$ such that $\{s_1, s_2\} \cap \{t_1, t_2\} = \emptyset$ are defined to be either $s_1$-$t_1$ and $s_2$-$t_2$ paths or $s_1$-$t_2$ and $s_2$-$t_1$ paths. Two paths $P_1$ and $P_2$ in a graph $G$ are called *disjoint covering paths* if $V(P_1) \cap V(P_2) = \emptyset$ and $V(P_1) \cup V(P_2) = V(G)$, where $V(P_i)$ is the set of vertices in $P_i$.

**Lemma 4.** *For any four distinct vertices $s_1$, $s_2$, $t_1$, and $t_2$ in $G(8,4)$, there exists a vertex $z \notin \{s_1, s_2, t_1, t_2\}$ such that $G(8,4) \backslash z$ has two disjoint covering paths joining $\{s_1, s_2\}$ and $\{t_1, t_2\}$ with the unique exception up to symmetry that $\{s_1, s_2\} = \{v_0, v_1\}$ and $\{t_1, t_2\} = \{v_4, v_5\}$.*

*Proof.* The proof is by an immediate inspection and omitted here. □

**Lemma 5.** *Let $P_1$ and $P_2$ be two disjoint covering paths joining $\{s_1, s_2\}$ and $\{t_1, t_2\}$ in $G(8,4)$ such that $\{s_1, s_2\} \cap \{t_1, t_2\} = \emptyset$.*
*(a) When $\{s_1, s_2\} = \{v_0, v_1\}$, they exist unless $\{t_1, t_2\} = \{v_3, v_6\}$.*
*(b) When $\{s_1, s_2\} = \{v_0, v_2\}$, they exist unless $\{t_1, t_2\} = \{v_3, v_5\}$ or $\{v_5, v_7\}$.*
*(c) When $\{s_1, s_2\} = \{v_0, v_3\}$, they exist unless $\{t_1, t_2\} = \{v_1, v_6\}$, $\{v_2, v_5\}$, or $\{v_5, v_6\}$.*
*(d) When $\{s_1, s_2\} = \{v_0, v_4\}$, they exist unless $\{t_1, t_2\} = \{v_2, v_6\}$.*

*Proof.* The proof is enumerative. See Table 1. □

**Lemma 6.** *For any four distinct vertices $s_1$, $s_2$, $t_1$, and $t_2$ in $G(8,4) \oplus G(8,4)$, there exists a vertex $z \notin \{s_1, s_2, t_1, t_2\}$ such that $G(8,4) \oplus G(8,4) \backslash z$ has two disjoint covering paths joining $\{s_1, s_2\}$ and $\{t_1, t_2\}$.*

**Table 1.** Disjoint covering paths $P_1$ and $P_2$ in $G(8,4)$ joining $\{s_1, s_2\}$ and $\{t_1, t_2\}$

| $\{s_1, s_2\}$ | $\{t_1, t_2\}$: $P_1$, $P_2$ | |
|---|---|---|
| $\{v_0, v_1\}$ | $\{v_2, v_3\}$: $v_0$-$v_7$-$v_6$-$v_5$-$v_4$-$v_3$, $v_1$-$v_2$; | $\{v_2, v_4\}$: $v_0$-$v_7$-$v_3$-$v_4$, $v_1$-$v_5$-$v_6$-$v_2$; |
| | $\{v_2, v_5\}$: $v_0$-$v_4$-$v_3$-$v_7$-$v_6$-$v_5$, $v_1$-$v_2$; | $\{v_2, v_6\}$: $v_0$-$v_7$-$v_6$, $v_1$-$v_5$-$v_4$-$v_3$-$v_2$; |
| | $\{v_2, v_7\}$: $v_0$-$v_4$-$v_3$-$v_7$, $v_1$-$v_5$-$v_6$-$v_2$; | $\{v_3, v_4\}$: $v_0$-$v_7$-$v_6$-$v_5$-$v_4$, $v_1$-$v_2$-$v_3$; |
| | $\{v_3, v_5\}$: $v_0$-$v_4$-$v_5$, $v_1$-$v_2$-$v_6$-$v_7$-$v_3$; | $\{v_3, v_6\}$: does not exist; |
| | $\{v_3, v_7\}$: symmetric to $\{v_2, v_6\}$; | $\{v_4, v_5\}$: $v_0$-$v_7$-$v_6$-$v_5$, $v_1$-$v_2$-$v_3$-$v_4$; |
| | $\{v_4, v_6\}$: symmetric to $\{v_3, v_5\}$; | $\{v_4, v_7\}$: symmetric to $\{v_2, v_5\}$; |
| | $\{v_5, v_6\}$: symmetric to $\{v_3, v_4\}$; | $\{v_5, v_7\}$: symmetric to $\{v_2, v_4\}$; |
| | $\{v_6, v_7\}$: symmetric to $\{v_2, v_3\}$; | |
| $\{v_0, v_2\}$ | $\{v_1, v_3\}$: $v_0$-$v_7$-$v_6$-$v_5$-$v_4$-$v_3$, $v_2$-$v_1$; | $\{v_1, v_4\}$: $v_0$-$v_7$-$v_3$-$v_4$, $v_2$-$v_6$-$v_5$-$v_1$; |
| | $\{v_1, v_5\}$: $v_0$-$v_1$, $v_2$-$v_6$-$v_7$-$v_3$-$v_4$-$v_5$; | $\{v_1, v_6\}$: symmetric to $\{v_1, v_4\}$; |
| | $\{v_1, v_7\}$: symmetric to $\{v_1, v_3\}$; | $\{v_3, v_4\}$: $v_0$-$v_1$-$v_5$-$v_4$, $v_2$-$v_6$-$v_7$-$v_3$; |
| | $\{v_3, v_5\}$: does not exist; | $\{v_3, v_6\}$: $v_0$-$v_7$-$v_6$, $v_2$-$v_1$-$v_5$-$v_4$-$v_3$; |
| | $\{v_3, v_7\}$: $v_0$-$v_1$-$v_5$-$v_4$-$v_3$, $v_2$-$v_6$-$v_7$; | $\{v_4, v_5\}$: $v_0$-$v_1$-$v_5$, $v_2$-$v_6$-$v_7$-$v_3$-$v_4$; |
| | $\{v_4, v_6\}$: $v_0$-$v_7$-$v_3$-$v_4$, $v_2$-$v_1$-$v_5$-$v_6$; | $\{v_4, v_7\}$: symmetric to $\{v_3, v_6\}$; |
| | $\{v_5, v_6\}$: symmetric to $\{v_4, v_5\}$; | $\{v_5, v_7\}$: does not exist; |
| | $\{v_6, v_7\}$: symmetric to $\{v_3, v_4\}$; | |
| $\{v_0, v_3\}$ | $\{v_1, v_2\}$: $v_0$-$v_4$-$v_5$-$v_1$, $v_3$-$v_7$-$v_6$-$v_2$; | $\{v_1, v_4\}$: $v_0$-$v_7$-$v_6$-$v_5$-$v_4$, $v_3$-$v_2$-$v_1$; |
| | $\{v_1, v_5\}$: $v_0$-$v_7$-$v_6$-$v_2$-$v_1$, $v_3$-$v_4$-$v_5$; | $\{v_1, v_6\}$: does not exist; |
| | $\{v_1, v_7\}$: $v_0$-$v_7$, $v_3$-$v_4$-$v_5$-$v_6$-$v_2$-$v_1$; | $\{v_2, v_4\}$: symmetric to $\{v_1, v_7\}$; |
| | $\{v_2, v_5\}$: does not exist; | $\{v_2, v_6\}$: symmetric to $\{v_1, v_5\}$; |
| | $\{v_2, v_7\}$: symmetric to $\{v_1, v_4\}$; | $\{v_4, v_5\}$: $v_0$-$v_4$, $v_3$-$v_7$-$v_6$-$v_2$-$v_1$-$v_5$; |
| | $\{v_4, v_6\}$: $v_0$-$v_7$-$v_6$, $v_3$-$v_2$-$v_1$-$v_5$-$v_4$; | $\{v_4, v_7\}$: $v_0$-$v_4$, $v_3$-$v_2$-$v_1$-$v_5$-$v_6$-$v_7$; |
| | $\{v_5, v_6\}$: does not exist; | $\{v_5, v_7\}$: symmetric to $\{v_4, v_6\}$; |
| | $\{v_6, v_7\}$: symmetric to $\{v_4, v_5\}$; | |
| $\{v_0, v_4\}$ | $\{v_1, v_2\}$: $v_0$-$v_7$-$v_6$-$v_5$-$v_1$, $v_4$-$v_3$-$v_2$; | $\{v_1, v_3\}$: $v_0$-$v_7$-$v_3$, $v_4$-$v_5$-$v_6$-$v_2$-$v_1$; |
| | $\{v_1, v_5\}$: $v_0$-$v_7$-$v_6$-$v_5$, $v_4$-$v_3$-$v_2$-$v_1$; | $\{v_1, v_6\}$: $v_0$-$v_7$-$v_3$-$v_2$-$v_6$, $v_4$-$v_5$-$v_1$; |
| | $\{v_1, v_7\}$: $v_0$-$v_1$, $v_4$-$v_5$-$v_6$-$v_2$-$v_3$-$v_7$; | $\{v_2, v_3\}$: symmetric to $\{v_1, v_2\}$; |
| | $\{v_2, v_5\}$: symmetric to $\{v_1, v_6\}$; | $\{v_2, v_6\}$: does not exist; |
| | $\{v_2, v_7\}$: symmetric to $\{v_1, v_6\}$; | $\{v_3, v_5\}$: symmetric to $\{v_1, v_7\}$; |
| | $\{v_3, v_6\}$: symmetric to $\{v_1, v_6\}$; | $\{v_3, v_7\}$: symmetric to $\{v_1, v_5\}$; |
| | $\{v_5, v_6\}$: symmetric to $\{v_1, v_2\}$; | $\{v_5, v_7\}$: symmetric to $\{v_1, v_3\}$; |
| | $\{v_6, v_7\}$: symmetric to $\{v_1, v_2\}$; | |

*Proof.* We let $G_0$ and $G_1$ be graphs isomorphic to $G(8,4)$. We assume w.l.o.g. that the number of terminals in $G_0$ is at least that in $G_1$. When all the four terminals are contained in $G_0$, we first find a hamiltonian path $P_0$ in $G_0$ joining $s_1$ and $s_2$, and let $P_0 = (s_1, P_x, x, t_1, P_y, y, t_2, P_z, s_2)$. For a path $P = (v_1, v_2, \ldots, v_l)$, we denote by $P^R$ the reverse of a path $P$, that is, $P^R = (v_l, v_{l-1}, \ldots, v_1)$. Then, we have $P_1 = (s_1, P_x, x, HH[\bar{x}, \bar{y}|G_1, \emptyset], y, P_y^R, t_1)$ and $P_2 = (s_2, P_z^R, t_2)$. When there are three terminals in $G_0$, we assume w.l.o.g. that $s_1$, $s_2$, and $t_1$ are contained in $G_0$. We first find a hamiltonian path $P_0$ in $G_0$ joining $s_1$ and $s_2$ and let $P_0 = (s_1, P_x, x, t_1, y, P_y, s_2)$. Assuming w.l.o.g. that $\bar{x} \neq t_2$, we have $P_1 = (s_1, P_x, x, HH[\bar{x}, t_2|G_1, \emptyset])$ and $P_2 = (s_2, P_y^R, y, t_1)$.

Now we consider the case that there are two terminals in $G_0$. If there are one source and one sink in $G_0$, assuming w.l.o.g. that $s_1$ and $t_1$ are contained in $G_0$, we have $P_1 = HH[s_1, t_1|G_0, \emptyset]$ and $P_2 = H[s_2, t_2|G_1, \emptyset]$. Thus, we assume that $s_1$ and $s_2$ are contained in $G_0$ and $t_1$ and $t_2$ are contained in $G_1$. We will show that there exist a pair of free edges $(x, \bar{x})$ and $(y, \bar{y})$ with $x, y \in V(G_0)$ satisfying (A1) $G_0$ has disjoint covering paths joining $\{s_1, s_2\}$ and $\{x, y\}$ and (A2) for some $z \neq \bar{x}, \bar{y}$, $G_1 \backslash z$ also has disjoint covering paths joining $\{t_1, t_2\}$ and $\{\bar{x}, \bar{y}\}$. Once we have such a pair of free edges, merging the disjoint covering paths in $G_0$ and the disjoint covering paths in $G_1 \backslash z$ with the pairs of free edges results in disjoint covering paths in $G_0 \oplus G_1 \backslash z$ joining $\{s_1, s_2\}$ and $\{t_1, t_2\}$. There are at least 4 free edges joining vertices in $G_0$ and vertices in $G_1$, and thus there are at least $\binom{4}{2} = 6$ pairs of such edges. Among the 6 pairs, due to Lemma 5, at least 3 pairs satisfy the condition A1, and thus at least 2 pairs satisfy both conditions A1 and A2 by Lemma 4. Therefore, we have the lemma. □

*Remark 2.* Similar to the proof of Lemma 6, we can show that $G(8,4) \oplus G(8,4)$ has two disjoint covering paths joining every $\{s_1, s_2\}$ and $\{t_1, t_2\}$ with $\{s_1, s_2\} \cap \{t_1, t_2\} = \emptyset$.

**Lemma 7.** *Every 4-dimensional restricted HL-graph $G(8,4) \oplus G(8,4)$ is 1-fault 5-panconnected.*

*Proof.* Let $G_0$ and $G_1$ be graphs isomorphic to $G(8,4)$. By Theorem 1(c) and Corollary 1, it suffices to construct a path of every length 5 or more joining $s$ and $t$ in the case that there is one faulty element in $G_0$ and $s$ and $t$ are contained in $G_1$. In $G_1$, we have a path $P_0$ of length from 3 to 7 inclusive joining $s$ and $t$ by Lemma 3. It remains to construct a path $P_1$ of every length $l_1$, $8 \leq l_1 \leq 15 - f_v$. Since $G_0 \backslash F_0$ has a hamiltonian cycle $C_0$ by Theorem 4, we have a path $P'$ on $C_0$ of length every $l'$, $1 \leq l' \leq 7 - f_v$, such that (i) letting $x$ and $y$ be the two endvertices of $P'$, $\{s, t\} \cap \{\bar{x}, \bar{y}\} = \emptyset$ and (ii) there exist two disjoint covering paths in $G_1 \backslash z$ for some $z$ joining $\{s, t\}$ and $\{\bar{x}, \bar{y}\}$. Then, $P_1$ can be constructed by merging $P'$ and two disjoint covering paths in $G_1 \backslash z$ joining $\{s, t\}$ and $\{\bar{x}, \bar{y}\}$. The length $l_1$ of $P_1$ is in the range $8 \leq l_1 \leq 15 - f_v - 1$. A path of length $15 - f_v$ is a hamiltonian path, and its existence is due to Theorem 4. Thus, we have the lemma. □

**Lemma 8.** *Every 5-dimensional restricted HL-graph $[G(8,4) \oplus G(8,4)] \oplus [G(8,4) \oplus G(8,4)]$ is 2-fault 7-panconnected.*

*Proof.* The proof of the lemma is similar to that of Lemma 7. Let $G_0$ and $G_1$ be graphs isomorphic to $G(8,4) \oplus G(8,4)$. By Theorem 1(b) and Corollary 1, we assume that $s$ and $t$ are contained in $G_1$ and both $\bar{s}$ and $\bar{t}$ in $G_0$ are the faulty vertices. There exists a path $P_0$ in $G_1$ of every length $l_0$, $5 \le l_0 \le 15$, joining $s$ and $t$ by Lemma 7. Since $G_0 \backslash F_0$ has a hamiltonian cycle $C_0$, we can construct a path $P'$ of every length $l'$, $1 \le l' \le 13$. Letting $x$ and $y$ be the endvertices of $P'$, we can obtain a path $P_1$ by merging $P'$ and two disjoint covering paths in $G_1 \backslash z$ for some $z$ joining $\{s,t\}$ and $\{\bar{x},\bar{y}\}$ with edges $(x,\bar{x})$ and $(y,\bar{y})$. The length $l_1$ of $P_1$ is in the range $16 \le l_1 \le 28$. A hamiltonian path of length 29 exists due to Theorem 4. This completes the proof. □

By an inductive argument utilizing Theorem 1(a) and Lemmas 3, 7, and 8, we have Theorem 5. Note that for $n = 2^m$, $f = m-3$, and $q = 2m-3$, it holds true that for any $m \ge 3$, $n = 2^m \ge f + 2q + 1 = 5m - 8$ and $q = 2m - 3 \ge 2f + 3 = 2m - 3$.

**Theorem 5.** *Every $m$-dimensional restricted HL-graph, $m \ge 3$, is $m-3$-fault $2m-3$-panconnected.*

**Corollary 2.** *Every $m$-dimensional restricted HL-graph, $m \ge 3$, is $m-3$-fault hypohamiltonian-connected.*

*Remark 3.* Let $q^*_m$ be the minimum $q_m$ such that every $m$-dimensional restricted HL-graph is $m-3$-fault $q_m$-panconnected. An upper bound $2m-3$ on $q^*_m$ is suggested by Theorem 5. The graph product $G(8,4) \times Q_{m-3}$ of $G(8,4)$ and $m-3$-dimensional hypercube $Q_{m-3}$, which is an $m$-dimensional restricted HL-graph, is not 0-fault $m$-panconnected (even though $f = 0$) since there does not exist a path of length $m$ between the two vertices $(v_0, 00\cdots0)$ and $(v_0, 11\cdots1)$ of distance $m-3$. Therefore, we have $m+1 \le q^*_m \le 2m-3$.

A graph $G$ is called *$f$-fault $q$-edge-pancyclic* if for any faulty set $F$ with $|F| \le f$, there exists a cycle of every length from $q$ to $|V(G\backslash F)|$ that passes through an arbitrary fault-free edge. Of course, an $f$-fault $q$-panconnected graph is always $f$-fault $q+1$-edge-pancyclic. From Theorem 5, we have the following.

**Theorem 6.** *Every $m$-dimensional restricted HL-graph, $m \ge 3$, is $m-3$-fault $2m-2$-edge-pancyclic.*

### 3.2 Pancyclicity of restricted HL-graphs

To show that every $m$-dimensional restricted HL-graph is $m-2$-fault almost pancyclic, due to Lemmas 1 and 2, we assume that the faulty set $F$ contains $m-2$ faulty vertices.

**Lemma 9.** *The 3-dimensional restricted HL-graph $G(8,4)$ is 1-fault almost pancyclic.*

*Proof.* We assume $v_0$ is faulty. Since $G(8,4)$ is 1-fault hamiltonian, it is sufficient to construct a cycle $C_l$ of length $l$ for every $4 \le l \le 6$. We have $C_4 = (v_1, v_5, v_6, v_2)$, $C_5 = (v_1, v_2, v_3, v_4, v_5)$, $C_6 = (v_1, v_2, v_3, v_7, v_6, v_5)$. □

**Lemma 10.** *Every* 4*-dimensional restricted HL-graph* $G(8,4) \oplus G(8,4)$ *is* 2*-fault almost pancyclic.*

*Proof.* We let $G_0$ and $G_1$ be graphs isomorphic to $G(8,4)$. They are 0-fault hamiltonian-connected, 0-fault hypohamiltonian-connected, and 1-fault almost pancyclic by Lemmas 3 and 9. To show $G_0 \oplus G_1$ is 2-fault almost pancyclic, by Theorem 3(b), we assume that each $G_i$ has one faulty vertex. $G_0$ has cycles of length 4 through 7, and $G_0 \oplus G_1$ has a hamiltonian cycle of length 14. To construct a cycle of length $l$ for every $8 \le l \le 13$, we find a path $P_0$ of length $l-7$ in $G_0$ joining some pair of vertices $x$ and $y$ such that (B1) $\bar{x}$ and $\bar{y}$ are fault-free and (B2) there exists a hypohamiltonian path $P_1$ in $G_1 \backslash F_1$ between $\bar{x}$ and $\bar{y}$. Then, $P_0$ and $P_1$ are merged with $(x, \bar{x})$ and $(y, \bar{y})$ to obtain a cycle of length $l$. To see the existence of such $P_0$ and $P_1$, let $C_0$ be a hamiltonian cycle in $G_0 \backslash F_0$. On $C_0$, there are 7 different paths of length $l-7$. Among them, at least 5 satisfy the condition B1, and furthermore, by Lemma 11 given below, at least 2 also satisfy the condition B2. □

**Lemma 11.** *Let* $G(8,4)$ *have one faulty vertex* $v_0$*. There exists a hypohamiltonian path in* $G(8,4) \backslash v_0$ *between every pair of vertices* $s$ *and* $t$ *provided* $\{s,t\} \ne \{v_2, v_6\}$, $\{v_3, v_4\}$, *and* $\{v_4, v_5\}$.

*Proof.* The proof is enumerative. See Table 2. □

**Table 2.** Hypohamiltonian path $P$ in $G(8,4) \backslash v_0$ between $s$ and $t$

| $s$ | $t$: $P$ | | |
|---|---|---|---|
| $s = v_1$ | $v_2$: $v_1$-$v_5$-$v_6$-$v_7$-$v_3$-$v_2$; | $v_3$: $v_1$-$v_2$-$v_6$-$v_5$-$v_4$-$v_3$; | $v_4$: $v_1$-$v_5$-$v_6$-$v_2$-$v_3$-$v_4$; |
| | $v_5$: $v_1$-$v_2$-$v_3$-$v_7$-$v_6$-$v_5$; | $v_6$: $v_1$-$v_2$-$v_3$-$v_4$-$v_5$-$v_6$; | $v_7$: $v_1$-$v_5$-$v_6$-$v_2$-$v_3$-$v_7$; |
| $s = v_2$ | $v_3$: $v_2$-$v_1$-$v_5$-$v_6$-$v_7$-$v_3$; | $v_4$: $v_2$-$v_3$-$v_7$-$v_6$-$v_5$-$v_4$; | $v_5$: $v_2$-$v_6$-$v_7$-$v_3$-$v_4$-$v_5$; |
| | $v_6$: does not exist; | $v_7$: symm. to $(v_1, v_6)$; | |
| $s = v_3$ | $v_4$: does not exist; | $v_5$: $v_3$-$v_7$-$v_6$-$v_2$-$v_1$-$v_5$; | $v_6$: symm. to $(v_2, v_5)$; |
| | $v_7$: symm. to $(v_1, v_5)$; | | |
| $s = v_4$ | $v_5$: does not exist; | $v_6$: symm. to $(v_2, v_4)$; | $v_7$: symm. to $(v_1, v_4)$; |
| $s = v_5$ | $v_6$: symm. to $(v_2, v_3)$; | $v_7$: symm. to $(v_1, v_3)$; | |
| $s = v_6$ | $v_7$: symm. to $(v_1, v_2)$; | | |

From Lemmas 9 and 10, Corollary 2, and Theorem 3(a), we have Theorem 7.

**Theorem 7.** *Every* $m$*-dimensional restricted HL-graph,* $m \ge 3$*, is* $m-2$*-fault almost pancyclic.*

**Corollary 3.** *(a) Twisted cube $TQ_m$, $m \geq 3$, is $m-2$-fault almost pancyclic[29].*
*(b) Crossed cube $CQ_m$, $m \geq 3$, is $m-2$-fault almost pancyclic[28].*
*(c) Multiply twisted cube $MQ_m$, $m \geq 3$, is $m-2$-fault almost pancyclic.*
*(d) Both 0-Möbius cube and 1-Möbius cube of dimension $m$, $m \geq 3$, are $m-2$-fault almost pancyclic[14].*
*(e) The $m$-Mcube, $m \geq 3$, is $m-2$-fault almost pancyclic.*
*(f) Generalized twisted cube $GQ_m$, $m \geq 3$, is $m-2$-fault almost pancyclic.*
*(g) Locally twisted cube $LTQ_m$, $m \geq 3$, is $m-2$-fault almost pancyclic.*
*(h) $G(2^m, 4)$, $m$ odd and $m \geq 3$, is $m-2$-fault almost pancyclic[20].*

We note that recursive circulant $G(2^m, 4)$ for an odd $m$ is a restricted HL-graph although not every $G(2^m, 4)$ is a restricted HL-graph. One can check without difficulty that $G(16, 4)$ is not isomorphic to $G(8, 4) \oplus_M G(8, 4)$ for any $M$, and even $G(16, 4)$ does not have $G(8, 4)$ as a subgraph.

## 4 Concluding Remarks

In this paper, we studied the problems of how fault-panconnectivity and fault-pancyclicity of two graphs $G_0$ and $G_1$ are translated into fault-panconnectivity and fault-pancyclicity of $G_0 \oplus G_1$, respectively. It was proved that if $G_0$ and $G_1$ are $f$-fault $q$-panconnected and $f+1$-fault hamiltonian (with additional conditions $n \geq f + 2q + 1$ and $q \geq 2f + 3$), then $G_0 \oplus G_1$ is $f+1$-fault $q+2$-panconnected for any $f \geq 2$, and that if $G_0$ and $G_1$ are $f$-fault hamiltonian-connected, $f$-fault hypohamiltonian-connected, and $f+1$-fault almost pancyclic, then $G_0 \oplus G_1$ is $f+2$-fault almost pancyclic for any $f \geq 1$. Applying these results to restricted HL-graphs, we concluded that every $m$-dimensional restricted HL-graph with $m \geq 3$ is $m-3$-fault $2m-3$-panconnected and $m-2$-fault almost pancyclic.

According to the constructions presented in this paper, we can design efficient algorithms for finding an $s$-$t$ path and a fault-free cycle of specified length in a faulty restricted HL-graph. The work on almost pancyclicity of restricted HL-graphs with faulty elements is a generalization of some works on individual interconnection networks such as crossed cubes[28], Möbius cubes[14], and twisted cubes[29]. As the authors know, no results on fault-panconnectivity and fault-edge-pancyclicity of interconnection networks appeared in the literature. It is worthwhile to investigate fault-panconnectivity and fault-edge-pancyclicity of individual interconnection networks such as recursive circulants, crossed cubes, twisted cubes, etc.

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