Embedding Starlike Trees into Hypercube-Like Interconnection Networks

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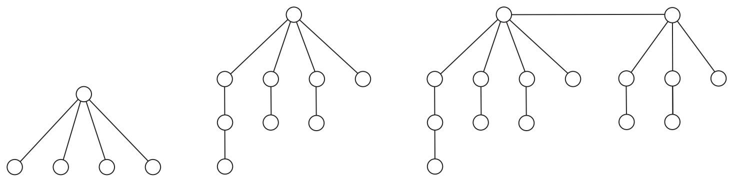
Abstract. A *starlike tree* (or a *quasistar*) is a subdivision of a star tree. A family of hypercube-like interconnection networks called *restricted HL-graphs* includes many interconnection networks proposed in the literature such as twisted cubes, crossed cubes, multiply twisted cubes, Möbius cubes, Mcubes, and generalized twisted cubes. We show in this paper that every starlike tree of degree at most m with 2^m vertices is a spanning tree of m-dimensional restricted HL-graphs. It is also proved that in an m-dimensional restricted HL-graph, there exist $k(\leq m-1)$ vertex-disjoint s_i-paths with l_i vertices each that cover all the vertices in the graph for any k sources $s_1, s_2, \ldots, s_k$ associated with positive integers $l_1, l_2, \ldots, l_k$ whose sum is equal to the number of vertices.

Key Words: Spanning trees, restricted HL-graphs, path partition, interconnection networks.

1 Introduction

Much research has been done to investigate whether an interconnection network contains a certain class of trees as spanning subgraphs. For spanning trees of hypercubes, one of well-known interconnection networks, various trees were investigated such as binomial trees, caterpillars[3, 8], double-rooted complete binary trees[10], starlike and double starlike trees[9]. Other containment results can be found in [10]. This paper deals with starlike trees for spanning trees of a family of interconnection networks called restricted HL-graphs proposed in [17].

A *d-star* is a tree of degree d which is isomorphic to a complete bipartite graph $K_{1,d}$ whose vertex set is $\{r, z_1, z_2, \ldots, z_d\}$ and whose edge set is $\{(r, z_i) | 1 \leq i \leq d\}$. Any graph formed from a graph G by the insertion of vertices of degree two into the edges of G is called a *subdivision* of G. A *d-starlike tree* (or a *d-quasistar*) is a subdivision of a d-star. Here, r is the root of the tree and z_i's are leaves. A *d-double starlike tree* is a tree obtained by connecting via an edge the roots of a d-starlike tree and a d'-starlike tree with $d' \leq d$. Examples of a d-star, a d-starlike tree, and a d-double starlike tree are shown in Figure 1.



(a) 4-star (b) 4-starlike tree (c) 4-double starlike tree

Fig. 1. a 4-star, a 4-starlike tree, and a 4-double starlike tree.

For two graphs G and H, G *spans* H if there is a one-to-one mapping ϕ of $V(G)$ into $V(H)$ such that if $(u, v) \in E(G)$ then $(\phi(u), \phi(v)) \in E(H)$. A bipartite graph G is called *equitable* if G has a proper bicoloring such that both color sets have the same cardinality. Nebeský[13] showed that every equitable d-starlike tree with 2^m vertices, $d \leq m$, spans m-dimensional hypercube Q_m. No non-equitable d-starlike tree with 2^m vertices spans Q_m since Q_m itself is equitable. Kobeissi and Mollard[9] showed that every equitable d-double starlike tree with 2^m vertices, $d \leq 5$ and $d + 1 \leq m$, spans Q_m.

Many interconnection networks can be expanded into higher dimensional networks by connecting two lower dimensional networks. As a graph modeling of the expansion, we consider the graph obtained by connecting two graphs G_0 and G_1 with n vertices. We denote by V_i and E_i the vertex set and edge set of G_i, $i = 0, 1$, respectively. We let $V_0 = \{v_1, v_2, \ldots, v_n\}$ and $V_1 = \{w_1, w_2, \ldots, w_n\}$. With respect to a permutation $M = (i_1, i_2, \ldots, i_n)$ of $\{1, 2, \ldots, n\}$, we can "merge" the two graphs into a graph $G_0 \oplus_M G_1$ with $2n$ vertices in such a way that the vertex set $V = V_0 \cup V_1$ and the edge set $E = E_0 \cup E_1 \cup E_2$, where $E_2 = \{(v_j, w_{i_j}) | 1 \leq j \leq n\}$. We denote by $G_0 \oplus G_1$ a graph obtained by merging G_0 and G_1 w.r.t. an arbitrary permutation M. Here, G_0 and G_1 are called *components* of $G_0 \oplus G_1$. Figure 2 shows an example of $G_0 \oplus G_1$.

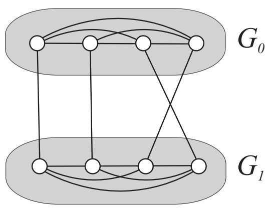
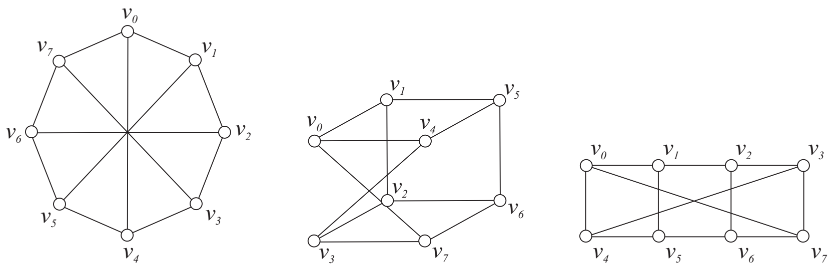


Fig. 2. An example of $G_0 \oplus G_1$.

Vaidya *et al.*[20] introduced a class of hypercube-like interconnection networks, called *HL-graphs*, which can be defined by applying the $\oplus$ operation repeatedly as follows: $HL_0 = \{K_1\}$; for $m \geq 1$, $HL_m = \{G_0 \oplus G_1 | G_0, G_1 \in HL_{m-1}\}$. Then, $HL_1 = \{K_2\}$; $HL_2 = \{C_4\}$; $HL_3 = \{Q_3,\ G(8,4)\}$. Here, C_4 is a cycle graph with 4 vertices, Q_3 is a 3-dimensional hypercube, and $G(8,4)$ is a recursive circulant which is isomorphic to twisted cube TQ_3 and Möbius ladder[12] with four spokes as shown in Figure 3.



(a) $G(8,4)$ (b) TQ_3 (c) Möbius ladder

Fig. 3. Isomorphic graphs.

In [17], a subclass of nonbipartite HL-graphs, called *restricted HL-graphs* was introduced by the authors which is defined recursively as follows: $RHL_m = HL_m$ for $0 \leq m \leq 2$; $RHL_3 = HL_3 \backslash Q_3 = \{G(8,4)\}$; $RHL_m = \{G_0 \oplus G_1 | G_0, G_1 \in RHL_{m-1}\}$ for $m \geq 4$. A graph which belongs to RHL_m is called an *m-dimensional restricted HL-graph.* Many of the nonbipartite hypercube-like interconnection networks such as crossed cube[5], Möbius cube[2], twisted cube[7], multiply twisted cube[4], Mcube[19], generalized twisted cube[1], etc. proposed in the literature are restricted HL-graphs with the exception of recursive circulant $G(2^m, 4)$[16] and "near" bipartite interconnection networks such as twisted m-cube[6]. In fact, every $G(2^m, 4)$ with odd m is an m-dimensional restricted HL-graph. Some works on HL-graphs and restricted HL-graphs were appeared in the literature; for example, hamiltonicity of HL-graphs[14], fault-hamiltonicity of restricted HL-graphs[17], fault-pancyclicity of restricted HL-graphs[11], and disjoint path cover, a set of disjoint paths that cover all the vertices in restricted HL-graphs[15, 18].

Concerning starlike trees as spanning trees of restricted HL-graphs, this paper shows two main theorems in the following. Theorem 1 cannot be extended to HL-graphs since every bipartite graph in HL-graphs is equitable and no equitable bipartite graph has a non-equitable starlike tree as a spanning tree.

Theorem 1. *Every d-starlike tree with 2^m vertices is a spanning tree of m-dimensional restricted HL-graphs for any $1 \leq d \leq m$. Furthermore, the root of the starlike tree can be embedded into any vertex of the restricted HL-graph.*

To prove Theorem 1, we rely on path partitionability of restricted HL-graphs. The path partition problem with which we are concerned in this paper is defined as follows. Given k distinct sources $s_1, s_2, \ldots, s_k$ in a graph G and k positive integers $l_1, l_2, \ldots, l_k$ with $\sum_{1 \le i \le k} l_i = |V(G)|$, *a k-path partition* is a set of k vertex-disjoint paths $\{P_1, P_2, \ldots, P_k\}$ such that each P_i is an s_i-path (that is, s_i is an endvertex of P_i) with l_i vertices and $\bigcup_{1 \le i \le k} V(P_i) = V(G)$. A graph G is called to be *k-path partitionable* if for any k distinct sources associated with positive integers l_i's with $\sum_{1 \le i \le k} l_i = |V(G)|$, G has a k-path partition.

Theorem 2. *Every m-dimensional restricted HL-graph is k-path partitionable for any* $1 \le k \le m-1$, $m \ge 2$.

The two main theorems utilize fault-hamiltonicity of restricted HL-graphs. A graph G is called *f-fault hamiltonian* (resp. *f-fault hamiltonian-connected*) if there exists a hamiltonian cycle (resp. if each pair of vertices are joined by a hamiltonian path) in $G \backslash F$ for any set F of faulty elements (vertices and/or edges) with $|F| \le f$. It was shown in [17] that every m-dimensional restricted HL-graph, $m \ge 3$, is $m-3$-fault hamiltonian-connected and $m-2$-fault hamiltonian.

Throughout this paper, a path in a graph is represented as a sequence of vertices. We denote by $H[v, w|G, F]$ a hamiltonian path in $G \backslash F$ joining a pair of fault-free vertices v and w in a graph G with a set F of faulty elements. G^m denotes an arbitrary m-dimensional restricted HL-graph. By definition, G^m is isomorphic to $G_0 \oplus G_1$ for some $m-1$-dimensional restricted HL-graphs G_0 and G_1. For a vertex v in $G_0 \oplus G_1$, we denote by $\bar{v}$ the vertex adjacent to v which is in a component different from the component in which v is contained. In the rest of this paper, two main theorems will be proved.

2 Proof of Theorem 1

Each subtree of a starlike tree forms a path. We denote by $T(a_1, a_2, \ldots, a_d)$ a d-starlike tree with root r and r-paths of length a_i, $1 \le i \le d$. The starlike tree has $\sum_{1 \le i \le d} a_i + 1$ vertices. We assume without loss of generality that $a_1 \ge a_2 \ge \cdots \ge a_d$. Some cases of Lemmas 3 and 4 rely on path partition, considered in the next section.

Lemma 1. *For* $d = 1, 2$*, every d-starlike tree with* 2^m *vertices is a spanning tree of* G^m, $m \ge 2$.

Proof. Let $C = (x_0, x_1, \ldots, x_{2^m-1})$ be a hamiltonian cycle in G^m. The d-starlike trees rooted at x_0 can be constructed by removing an appropriate edge from C. Precisely speaking, $C \backslash (x_0, x_{2^m-1})$ and $C \backslash (x_{a_1}, x_{a_1+1})$ are the desired trees for $d = 1$ and $d = 2$, respectively. □

Lemma 2. *Every* 3*-starlike tree with* 2^m *vertices is a spanning tree of* G^m, $m \ge 3$.

Proof. For $m \geq 4$, we first find $T(b, a_3)$ rooted at an arbitrary vertex in G_0 (or symmetrically in G_1), where $b = 2^{m-1} - a_3 - 1$. We have $b \geq 1$ since $a_3 + 1 \leq (2^m - 1)/3 + 1 < 2^{m-1}$ for any $m \geq 4$. Let the r-path of length b be r-z path, that is, z is an endvertex of the path. And then, we find a hamiltonian path $P = H[\bar{z}, \bar{r}|G_1, \emptyset]$ in G_1 between $\bar{z}$ and $\bar{r}$. We merge the r-z path and P with two edges $(z, \bar{z})$ and $(\bar{r}, r)$ into a cycle C of length $a_1 + a_2 + 1$. The tree can be obtained by removing an appropriate edge from C. Now, let $m = 3$. G^m is isomorphic to $G(8, 4)$. We construct a starlike tree rooted at v_0. When $a_3 = 1$, letting x be any vertex in $G(8, 4)$ adjacent to v_0, (v_0, x) is a path of length a_3 and the other two paths are obtained by removing an appropriate edge in a hamiltonian cycle in $G(8, 4)\backslash x$. When $a_3 \geq 2$ (by assumption, $a_1 = 3$ and $a_2, a_3 = 2$), we explicitly construct three paths (v_0, v_7, v_6, v_5), (v_0, v_4, v_3), and (v_0, v_1, v_2). This completes the proof. □

Lemma 3. *Every 4-starlike tree with 2^m vertices is a spanning tree of G^m, $m \geq 4$.*

Proof. Let us consider the case of $a_3 + a_4 \leq 2^{m-1} - 2$ first. Let $T(b, a_3, a_4)$ be a starlike tree rooted at any vertex r in G_0, where $b = 2^{m-1} - a_3 - a_4 - 1$, and the r-path of length b be r-z path. The r-z path and $H[\bar{z}, \bar{r}|G_1, \emptyset]$ are merged with $(z, \bar{z})$ and $(\bar{r}, r)$ into a cycle of length $a_1 + a_2 + 1$. Removing an appropriate edge from the cycle results in a desired tree. Now, we assume that $a_3 + a_4 \geq 2^{m-1} - 1$, that is, $a_1 = a_2 = a_3 = 2^{m-2}$ and $a_4 = 2^{m-2} - 1$. When $m \geq 5$, similar to the previous case, the tree can be constructed by using $T(b, a_3 - 1, a_4)$ and $H[\bar{z}, \bar{r}|G_1, \{\bar{x}\}]$, where $b = 1$ and x is the endvertex of r-path of length $a_3 - 1$, $x \neq r$. The hamiltonian path exists since G_1 is 1-fault hamiltonian-connected. When $m = 4$, we find $T(3, 2, 2)$ in G_0. Let r-paths of length 2 be r-x path and r-y path, respectively. And then, we find a 3-path partition for $\bar{r}$, $\bar{x}$, and $\bar{y}$ with associated weights 4, 2, 2, respectively. The tree in G_0 and the 3-path partition in G_1 are merged into the desired tree. The existence of path partition is due to Lemma 5 in Section 3. □

Lemma 4. *For $d \geq 5$, every d-starlike tree with 2^m vertices is a spanning tree of G^m, $m \geq d$.*

Proof. For the case of $a_1 > 2^{m-1}$, similar to the proof of Lemma 3, we find $T(b, a_3, a_4, \ldots, a_d)$ in G_0 with $b = 2^{m-1} - \sum_{3 \leq i \leq d} a_i - 1$. The r-z path, the r-path of length b is merged with $H[\bar{z}, \bar{r}|G_1, \emptyset]$ into a cycle of length $a_1 + a_2 + 1$. It suffices to remove an appropriate edge from the cycle. Now, we assume $a_1 \leq 2^{m-1}$. We let $a'_2, a'_3, \ldots, a'_d$ be positive integers satisfying (i) $1 + \sum_{2 \leq i \leq d} a'_i = 2^{m-1}$, (ii) $a'_i = a_i$ for $i = d-1, d$, and (iii) $a'_i \leq a_i$ for every $2 \leq i \leq d-2$. To show such a'_i's exist, we claim that $a_{d-1} + a_d + (d-2) \leq 2^{m-1}$ for any $5 \leq d \leq m$. The proof of the claim is by a simple calculation using $a_{d-1} + a_d \leq 2(2^m - 1)/d$, and omitted here. Moreover, we can see that there exist a'_i's such that for some p, $2 \leq p \leq d-1$, $a'_i < a_i$ for all $2 \leq i < p$ and $a'_i = a_i$ for all $p \leq i \leq d$. Then, we find $T(a'_2, a'_3, \ldots, a'_d)$ in G_0. Let the r-path of length a'_i be r-z_i path for each $2 \leq i < p$. To obtain a desired tree, it suffices to construct a $p-1$-path partition for $\bar{r}$ and $\bar{z}_i$ for all $2 \leq i < p$ with associated weights a_1 and $a_i - a_i'$'s, respectively. We have $p - 1 \leq d - 2 \leq m - 2$. The existence of path partition is due to Theorem 2, which will be proved later in Section 3. □

3 Path Partitions

Given k distinct sources $s_1, s_2, \ldots, s_k$ in a graph G associated with k positive integers $l_1, l_2, \ldots, l_k$, respectively, satisfying $\sum_{1 \leq i \leq k} l_i = |V(G)|$, a k-path partition consists of k disjoint paths P_i with l_i vertices, $1 \leq i \leq k$, where each P_i is an s_i-path. The *sink* of P_i is the endvertex of P_i different from s_i if $l_i \geq 2$; if $l_i = 1$, s_i is the sink as well as the source of P_i.

For $m = 2, 3$, Theorem 2 holds true since the path partitions are constructed straightforwardly from the hamiltonian cycles/paths. For some l_i's, the 3-dimensional restricted HL-graph $G(8, 4)$ has a 3-path partition for any 3 sources as follows.

Lemma 5. *$G(8, 4)$ has a 3-path partition for any three sources if $(l_1, l_2, l_3) = (3, 3, 2)$, $(4, 2, 2)$, or $(5, 2, 1)$.*

Proof. The proof is by an immediate inspection. □

For $m \geq 4$, we will prove a stronger result than Theorem 2 claims. We are to pose an additional constraint on the k-path partition that for any vertex subsets W_i with $|W_i| \leq m - k$, $1 \leq i \leq k$, the sink of each s_i-path should never be contained in W_i. Here, we assume $s_i \notin W_i$ whenever $l_i = 1$. Otherwise, no graph has such a k-path partition. Moreover, we assume without loss of generality that no sources are contained in W_i for all i. A graph G is called *strongly k-path partitionable* if G has a k-path partition satisfying the additional constraint for any s_i, l_i, and W_i, $1 \leq i \leq k$. Hereafter in this section, we will prove Theorem 3 by an induction on m.

Theorem 3. *Every m-dimensional restricted HL-graph G^m is strongly k-path partitionable for any $1 \leq k \leq m - 1$, $m \geq 4$.*

Obviously, the theorem holds true for $k = 1$ since G^m is hamiltonian-connected. From now on, we assume $k \geq 2$.

Lemma 6. *Every 4-dimensional restricted HL-graph $G(8, 4) \oplus G(8, 4)$ is strongly k-path partitionable for any $2 \leq k \leq 3$.*

Proof. The proof will be given in Appendix. □

Let $m \geq 5$. We denote by k_0 and k_1 the numbers of sources in G_0 and G_1, respectively. Of course, $k_0 + k_1 = k$. Let $I_0 = \{1, 2, \ldots, k_0\}$ and $I_1 = \{k_0 + 1, k_0 + 2, \ldots, k_0 + k_1\}$. We assume that $S_0 = \{s_i | i \in I_0\}$ and $S_1 = \{s_j | j \in I_1\}$ are sets of sources contained in G_0 and G_1, respectively, and that $l_1 \geq l_2 \geq \cdots \geq l_{k_0}$ and $l_{k_0+1} \geq \cdots \geq l_{k_0+k_1}$. Let $W_i^0 = W_i \cap V(G_0)$ and $W_i^1 = W_i \cap V(G_1)$ for each $i \in I_0 \cup I_1$. We denote by k-PP$[\{(s_1, l_1, W_1), \ldots, (s_k, l_k, W_k)\}|G]$ a k-path partition in a graph G for s_i, l_i, and W_i, $1 \le i \le k$, if any. Let P_i be the s_i-path in the path partition, and let $t(P_i)$ be the sink of P_i. We let $L_0 = \sum_{i \in I_0} l_i$ and $L_1 = \sum_{j \in I_1} l_j$. If $L_0 = L_1$, we are done since the union of k_0-path partition in G_0 and k_1-path partition in G_1 results in a k-path partition in G^m. We assume without loss of generality $L_0 > L_1$.

In proving Theorem 3, hamiltonian properties of 4-dimensional restricted HL-graphs with faulty vertices are utilized. They are listed in Subsection 3.1. The construction of a k-PP for $k = 2$ will be investigated in Subsection 3.2, and the construction for $k \ge 3$ will be addressed in Subsections 3.3 through 3.5.

3.1 Fault-hamiltonicity of $G(8, 4) \oplus G(8, 4)$

All the lemmas given in this subsection will be proved in Appendix.

Lemma 7. *Let $G(8, 4) \oplus G(8, 4)$ have two faulty vertices v_f and v'_f. For any fault-free vertex s, there exist three fault-free vertices t_1, t_2, and t_3 such that $G(8, 4) \oplus G(8, 4) \backslash \{v_f, v'_f\}$ has an s-t_i hamiltonian path for each i.*

Lemma 8. *Let $G(8, 4) \oplus G(8, 4)$ have two faulty vertices v_f and v'_f. There exist two fault-free vertices x_1 and x_2 such that $G(8, 4) \oplus G(8, 4) \backslash \{v_f, v'_f, x_i\}$ has a hamiltonian cycle for each i. Furthermore, if v_f is adjacent to v'_f, there exist four such vertices x_i, $i = 1, \ldots, 4$.*

Lemma 9. *Let $G(8, 4) \oplus G(8, 4)$ have one faulty vertex v_f. For any pair of fault-free vertices s and t, there is a vertex $x \ne s, t$ adjacent to v_f such that $G(8, 4) \oplus G(8, 4) \backslash \{v_f, x\}$ has an s-t hamiltonian path.*

3.2 $k = 2$

When $l_2 = 1$, we have $P_2 = (s_2)$ and $P_1 = H[s_1, x|G^m, \{s_2\}]$ for some vertex $x \notin W_1 \cup \{s_1, s_2\}$. When $l_2 = 2$, for some vertex y adjacent to s_2 with $y \notin W_2 \cup \{s_1\}$, we have $P_2 = (s_2, y)$ and $P_1 = H[s_1, x|G^m, \{s_2, y\}]$ for some vertex $x \notin W_1 \cup \{s_1, s_2, y\}$. Note that G^m is 2-fault hamiltonian-connected. Let $l_2 \ge 3$. We have two cases.

Case 1 $s_1, s_2 \in V(G_0)$.
When $|W_2^0| \le m - 3$ and $l_2 < 2^{m-1}$, we find 2-PP$[\{(s_1, l'_1, \emptyset), (s_2, l_2, W_2^0)\}|G_0]$, where $l'_1 = 2^{m-1} - l_2$. Let P'_i be the s_i-path in the 2-PP. Then, $P_2 = P'_2$ and $P_1 = (P'_1, H[\bar{x}, y|G_1, \emptyset])$, where $x = t(P'_1)$ and y is a vertex in G_1 with $y \notin W_1 \cup \{\bar{x}\}$. When $|W_2^0| \le m - 3$ and $l_2 = 2^{m-1}$ ($l_1 = 2^{m-1}$), we let $P_2 = (H[s_2, \bar{x}|G_0, \{s_1\}], x)$ for some vertex x in G_1 with $x \notin W_2$ and $\bar{x} \ne s_1, s_2$, and let $P_1 = (s_1, H[\bar{s_1}, y|G_1, \{x\}])$ for some vertex y in G_1 with $y \notin W_1 \cup \{x, \bar{s_1}\}$. Finally when $|W_2^0| = m - 2$, we find 2-PP$[\{(s_1, l'_1, \emptyset), (s_2, l'_2, \emptyset)\}|G_0]$, where $l'_2 = l_2 - 1$ and $l'_1 = 2^{m-1} - l'_2$. Let P'_i be the s_i-path in the 2-PP. Then, $P_2 = (P'_2, \bar{x})$, where $x = t(P'_2)$, and $P_1 = (P'_1, H[\bar{y}, z|G_1, \{\bar{x}\}])$, where $y = t(P'_1)$ and z is a vertex in G_1 with $z \notin W_1 \cup \{\bar{x}, \bar{y}\}$.

Case 2 $s_1 \in V(G_0)$ and $s_2 \in V(G_1)$.
When $|W_2^1| \le m-3$ and $l_1 \ge 2^{m-1}+2$, we let x be a vertex in G_1 with $x \notin \{s_2, \bar{s_1}\}$ and assume $x \in W_1^1$ if $|W_1^1| = m-2$. Find 2-PP$[\{(x, l_1', W_1'), (s_2, l_2, W_2^1)\}|G_1]$, where $l_1' = 2^{m-1} - l_2$ and $W_1' = W_1^1 \backslash x$. Let P_1' and P_2' be the x-path and s_2-path in the partition, respectively. Then, $P_1 = (H[s_1, \bar{x}|G_0, \emptyset], P_1')$ and $P_2 = P_2'$. When $|W_2^1| \le m-3$ and $l_1 = 2^{m-1}+1$, letting x be a vertex in G_1 with $x \notin W_1 \cup \{s_2, \bar{s_1}\}$, we have $P_1 = (H[s_1, \bar{x}|G_0, \emptyset], x)$ and $P_2 = H[s_2, y|G_1, \{x\}]$ for some vertex y in G_1 with $y \notin W_2 \cup \{x, s_2\}$. When $|W_2^1| = m-2$ and $\bar{s_2} \ne s_1$, let $(x_1, x_2, \ldots, x_{2^{m-1}})$ be an $\bar{s_2}$-s_1 hamiltonian path in G_0. Then $P_2 = (s_2, x_1, x_2, \ldots, x_{l_2-1})$ and $P_1 = (x_{2^{m-1}}, x_{2^{m-1}-1}, \ldots, x_{l_2}, H[\bar{x_{l_2}}, y|G_1, \{s_2\}])$ for some vertex y in G_1 with $y \notin W_1 \cup \{s_2, \bar{x_{l_2}}\}$. When $|W_2^1| = m-2$ and $\bar{s_2} = s_1$, for a vertex y in G_1 with $y \notin W_1 \cup \{s_2\}$, let $(x_1, x_2, \ldots, x_{2^{m-1}})$ be an s_2-y hamiltonian path in G_1. Then, $P_2 = (x_1, x_2, \ldots, x_{l_2-1}, \bar{x_{l_2-1}})$ and $P_1 = (H[s_1, \bar{x_{l_2}}|G_0, \{\bar{x_{l_2-1}}\}], x_{l_2}, \ldots, x_{2^{m-1}})$.

3.3 $1 \le k_1 \le k-2$ ($k_0 \ge 2$)

First, we will develop a basic procedure PP-A for constructing a k-path partition. The procedure is applicable to the most of the subcases. Let $\bar{X} = \{\bar{x}|x \in X\}$ for a vertex subset X of $G_0 \oplus G_1$.

Procedure PP-A$(\{(s_1, l_1, W_1), \ldots, (s_k, l_k, W_k)\}, G_0 \oplus G_1)$

1. Find l_i' and l_i'', $i \in I_0$, satisfying (A1) $l_i' + l_i'' = l_i$, $1 \le l_i' \le l_i$, (A2) $\sum_{i \in I_0} l_i' = 2^{m-1}$, and (A3) $l_i' = l_i$ for some $i \in I_0$. Let $I_0' = \{i \in I_0 | l_i'' \ge 1\}$ and $k_0' = |I_0'|$.
2. Find k_0-PP$[\{(s_1, l_1', W_1'), \ldots, (s_{k_0}, l_{k_0}', W_{k_0}')\}|G_0]$, where $W_i' = W_i^0$ if $l_i'' = 0$; $W_i' = (\bar{W_i^1} \cup \bar{S_1}) \backslash S_0$ if $l_i'' = 1$; $W_i' = \bar{S_1}$ if $l_i'' \ge 2$. Let x_i be the sink of s_i-path in the k_0-PP.
3. Find $k_0' + k_1$-PP$[\{(\bar{x_i}, l_i'', W_i^1) | i \in I_0'\} \cup \{(s_j, l_j, W_j^1) | j \in I_1\}|G_1]$.
4. The two path partitions are merged with edges $(x_i, \bar{x_i})$ for $i \in I_0'$.

Lemma 10. *When $L_0 \ge 2^{m-1}+2$, Procedure PP-A constructs a k-PP unless (a) $k_0 = 2$, $k_1 = 1$, $l_1 = 2^{m-1}$, $l_2 = 2^{m-1}-1$, $l_3 = 1$, and $\bar{s_1} = s_3$, or (b) $k_0 = k_1 = 2$, $l_1 = l_2 = 2^{m-1}-1$, $l_3 = l_4 = 1$, and $\{\bar{s_1}, \bar{s_2}\} = \{s_3, s_4\}$. There also exist k-PP's for the two exceptional cases.*

Proof. Unless $k_0 = 2$ and $l_2 = 2^{m-1}-1$, we claim that there exist l_i' and l_i'', $i \in I_0$, satisfying additional two conditions (A4) $l_i' = l_i$ if $l_i \le 3$ and (A5) if $l_i \ge 4$, either $l_i' = l_i$ or $l_i = l_i' + l_i''$ with $l_i', l_i'' \ge 2$, as well as A1, A2, and A3. The proof of the claim has two cases: $l_{k_0} \le 3$ and $l_{k_0} \ge 4$. For the first case, let $P = \{i \in I_0 | l_i \ge 4\}$ and $p = |P|$. We show that for any l_i''s, $i \in P$, with $l_i' \le l_i$ and $L_0' \ge 2^{m-1}+2$, where L_0' is defined as $\sum_{i \in P} l_i' + \sum_{i \in I_0 \backslash P} l_i$, there exists j in P such that $l_j' \ge 5$. Suppose otherwise, $L_0' \le 4p + 3(k_0 - p) \le 4k_0 \le 4(m-2) < 2^{m-1}+2 \le L_0'$ for any $m \ge 5$, which is a contradiction. We decrement l_j' and L_0' by 2 if L_0' is even; otherwise, we decrement l_j' and L_0' by 3. Starting from $l_i' = l_i$ for all $i \in P$, the process is repeated until $L_0' = 2^{m-1}$. Letting $l_i' = l_i$ for all $i \in I_0 \backslash P$ and

$l''_i = l_i - l'_i$ for all $i \in I_0$, we have l'_i's and l''_i's satisfying all the five conditions. For the second case of $l_{k_0} \geq 4$, letting $Q = I_0 \backslash k_0$, we show that for any l'_i, $i \in Q$, with $l'_i \leq l_i$, L'_0 being defined as $\sum_{i \in Q} l'_i + l_{k_0}$, there exists j in Q such that $l'_j \geq 5$ if $L'_0 \geq 2^{m-1}+3$ and there exists j in Q such that $l'_j \geq 4$ if $L'_0 = 2^{m-1}+2$. Suppose otherwise, we can conclude a contradiction for each subcase. When $k_0 = 2$ and $L'_0 \geq 2^{m-1} + 3$, we have $L'_0 \leq l_{k_0} + 4 \leq (2^{m-1} - 2) + 4 < 2^{m-1} + 3 \leq L'_0$. When $k_0 = 2$ and $L'_0 = 2^{m-1} + 2$, we have $L'_0 \leq l_{k_0} + 3 < 2^{m-1} + 2 = L'_0$. Similarly, when $k_0 \geq 3$ and $L'_0 \geq 2^{m-1} + 3$, we show $L'_0 \leq l_{k_0} + 4(k_0 - 1) \leq (2^m - 1)/k_0 + 4(k_0 - 1) < 2^{m-1} + 3 \leq L'_0$. It suffices to show that the function $f(k_0, m) = (2^m - 1)/k_0 + 4(k_0 - 1) - 2^{m-1} - 3 < 0$ for any $m \geq 5$ and $3 \leq k_0 \leq m-2$. First, $f(3, m) = (2^m-1)/3+8-2^{m-1}-3 = -2^m/6+5-1/3 < 0$. Secondly, we show $f(k_0, m) \geq f(k_0 + 1, m)$ for any $m \geq 5$ and $3 \leq k_0 < m - 2$. We have $f(k_0, m) - f(k_0+1, m) = (2^m-1)\{1/k_0 - 1/(k_0+1)\} - 4 \geq (2^m-1)\{1/(m-3) - 1/(m-2)\} - 4 = (2^m-1)/(m-3)(m-2) - 4 \geq 0$. Finally, when $k_0 \geq 3$ and $L'_0 = 2^{m-1}+2$, we have $L'_0 \leq l_{k_0}+3(k_0-1) \leq (2^m-1)/k_0+3(k_0-1) < 2^{m-1}+2 = L'_0$. Now starting from $l'_i = l_i$ for all $i \in Q$, the process of decrementing l'_j and L'_0 as in the first case is applied repeatedly until $L'_0 = 2^{m-1}$. This completes proof of the claim. The k_0-PP exists in G_0 since $|W'_i| \leq (m-1)-k_0$ for every i. Note that $k_1 = k - k_0 \leq (m-1) - k_0$. The existence of $k'_0 + k_1$-PP in G_1 is straightforward.

For the case of $k_0 = 2$ and $l_2 = 2^{m-1} - 1$, in a very similar way, we will construct a k-PP excluding the two exceptional cases. The k-PP's for the exceptional cases will be obtained by using fault-hamiltonicity of G_0 and G_1. When $k_1 = 1$ and $l_1 = 2^{m-1} - 1$ ($l_3 = 2$), assuming w.l.o.g. $\bar{s_1} \neq s_3$, we let $l'_1 = 1$ and $l'_2 = l_2$, and apply Procedure PP-A. Then, we obtain a desired k-PP. Let $k_1 = 1$ and $l_1 = 2^{m-1}$ ($l_3 = 1$). Unless $\bar{s_1} = s_3$, letting $l'_1 = 1$ and $l'_2 = l_2$, Procedure PP-A is applied. For the exceptional case of (a), we first find 2-PP$[\{(s_1, l'_1, W'_1), (s_2, l'_2, W'_2)\}|G_0]$, where $l'_1 = 2$ and $l'_2 = l_2 - 1$. Let P'_i be the s_i-path in the 2-PP of G_0 and let $x_i = t(P'_i)$. Then, we have $P_2 = (P'_2, \bar{x_2})$. To construct P_1, we show that there exists a vertex $y \notin W_1^1 \cup \{s_3, \bar{x_2}\}$ in G_1 such that $\bar{x_1}$ and y are joined by a hamiltonian path in $G_1 \backslash \{s_3, \bar{x_2}\}$. If $m \geq 6$, the existence of y is obvious since G_1 is 2-fault hamiltonian-connected. If $m = 5$, remembering $|W_1^1| \leq 2$, the existence is due to by Lemma 7. Now, we have $P_1 = (P'_1, H[\bar{x_1}, y|G_1, \{s_3, \bar{x_2}\}])$.

When $k_1 = 2$, we have $l_1 = l_2 = 2^{m-1} - 1$ and $l_3 = l_4 = 1$. Unless $\{\bar{s_1}, \bar{s_2}\} \neq \{s_3, s_4\}$, assuming $\bar{s_1} \neq s_3, s_4$, Procedure PP-A with $l'_1 = 1$ and $l'_2 = l_2$ produces a desired 2-PP. Now, we consider the case of $\{\bar{s_1}, \bar{s_2}\} = \{s_3, s_4\}$ and assume w.l.o.g. $\bar{s_1} = s_3$ and $\bar{s_2} = s_4$. When $m \geq 6$, we first choose a vertex y in G_1 with $y \notin W_2^1 \cup \{s_3, s_4\}$. Let $C_1 = (w_0, w_1, \ldots, w_{2^{m-1}-4})$ be a hamiltonian cycle in $G_1 \backslash \{s_3, s_4, y\}$. For each vertex v in G_0 adjacent to s_1 such that $v \neq s_2, \bar{y}$, we can construct a path P_v of length $2^{m-1} - 1$ in such a way that $P_v = (s_1, v, w_i, w_{(i+1) \bmod 2^{m-1}}, \ldots, w_{(i-1) \bmod 2^{m-1}})$, where $w_i = \bar{v}$. There are at least $m - 3$ such paths P_v. Among them, at least one path have the endvertex $w_{(i-1) \bmod 2^{m-1}} \notin W_1^1$. Note that $|W_1^1| \leq |W_1| \leq m - k = m - 4$. Let P_v be such a path. Then, we have $P_1 = P_v$ and $P_2 = (H[s_2, \bar{y}|G_0, \{s_1, v\}], y)$. When $m = 5$, there exists y in G_1 such that $y \notin W_2^1 \cup \{s_3, s_4\}$ and $G_1 \backslash \{s_3, s_4, y\}$

has a hamiltonian cycle by Lemma 8. Let $C_1 = (w_0, w_1, \ldots, w_{12})$ be a hamiltonian cycle in $G_1 \backslash \{s_3, s_4, y\}$. By Lemma 9, there exist a vertex v in G_0 adjacent to s_1 such that s_2 and $\bar{y}$ are joined by a hamiltonian path in $G_0 \backslash \{s_1, v\}$. Then, we have $P_2 = (H[s_2, \bar{y}|G_0, \{s_1, v\}], y)$. Let $\bar{v} = w_i$. Assuming w.l.o.g. $w_{(i-1) \bmod 16} \notin W_1^1$, we have $P_1 = (s_1, v, w_i, w_{(i+1) \bmod 16}, \ldots, w_{(i-1) \bmod 16})$. Note that $|W_1|, |W_2| \leq 1$. Therefore, we have the lemma. □

Lemma 11. *When $L_0 = 2^{m-1}+1$, Procedure PP-A constructs a k-PP (a) if for some $j \in I_1$, $\bar{s_j} \in S_0$, or (b) if for some $i \in I_0$, either $l_i \geq 3$ and $|W_i^1| < m-k$ or $l_i = 2$ and $\bar{s_i} \notin S_1 \cup W_i^1$. For the remaining cases, there also exist k-PP's.*

Proof. If $\bar{s_j} \in S_0$ for some $j \in I_1$, let s_a be a source in S_0 with $l_a \geq 3$ and let $W'_a = \bar{W}_a^1 \cup \bar{S}_1 \backslash \bar{s_j}$. Otherwise, let $s_a \in S_0$ satisfy the condition (b), and let $W'_a = \bar{W}_a^1 \cup \bar{S}_1$ if $l_a \geq 3$; $W'_a = \emptyset$ if $l_a = 2$. In both cases, let $l'_a = l_a - 1$ and $l''_a = 1$, and let $l'_i = l_i$ for all $i \in I_0 \backslash a$. Then, $W'_i \leq (m-1) - k_0$ for all $i \in I_0$, and thus a k_0-PP exists in G_0. Obviously, a k_1+1-PP exists and we are done. Now, we consider the remaining cases that $\bar{S}_1 \cap S_0 = \emptyset$, $|W_i^1| = m-k$ ($|W_i^0| = 0$) for all $i \in I_0$ with $l_i \geq 3$, and $\bar{s_i} \in W_i^1$ ($|W_i^0| < m-k$) for all $i \in I_0$ with $l_i = 2$. When $k_0 \geq 3$, let $b = k_0+1$. By assumption, $s_b \in S_1$ and $l_b \geq l_j$ for all $j \in I_1$. We find k_0-PP$[\{(s_i, l'_i, W'_i) | i \in I_0\} | G_0]$, where $l'_1 = l_1 - 1$, $W'_1 = \bar{W}_1^1 \cup \bar{S}_1 \backslash \bar{s}_b$, $l'_i = l_i$ and $W'_i = W_i^0$ for all $i \neq 1$. Observe that $l_i = 1$, or $|W'_i| < m-k$ and thus $|W'_i| \leq (m-1) - k_0$ for all $i \in I_0$. Let x_1 be the sink of s_1-path. If $\bar{x_1} \neq s_b$, by finding k_1+1-PP for the sources $\{\bar{x_1}\} \cup S_1$ in G_1, we can obtain a k-PP. Let $\bar{x_1} = s_b$. We claim there exists an edge (u, v) on the s_1-x_1 path $P_1 = (s_1, P'_1, u, v, P''_1, x_1)$ such that $\bar{u}, \bar{v} \notin S_1$ and l_b is greater than the number of vertices on the subpath $(\bar{v}, v, P''_1, x_1, s_b)$. The proof of the claim is omitted. Then, finding k_1+2-PP for sources $\{\bar{u}, \bar{v}\} \cup S_1$ and merging it with the k_0-PP results in a k-PP. When $k_0 = 2$, a k-PP can be constructed by using fault-hamiltonicity of G_0 and G_1. The construction is omitted here. □

3.4 $k_1 = k-1$ ($k_0 = 1$)

Let $\Delta = l_1 - 2^{m-1}$. Then, $L_0 - \Delta = L_1 + \Delta = 2^{m-1}$. We denote by P^R the reverse of a path P, that is, $P^R = (v_l, v_{l-1}, \ldots, v_1)$ for $P = (v_1, v_2, \ldots, v_l)$. A *concatenation* of two paths $(x_1, \ldots, x_p)$ and $(y_1, \ldots, y_q)$ is the path $(x_1, \ldots, x_p, y_1, \ldots, y_q)$.

Case 1 for all $j \in I_1$, $l_j = 1$ or $l_j \geq 2$ and $|W_j^1| < m-k$.

When $k < m-1$, we let α be a vertex in G_1 with $\bar{\alpha} \neq s_1$ and $\alpha \notin S_1 \cup W_1^1$ if $|W_1^1| < m-k$ or $\Delta = 1$; if $|W_1^1| = m-k$ and $\Delta \geq 2$, let α be a vertex in W_1^1 with $\bar{\alpha} \neq s_1$ and $\alpha \notin S_1$. We find k-PP$[\{(\alpha, \Delta, W'_1)\} \cup \{(s_j, l_j, W_j^1) | j \in I_1\} | G_1]$, where $W'_1 = W_1^1 \backslash \alpha$. And then, the k-PP in G_1 and $H[s_1, \bar{\alpha}|G_0, \emptyset]$ are merged with $(\bar{\alpha}, \alpha)$. Now, let $k = m-1$. Notice that for all $j \in I_1$, $l_j = 1$ or $l_j \geq 2$ and $W_j^1 = \emptyset$. Letting $l'_2 = l_2 + \Delta$, we find k_1-PP$[\{(s_2, l'_2, W_1^1)\} \cup \{(s_j, l_j, W_j^1) | j \in I_1 \backslash 2\} | G_1]$. Let the s_2-path in the k_1-PP be $(v_1, v_2, \ldots, v_{l'_2})$ with $v_1 = s_2$, and let $x = v_{l_2+1}$ and $y = v_{l'_2}$. To obtain a k-PP, it suffices to construct P_1 and P_2. Let $P_2 = (v_1, \ldots, v_{l_2})$. If $\bar{x} \neq s_1$, we have $P_1 = (H[s_1, \bar{x}|G_0, \emptyset], v_{l_2+1}, \ldots, v_{l'_2})$. Assume $\bar{x} = s_1$. If $\Delta \geq 2$, we have $P_1 = (s_1, v_{l_2+1}, \ldots, v_{l'_2}, H[\bar{y}, z|G_0, \{s_1\}])$ for some vertex z

in G_0 with $z \notin \{s_1, \bar{y}\} \cup W_1^0$. If $\Delta = 1$, we observe $l_2 \geq 5$. Letting $u = v_{l_2-2}$ and $v = v_{l_2-1}$, we find 3-PP$[\{(\bar{u}, 2, W_2^0), (\bar{v}, 2^{m-1} - 3, W_1^0), (s_1, 1, \emptyset)\}|G_0]$. Then, P_2 is a concatenation of $(v_1, \ldots, v_{l_2-2})$ and the $\bar{u}$-path, and P_1 is a concatenation of $(s_1, v_{l_2'}, \ldots, v_{l_2}, v_{l_2-1})$ and the $\bar{v}$-path.

Case 2 for some $j \in I_1$, $l_j \geq 2$ and $|W_j^1| = m - k$.
Let $a \in I_1$ be an index such that $l_a \geq 2$, $|W_a^1| = m - k$ ($|W_a^0| = 0$), and $l_a \geq l_j$ for any $j \in I_1$ with $l_j \geq 2$ and $|W_j^1| = m - k$. Furthermore, we assume $\bar{s_a} \neq s_1$ if $l_a = 2$ and for some $j \in I_1$ with $j \neq a$, $l_j \geq 2$ and $|W_j^1| = m - k$. We first find k_1-PP$[\{(s_a, l_a', W_1^1)\} \cup \{(s_j, l_j, W_j^1)|j \in I_1 \backslash a\}|G_1]$, where $l_a' = l_a + \Delta$. Let the s_a-path in the k_1-PP be $(v_1, v_2, \ldots, v_{l_a'})$. Let $z = v_{l_a}$, $x = v_{l_a+1}$, $y = v_{l_a'}$, and $v = v_{l_a-1}$. And let $u = v_{l_a-2}$ if $l_a \geq 3$. We denote by Q_z the s_a-z subpath $(v_1, \ldots, v_{l_a})$ of the s_a-path. Let $R_z = (v_{l_a+1}, \ldots, v_{l_a'})$ so that the s_a-path is a concatenation (Q_z, R_z) of Q_z and R_z. Similarly, let $Q_v = (v_1, \ldots, v_{l_a-1})$ and $R_v = (v_{l_a}, \ldots, v_{l_a'})$, etc. To obtain a k-PP, it remains to construct P_a and P_1. When $z \notin W_a^1$, we have $P_a = Q_z$ and $P_1 = (H[s_1, \bar{x}|G_0, \emptyset], R_z)$ if $\bar{x} \neq s_1$; if $\bar{x} = s_1$, $P_a = (Q_v, \bar{v})$ and $P_1 = (H[s_1, \bar{z}|G_0, \{\bar{v}\}], R_v)$.

Let $z \in W_a^1$. When $\bar{v} \neq s_1$, let $P_a = (Q_v, \bar{v})$. If $\bar{z} \neq s_1$, $P_1 = (H[s_1, \bar{z}|G_0, \{\bar{v}\}], R_v)$; otherwise, $P_1 = (s_1, R_v, H[\bar{y}, w|G_0, \{s_1, \bar{v}\}])$ for some w in G_0 with $w \notin \{s_1, \bar{v}, \bar{y}\} \cup W_1^0$. The existence of $\bar{y}$-w hamiltonian path is due to fault-hamiltonicity of G_0 for $m \geq 6$ and Lemma 7 for $m = 5$. When $\bar{v} = s_1$ and $l_a \geq 3$, we have two constructions. If $k \geq 4$ or $k = 3$ and $|W_1^0| < m - k$, we find 3-PP$[\{(\bar{u}, 2, \emptyset), (s_1, 1, \emptyset), (\bar{y}, 2^{m-1} - 3, W_1^0)\}|G_0]$. Then, P_a is a concatenation of Q_u and the $\bar{u}$-path, and P_1 is a concatenation of (s_1, R_u) and the $\bar{y}$-path. If $k = 3$ and $|W_1^0| = m - k$, letting $w \notin \{s_1, \bar{y}\}$ be a vertex in G_0 adjacent to $\bar{u}$, we have $P_a = (Q_u, \bar{u}, w)$ and $P_1 = (H[s_1, \bar{y}|G_0, \{\bar{u}, w\}], R_u^R)$. The existence of the s_1-$\bar{y}$ hamiltonian path in G_0 is due to fault-hamiltonicity of G_0 and Lemma 9.

When $\bar{v} = s_1$ and $l_a = 2$ ($s_a = v$, $\bar{s_a} = s_1$), by the choice of s_a, for every $j \in I_1 \backslash a$, $l_j = 1$ or $l_j \geq 2$ and $|W_j^1| < m - k$. Let $k < m - 1$ first. If $W_a^1 \backslash W_1^1 \neq \emptyset$, let t_1 be a vertex in $W_a^1 \backslash W_1^1$ and let $W_a' = W_a^1 \backslash t_1$ and $W_1' = \emptyset$; otherwise ($W_a^1 = W_1^1$), let t_1 be a vertex in W_a^1 and let $W_a' = W_a^1 \backslash t_1$ and $W_1' = W_1^1 \backslash t_1$. Find $k_1 + 1$-PP$[\{(t_1, \Delta, W_1'), (s_a, l_a, W_a')\} \cup \{(s_j, l_j, W_j^1)|j \in I_1 \backslash a\}|G_1]$. If $W_a^1 \backslash W_1^1 \neq \emptyset$, P_1 is obtained by concatenating $H[s_1, \bar{w}|G_0, \emptyset]$ and the reverse of t_1-path in $k_1 + 1$-PP, where w is the sink of t_1-path. If $W_a^1 = W_1^1$ and $\Delta \geq 2$, P_1 is a concatenation of $H[s_1, \bar{t_1}|G_0, \emptyset]$ and the t_1-path in the $k_1 + 1$-PP.

If $W_a^1 = W_1^1$ and $\Delta = 1$, we have two constructions. For $m \geq 6$, we let b be an index in I_1 such that $l_b \geq l_j$ for any $j \in I_1 \backslash a$. Obviously, $l_b \geq 5$. Letting $l_b' = l_b + 1$, we find k_1-PP$[\{(s_b, l_b', W_1^1)\} \cup \{(s_j, l_j, W_j^1)|j \in I_1 \backslash b\}|G_1]$. Let the s_b-path in the k_1-PP be $(w_1, \ldots, w_{l_b}, w_{l_b+1})$, and let $p = w_{l_b-2}$ and $q = w_{l_b-1}$. There exists a vertex w in G_0 adjacent to $\bar{p}$ with $w \notin \{s_1, \bar{q}\} \cup W_b^0$. Then, we have $P_b = (w_1, \ldots, w_{l_b-2}, \bar{p}, w)$ and $P_1 = (H[s_1, \bar{q}|G_0, \{\bar{p}, w\}], w_{l_b-1}, w_{l_b}, w_{l_b+1})$. For $m = 5$ ($k = 3$), let t_a be a vertex in G_1 adjacent to s_a with $t_a \notin \{s_b\} \cup W_a^1$. Let $P_a = (s_a, t_a)$. By Lemma 8, there exists a vertex w in G_1 with $w \notin \{s_a, t_a, s_b\} \cup W_1^1$ such that $G_1 \backslash \{s_a, t_a, w\}$ has a hamiltonian cycle C. Then, $P_1 = (H[s_1, \bar{w}|G_0, \emptyset], w)$. P_b is obtained by removing one of the two edges on C incident to s_b since $|W_b^1| < m - k = 1$. Finally, let $k = m - 1$. Note that $W_j^1 = \emptyset$ for any $j \in I_1 \backslash a$.

Letting $l'_b = l_b + \varDelta$, we find k_1-PP$[\{(s_b, l'_b, W_1^1)\} \cup \{(s_j, l_j, W_j^1) | j \in I_1 \backslash b\} | G_1]$. Let the s_b-path in the k_1-PP be $(w_1, \ldots, w_{l'_b})$ and let $x = w_{l_b+1}$. Then, we have $P_b = (w_1, \ldots, w_{l_b})$ and $P_1 = (H[s_1, \bar{x} | G_0, \emptyset], w_{l_b+1}, \ldots, w_{l'_b})$.

3.5 $k_1 = 0$ ($k_0 = k$)

Case 1 $k \leq m - 2$.
Let us first consider the case that there exists $i \in I_0$ such that $l_i = 1$ or $l_i \geq 2$ and $|W_i^0| < m - k$. We are to define l'_i and l''_i, $i \in I_0$, satisfying (i) $l'_i + l''_i = l_i$, $1 \leq l'_i \leq l_i$, (ii) $\sum_{i \in I_0} l'_i = 2^{m-1}$, (iii) $l'_i = l_i$ for some $i \in I_0$ such that $l_i = 1$ or $l_i \geq 2$ and $|W_i^0| < m - k$, (iv) $l'_i < l_i$ for all i with $l_i \geq 2$ and $|W_i^0| = m - k$, and (v) $l'_i = l_i$ for all i with $l_i = 2$ and $|W_i^0| < m - k$, and either $l'_i = l_i$ or $l'_i \leq l_i - 2$ for all i with $l_i \geq 3$ and $|W_i^0| < m - k$. It is not difficult to see that there exist l'_i's and l''_i's satisfying all the above five conditions unless (a) $k = 3$, $l_1 = l_2 = 2^{m-1} - 1$, $l_3 = 2$, $|W_1^0|, |W_2^0| < m - k$, and $|W_3^0| = m - k$, or (b) for some $p \in I_0$, $l_p \geq 2^{m-1} - (k - 2)$, $|W_p^0| < m - k$, $l_i \geq 2$ and $|W_i^0| = m - k$ for any $i \in I_0 \backslash p$.

We let $W'_i = \emptyset$ if $l'_i = 1$; $W'_i = W_i^0$ if $2 \leq l'_i = l_i$; $W'_i = \emptyset$ if $2 \leq l'_i < l_i$. Then, $|W'_i| < m - k$ for every i. There exists a k_0-PP$[\{(s_i, l'_i, W'_i) | i \in I_0\} | G_0]$. Letting $I'_0 = \{i \in I_0 | l'_i < l_i\}$ and $k'_0 = |I'_0|$, we find k'_0-PP$[\{(\bar{x_j}, l''_j, W_j^1) | j \in I'_0\} | G_1]$, where x_j is the sink of s_j-path in the k_0-PP. Merging the two PP's results in a desired k-PP. We omit the constructions of k-PP's using fault-hamiltonicity of G_0 and G_1 for the two exceptional cases (a) and (b).Now, we assume that for all $i \in I_0$, $l_i \geq 2$ and $|W_i^0| = m - k$ ($|W_i^1| = 0$). There exist l'_i's and l''_i's satisfying (i) $l'_i + l''_i = l_i$, $1 \leq l'_i < l_i$ and (ii) $\sum_{i \in I_0} l'_i = 2^{m-1}$. We find a k_0-PP$[\{(s_i, l'_i, \emptyset) | i \in I_0\} | G_0]$. And then, letting x_i be the sink of s_i-path in the k_0-PP in G_0, we find k_0-PP$[\{(\bar{x_i}, l''_i, \emptyset) | i \in I_0\} | G_1]$ and merge the two PP's.

Case 2 $k = m - 1$.
Recall the assumption $l_1 \geq l_2 \geq \cdots \geq l_k$. We first consider the case that $(k - 3) + l_{k-1} + l_k \leq 2^{m-1}$ and $l_1 \leq 2^{m-1}$. There are l'_i and l''_i, $i \in I_0 \backslash 1$, satisfying (i) $l'_i + l''_i = l_i$, $1 \leq l'_i \leq l_i$, (ii) $\sum_{i \in I_0 \backslash 1} l'_i = 2^{m-1}$, and (iii) $l'_{k-1} = l_{k-1}$ and $l'_k = l_k$. We let $W'_i = W_i^0 \cup \{s_1\}$ if $l'_i = l_i$ and $|W_i^0| < m - k$; $W'_i = \bar{W}_i^1$ if $l'_i = l_i - 1$; $W'_i = W_i^0$, otherwise. Regarding s_1 as a non-source vertex virtually, we find $k_0 - 1$-PP$[\{(s_i, l'_i, W'_i) | i \in I_0 \backslash 1\} | G_0]$. Let the s_a-path in the $k_0 - 1$-PP passes through s_1, that is, the s_a-path be $(v_1, \ldots, v_{l'_j})$ with $v_{i+1} = s_1$ for some i. Let $I'_0 = \{i \in I_0 | l'_i < l_i\} \cup \{1, a\}$ and $k'_0 = |I'_0|$. Clearly, $k'_0 < k_0$. Letting x_i be the sink of s_i-path in the $k_0 - 1$-PP, we find k'_0-PP$[\{(\bar{x_a}, l_1 - l'_j + i, W_1^1), (\bar{v}_i, l_a - i, W_a^1)\} \cup \{(\bar{x}_i, l''_i, W_i^1) | i \in I'_0 \backslash \{1, a\}\} | G_1]$. To obtain a k-PP, the two PP's are merged. For the case that $(k - 3) + l_{k-1} + l_k \leq 2^{m-1}$ and $l_1 > 2^{m-1}$, regarding s_2 as a virtual non-source vertex, we can construct a k-PP in a similar to the previous case. For any $k = m - 1 \geq 4$, it holds true that $(k-3) + l_{k-1} + l_k \leq 2^{m-1}$ unless $k = 4$ and $l_1 = l_2 = l_3 = l_4 = 2^{m-2}$. For the last subcase, a k-PP can be obtained using fault-hamiltonicity of G_0 and G_1. The construction is also omitted here.

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Appendix

Recursive circulant $G(8,4)$ is a graph defined as follows: vertex set is $\{v_i | 0 \le i \le 7\}$ and the edge set is $\{(v_i, v_j) | i+1 \text{ or } i+4 \equiv j \pmod 8\}$. $G(8,4)$ is isomorphic to twisted cube TQ_3 and Möbius ladder with four spokes as shown in Figure 3.

To prove strong path partitionability of $G(8,4) \oplus G(8,4)$ in Lemma 6 and fault-hamiltonicity of $G(8,4) \oplus G(8,4)$ in Lemmas 7, 8, and 9, several properties on paths and path partitions of $G(8,4)$ will be utilized. Thus, they are considered first.

Lemma 12. *For any pair of vertices s and t in $G(8,4)$, there exist two hamiltonian paths, P_1 and P_2, from s to t such that the i-th vertex of P_1 and the i-th vertex of P_2 are different for each $2 \le i \le 7$.*

Proof. We assume w.l.o.g. $s = v_0$. For each t, we can construct P_1 and P_2.
Case $t = v_1$: $P_1 = (v_0, v_4, v_5, v_6, v_7, v_3, v_2, v_1)$; $P_2 = (v_0, v_7, v_6, v_2, v_3, v_4, v_5, v_1)$.
Case $t = v_2$: $P_1 = (v_0, v_1, v_5, v_4, v_3, v_7, v_6, v_2)$; $P_2 = (v_0, v_4, v_3, v_7, v_6, v_5, v_1, v_2)$.
Case $t = v_3$: $P_1 = (v_0, v_4, v_5, v_1, v_2, v_6, v_7, v_3)$; $P_2 = (v_0, v_7, v_6, v_2, v_1, v_5, v_4, v_3)$.
Case $t = v_4$: $P_1 = (v_0, v_1, v_2, v_3, v_7, v_6, v_5, v_4)$; $P_2 = (v_0, v_7, v_6, v_5, v_1, v_2, v_3, v_4)$. □

Lemma 13. *Let G be $G(8,4)$ with one faulty element.*
(a) When the fault set $F = \{(x,y)\}$, there exists a hamiltonian path in $G \backslash F$ between every pair of vertices $s \in \{x,y\}$ and $t (\ne s)$.
(b) Let $F = \{v_0\}$. There exists a hamiltonian path in $G \backslash F$ between every pair of vertices $s = v_4$ and $t (\ne s)$. For each $s = v_1$, v_2, v_3, there is a hamiltonian path in $G \backslash F$ between s and $t \in \{v_2, v_4, v_5, v_7\}$, $t \in \{v_1, v_4, v_6\}$, $t \in \{v_4, v_7\}$, respectively.

Proof. The proof is enumerative and omitted here. □

In a graph, a path of length shorter by one than a hamiltonian path is called a *hypohamiltonian path*.

Lemma 14. *[11] Let G be $G(8,4)$ with one faulty element.*
(a) When the faulty set $F = \{v_0\}$, there exists a hypohamiltonian path in $G \backslash F$ between every pair of vertices s and t provided $\{s,t\} \ne \{v_2, v_6\}$, $\{v_3, v_4\}$, $\{v_4, v_5\}$.
(b) When $F = \{(v_0, v_4)\}$, there exists a hypohamiltonian path in $G \backslash F$ between every pair of vertices s and t.
(c) When $F = \{(v_7, v_0)\}$, there exists a hypohamiltonian path in $G \backslash F$ between every pair of vertices s and t provided $\{s,t\} \ne \{v_1, v_6\}$, $\{v_2, v_5\}$.

Two paths joining $\{s_1, s_2\}$ and $\{t_1, t_2\}$ such that $\{s_1, s_2\} \cap \{t_1, t_2\} = \emptyset$ are defined to be either s_1-t_1 and s_2-t_2 paths or s_1-t_2 and s_2-t_1 paths. Two paths P_1 and P_2 in a graph G are called *disjoint covering paths* if $V(P_1) \cap V(P_2) = \emptyset$ and $V(P_1) \cup V(P_2) = V(G)$, where $V(P_i)$ is the set of vertices in P_i.

Lemma 15. *[11] Let P_1 and P_2 be two disjoint covering paths joining $\{s_1, s_2\}$ and $\{t_1, t_2\}$ in $G(8,4)$ such that $\{s_1, s_2\} \cap \{t_1, t_2\} = \emptyset$.*

(a) When $\{s_1, s_2\} = \{v_0, v_1\}$*, they exist unless* $\{t_1, t_2\} = \{v_3, v_6\}$*.*
(b) When $\{s_1, s_2\} = \{v_0, v_2\}$*, they exist unless* $\{t_1, t_2\} = \{v_3, v_5\}, \{v_5, v_7\}$*.*
(c) When $\{s_1, s_2\} = \{v_0, v_3\}$*, they exist unless* $\{t_1, t_2\} = \{v_1, v_6\}, \{v_2, v_5\}, \{v_5, v_6\}$*.*
(d) When $\{s_1, s_2\} = \{v_0, v_4\}$*, they exist unless* $\{t_1, t_2\} = \{v_2, v_6\}$*.*

The following two lemmas are about 2-path partitions in $G(8, 4)$. Note that Lemma 16 does not say that $G(8, 4)$ is strongly 2-path partitionable. However, for almost all the cases there exist 2-PP's as follows.

Lemma 16. *For* $|W_1|, |W_2| \leq 1$*, there exists* 2-*PP*$[\{(s_1, l_1, W_1), (s_2, l_2, W_2)\}|G(8, 4)]$ *with the unique exception up to symmetry that* $s_1 = v_0$*,* $s_2 = v_4$*,* $l_1 = l_2 = 4$*,* $W_1 = \{v_3\}$*, and* $W_2 = \{v_1\}$*.*

Proof. The proof is by a case analysis and omitted here. □

Lemma 17.
(a) For any W_1 *with* $|W_1| \leq 3$*, there exist* 2-*PP*$[\{(s_1, 6, W_1), (s_2, 2, \emptyset)\}|G(8, 4)]$ *and* 2-*PP*$[\{(s_1, 5, W_1), (s_2, 3, \emptyset)\}|G(8, 4)]$*.*
(b) For any W_1 *with* $|W_1| \leq 2$*, there exists* 2-*PP*$[\{(s_1, 3, W_1), (s_2, 5, \emptyset)\}|G(8, 4)]$*.*

Proof. The proof is by an immediate inspection. □

Hereafter in this Appendix, when we are concerned with $G(8, 4) \oplus G(8, 4)$, we let G_0 be one copy of $G(8, 4)$ and G_1 be the other copy. Let the vertex set of G_0 and G_1 be $\{v_i|0 \leq i \leq 7\}$ and $\{w_i|0 \leq i \leq 7\}$, respectively. Here, the edge sets of G_0 and G_1 are $\{(v_i, v_j)|i + 1 \text{ or } i + 4 \equiv j \pmod 8\}$ and $\{(w_i, w_j)|i + 1 \text{ or } i + 4 \equiv j \pmod 8\}$, respectively.

A Proof of Lemma 6

Lemma 6 will be proved in Lemmas 18 and 19.

Lemma 18. $G(8, 4) \oplus G(8, 4)$ *is strongly* 2*-path partitionable.*

Proof. Without loss of generality, it is assumed that s_1 is in G_0 and $l_1 \geq l_2$. Remember $|W_i| \leq 2$. When $l_2 = 1$, the lemma holds true since $G(8, 4) \oplus G(8, 4)$ is 1−fault hamiltonian-connected. When $l_2 = 2$, letting $x \notin W_2 \cup \{s_1\}$ be a vertex adjacent to s_2, we have s_2-path (s_2, x). Regarding s_2 and x as virtual faults, an s_1-path can be found by Lemma 7. From now on, we assume $l_2 \geq 3$.

Case 1: $s_1 \in V(G_0)$ and $s_2 \in V(G_1)$.
We assume w.l.o.g. $s_1 = v_0$ and $s_2 = w_0$. If $l_1 = l_2$, the lemma holds true from the fact that each G_i is hamiltonian-connected. It is assumed that $l_1 > l_2$.

Subcase 1.1: $|W_2^1| \leq 1$.
Unless $l_1 = 12$ and $W_1 = \{\bar{s_1}, w_4\}$, we claim that for some $x \in V(G_1)$ with $x \neq s_2$ and $\bar{x} \neq s_1$, $H[s_1, \bar{x}|G_0, \emptyset]$ and 2-PP$[\{(x, l_1 - 8, W_1^1 \backslash x), (s_2, l_2, W_2^1)\}|G_1]$ exist and they can be merged with $(x, \bar{x})$ into a 2-PP in $G(8, 4) \oplus G(8, 4)$. To show the claim, by Lemma 16, it suffices to choose x in G_1 satisfying (i) if $l_1 = 9$,

then $x \notin W_1$, (ii) if $l_1 \geq 10$ and $|W_1^1| \leq 1$, then $x \neq w_4$, and (iii) if $l_1 \geq 10$ and $|W_1^1| = 2$, then $x \in W_1^1$, and moreover, if $l_1 = 12$, then $x \neq w_4$. For the case $l_1 = 12$ and $W_1 = \{\bar{s_1}, w_4\}$, we find 2-PP$[\{(\bar{s_1}, 4, \emptyset), (s_2, 4, W_2^1)\}|G_1]$, and letting y be the sink of $\bar{s_1}$-path, find 2-PP$[\{(\bar{y}, 7, \emptyset), (s_1, 1, \emptyset)\}|G_0]$. They are merged into a 2-PP in $G(8,4) \oplus G(8,4)$. Note $\bar{s_1} \neq s_2$ since $\bar{s_1}$ is contained in W_1.

Subcase 1.2: $|W_2^1| = 2$.

If $\bar{s_1} \neq s_2$, find two hamiltonian paths P and Q from s_1 to $\bar{s_2}$ in G_0 such that i-th vertex of P and i-th vertex of Q are different for each $2 \leq i \leq 7$. The existence of such P and Q follows from Lemma 12. First, P is partitioned into an s_1-path P_1' and an $\bar{s_2}$-path P_2' such that the length of P_2' is $l_2 - 1$. An s_2-path of length l_2 is constructed by connecting s_2 and P_2' with the edge $(s_2, \bar{s_2})$. Let x be the sink of s_1-path P_1'. From P_1', an edge $(x, \bar{x})$, and a hamiltonian cycle C in $G_1 \backslash s_2$, we have two s_1-paths of length l_1 whose sinks are different. When the sinks of these s_1-paths are all in W_1, it is easy to see that s_1-path P_1 and s_2-path P_2 are constructed from Q instead of P in such a way that the sinks of P_1 and P_2 are not in W_1 and W_2, respectively.

If $\bar{s_1} = s_2$, let $x = w_4$. Observe that there is a hamiltonian cycle C of length six in $G_1 \backslash \{s_2, x\}$. Find two hamiltonian paths P and Q from s_1 to $\bar{x}$ in G_0 such that i-th vertex of P and i-th vertex of Q are different for each $2 \leq i \leq 7$. An s_1-path of length l_1 whose sink is not in W_1 and an s_2-path of length l_2 whose sink is not in W_2 can be constructed from P, Q, and C as in the previous case $\bar{s_1} \neq s_2$. In this case, s_2-path contains x as the vertex which follows s_2.

Case 2: $s_1, s_2 \in V(G_0)$.

For the case $|W_2^1| = 0$, we find 2-PP$[\{(s_1, 10-l_2, \emptyset), (s_2, l_2-2, \emptyset)\}|G_0]$, and letting x and y be sinks of s_1-path and s_2-path, respectively, find 2-PP$[\{(\bar{x}, 6, W_1^1), (\bar{y}, 2, \emptyset)\}|G_1]$. The 2-PP in G_1 exists by Lemma 17. The two partitions are merged into a 2-PP in $G(8,4) \oplus G(8,4)$. For the second case $|W_2^1| \geq 1$ and $l_2 \leq 7$, we can construct a 2-PP from 2-PP$[\{(s_1, 8 - l_2, \emptyset), (s_2, l_2, W_2^0)\}|G_0]$ and $H[\bar{x}, y|G_1, \emptyset]$ for some $y \in V(G_1) \backslash W_1^1$, where x is the sink of s_1-path. From now on, we assume $|W_2^1| \geq 1$ and $l_1 = l_2 = 8$. We are to construct s_1-path and s_2-path whose sinks are contained in G_1. Thus, we can assume $|W_1^1|, |W_2^1| = 2$.

Subcase 2.1: $W_1^1 \cap W_2^1 \neq \emptyset$.

Let $\alpha \in W_1^1 \cap W_2^1$. When $\bar{\alpha} = s_1$ (or symmetrically $\bar{\alpha} = s_2$), there is a hamiltonian cycle $C = (s_2, x, \dots, y)$ in $G_0 \backslash s_1$. Letting $\bar{y} \notin W_2^1$, we have an s_2-path $(C \backslash (s_2, y), \bar{y})$. An s_1 path is $(s_1, H[\alpha, z|G_1, \{\bar{y}\}])$ for some $z \notin W_1^1$. Now, $\bar{\alpha} \neq s_1, s_2$. There exists an s_1-s_2 hamiltonian path $P = (s_1, u_1, u_2, u_3, u_4, u_5, u_6, s_2)$ in G_0. When $\bar{\alpha} = u_1$, two path segments (s_1, u_1) and $(s_2, u_6, \dots, u_2)$ of P are merged with 2-PP$[\{(\bar{u_1}, 6, W_1^1 \backslash \alpha), (\bar{u_2}, 2, W_2^1 \backslash \alpha)\}|G_1]$ and we are done. When $\bar{\alpha} = u_2$ or $\bar{\alpha} = u_3$, similarly, (s_1, u_1, u_2) and $(s_2, u_6, \dots, u_3)$ of P are merged with 2-PP$[\{(\bar{u_2}, 5, W_1^1 \backslash \alpha), (\bar{u_3}, 3, W_2^1 \backslash \alpha)\}|G_1]$.

Subcase 2.2: $W_1^1 \cap W_2^1 = \emptyset$ and $\bar{s_1} \in W_1^1$.

If there exists an s_2-t_2 hamiltonian path P in $G_0 \backslash s_1$ for some t_2 with $\bar{t_2} \notin W_2^1$, then we have an s_2-path $(P, \bar{t_2})$ and an s_1-path $(s_1, H[\bar{s_1}, z|G_1, \{\bar{t_2}\}])$ for some $z \notin W_1^1$. If there does not exist such a hamiltonian path in $G_0 \backslash s_1$, by Lemma 13, we can assume w.l.o.g. $s_1 = v_0$, $s_2 = v_3$ and $W_2^1 = \{\bar{v_4}, \bar{v_7}\}$. In this case, two paths (s_1, v_4) and $(s_2, v_2, v_1, v_5, v_6, v_7)$ in G_0 are merged with 2-PP$[\{(\bar{v_4}, 6, W_1^1), (\bar{v_7}, 2, \emptyset)\}|G_1]$. The 2-PP exists by Lemma 17.

Subcase 2.3: $W_1^1 \cap W_2^1 = \emptyset$ and $\bar{s_1} \in W_2^1$.

If there exists a vertex $t_2 \in V(G_0)$ such that (i) $P_0 = H[s_2, t_2|G_0, \{s_1\}]$ exists, (ii) $\bar{t_2} \notin W_2^1$, and (iii) $P_1 = H[\bar{s_1}, z|G_1, \{\bar{t_2}\}]$ exists for some $z \notin W_1^1$. Then, we have an s_1-path (s_1, P_1) and s_2-path $(P_0, \bar{t_2})$. If no such t_2 exists, by Lemma 13, we can assume w.l.o.g. that $s_1 = v_0$, $s_2 = v_3$, $\bar{s_1} = w_0$. Furthermore, letting x be a vertex in $\{v_4, v_7\}$ with $\bar{x} \notin W_2^1$, we have only one case up to symmetry that $\bar{x} = w_3$ and $W_1^1 = \{w_4, w_7\}$. Let $y = \{v_4, v_7\}\backslash x$. Remember $\bar{y} \in W_2^1$. We claim that in G_1, there exist an $\bar{x}$-path Q_x and $\bar{y}$-path Q_y of length 6 and 2, respectively, whose sinks are not contained in $W_1^1 \cup W_2^1$. If $\bar{y} = w_1$, then we have $Q_x = (w_3, w_7, w_0, w_4, w_5, w_6)$ and $Q_y = (w_1, w_2)$; if $\bar{y} = w_2$, $Q_x = (w_3, w_7, w_0, w_4, w_5, w_1)$, $Q_y = (w_2, w_6)$; if $\bar{y} = w_5$, $Q_x = (w_3, w_4, w_0, w_7, w_6, w_2)$, $Q_y = (w_5, w_1)$; if $\bar{y} = w_6$, $Q_x = (w_3, w_7, w_0, w_4, w_5, w_1)$, $Q_y = (w_6, w_2)$. The paths Q_x and Q_y are merged, depending on whether x is v_4 or v_7, either with $P_x = (s_1, v_4)$ and $P_y = (s_2, v_2, v_1, v_5, v_6, v_7)$ or $P_x = (s_2, v_7)$ and $P_y = (s_1, v_1, v_2, v_6, v_5, v_4)$. Thus, we have a 2-PP.

Subcase 2.4: $W_1^1 \cap W_2^1 = \emptyset$ and $\bar{s_1}, \bar{s_2} \notin W_1^1 \cup W_2^1$.

We let $W_1^1 \cup W_2^1 = \{z_1, z_2, z_3, z_4\}$ and let a, b be the two vertices in G_1 with $a, b \notin W_1^1 \cup W_2^1$ and $a, b \notin \{\bar{s_1}, \bar{s_2}\}$, so that $V(G_1) = \{z_1, z_2, z_3, z_4, a, b, \bar{s_1}, \bar{s_2}\}$. We claim that for some z_p and z_q, $p \neq q$, there exist two disjoint covering paths in G_0 joining $\{s_1, s_2\}$ and $\{\bar{z_p}, \bar{z_q}\}$ of lengths 5 and 3, with the unique exception up to symmetry that $s_1 = v_0$, $s_2 = v_1$, $\{\bar{a}, \bar{b}\} = \{v_3, v_6\}$. If the s_1-path is of length 5, then the s_2-path is of length 3, and vice versa. The proof of the claim is by an immediate inspection and omitted here. Then, the two disjoint covering paths in G_0 are merged with 2-PP$[\{(\bar{z_p}, l_p, W_p), (\bar{z_q}, l_q, W_q)\}|G_1]$. Here, if the s_1-path in G_0 is s_1-z_p path of length 3, then $l_p = 5$, $W_p = W_1^1\backslash\{\bar{z_p}, \bar{z_q}\}$, $l_q = 3$, and $W_q = W_2^1\backslash\{\bar{z_p}, \bar{z_q}\}$, and for the other three cases l_p, l_q, W_p, and W_q can be defined in a similar way. The 2-PP in G_1 exists due to Lemmas 16 and 17. For the exceptional case $s_1 = v_0$, $s_2 = v_1$, $\{\bar{a}, \bar{b}\} = \{v_3, v_6\}$, we let $\bar{v_4} \in W_i^1$, $i = 1, 2$, and let s_1-path be (s_1, v_4). At least one of $\bar{v_5}$ and $\bar{v_7}$ are contained in W_j^1, $j \neq i$. If $v_5 \in W_j^1$, we let s_2-path be $(s_2, v_2, v_3, v_7, v_6, v_5)$; otherwise, let s_2-path be $(s_2, v_5, v_6, v_2, v_3, v_7)$. Then, letting t_2 be the sink of s_2-path, the s_1-path and s_2-path in G_0 are merged with 2-PP$[\{(\bar{v_4}, 6, W_1^1\backslash\{\bar{v_4}, \bar{t_2}\}), (\bar{t_2}, 2, W_2^1\backslash\{\bar{v_4}, \bar{t_2}\})\}|G_1]$. By Lemma 16, the 2-PP in G_1 exists. □

Lemma 19. *$G(8, 4) \oplus G(8, 4)$ is strongly 3-path partitionable.*

Proof. We have three sources s_1, s_2, and s_3 with $|W_i| \leq 1$ for each i.

Case 1: $L_0 = L_1$.

Assume $s_1, s_2 \in V(G_0)$ and $s_3 \in V(G_1)$. We have $l_1 + l_2 = 8$ and $l_3 = 8$. If there exists 2-PP$[\{(s_1, l_1, W_1^0), (s_2, l_2, W_2^0)\}|G_0]$, then the 2-PP and $H[s_3, z|G_1, \emptyset]$ for some $z \notin W_3^1$ constitute a 3-PP in $G_0 \oplus G_1$. Suppose no such 2-PP in G_0 exists. Then, we assume w.l.o.g. $s_1 = v_0$, $s_2 = v_4$, $l_1 = l_2 = 4$, $W_1 = \{v_3\}$, and $W_2 = \{v_1\}$. We are to construct a 3-PP such that the sinks of s_1-path and s_2-path are not contained in $\{v_1, v_3, v_5, v_7\}$. We let $s_3 = w_0$.

When $\bar{w_4} \in \{v_1, v_3, v_5, v_7\}$, we assume w.l.o.g. $\bar{w_4} = v_3$. If $v_1 \neq \bar{s_3}$ and $v_1 \notin W_3$, we have s_1-path (s_1, v_7, v_3, w_4), s_2-path (s_2, v_5, v_6, v_2), and s_3-path $(H[s_3, \bar{v_1}|G_1, \{w_4\}], v_1)$. Similarly if $v_6 \neq \bar{s_3}$ and $v_6 \notin W_3$, we have s_1-path (s_1, v_7, v_3, w_4), s_2-path (s_2, v_5, v_1, v_2), and s_3-path $(H[s_3, \bar{v_6}|G_1, \{w_4\}], v_6)$. If $v_6 = \bar{s_3}$ and $W_3 = \{v_1\}$, we have s_2-path (s_2, v_5, v_1, v_2). We can obtain s_1-path and s_3-path from s_3-s_1 path $(s_3, v_6, v_7, v_3, H[w_4, \bar{s_1}|G_1, \{s_3\}], s_1)$ by removing an appropriate edge on the path. If $v_1 = \bar{s_3}$ and $W_3 = \{v_6\}$, we have s_1-path (s_1, v_7, v_6, v_2). There exists 3-PP$[\{(s_3, 1, \emptyset), (\bar{v_5}, 5, \emptyset), (w_4, 2, \emptyset)\}|G_1]$ by Lemma 5. Letting Q_1 and Q_2 be $\bar{v_5}$-path and w_4-path in the 3-PP, respectively, we have s_3-path (s_3, v_1, v_5, Q_1) and s_2-path (s_2, v_3, Q_2).

When $\bar{w_4} \in \{v_2, v_6\}$, we assume w.l.o.g. $\bar{w_4} = v_2$. If $v_5 \neq \bar{s_3}$ and $v_5 \notin W_3$, we have s_1-path (s_1, v_1, v_2, w_4), s_2-path (s_2, v_3, v_7, v_6), and s_3-path $(H[s_3, \bar{v_5}|G_1, \{w_4\}], v_5)$. If $v_7 \neq \bar{s_3}$ and $v_7 \notin W_3$, we have s_1-path (s_1, v_1, v_5, v_6), s_2-path (s_2, v_3, v_2, w_4), and s_3-path $(H[s_3, \bar{v_7}|G_1, \{w_4\}], v_7)$. If $v_5 = \bar{s_3}$ and $W_3 = \{v_7\}$, we have s_2-path (s_2, v_3, v_7, v_6). From s_3-s_1 path $(s_3, v_5, v_1, v_2, H[w_4, \bar{s_1}|G_1, \{s_3\}], s_1)$, s_3-path and s_1-path can be obtained. If $v_7 = \bar{s_3}$ and $W_3 = \{v_5\}$, we have s_1-path (s_1, v_1, v_5, v_6). From s_3-s_2 path $(s_3, v_7, v_3, v_2, H[w_4, \bar{s_2}|G_1, \{s_3\}], s_2)$, s_3-path and s_1-path can be constructed.

When $\bar{w_4} \in \{v_0, v_4\}$, we let w.l.o.g. $\bar{w_4} = v_0$. If $\bar{s_3} \notin \{v_3, v_6\}$, we let s_2-path (s_2, v_5, v_1, v_2) and let Q be (v_3, v_7, v_6); otherwise, we let s_2-path be (s_2, v_3, v_7, v_6) and let Q be (v_5, v_1, v_2). Observe $\bar{s_3}$ is different from the endvertices of Q. We claim that there exists 2-PP$[\{(w_0, 5, \emptyset), (w_4, 3, \emptyset)\}|G_1]$ such that the sink of w_0-path is w_i for any $i = 1, 2, 3, 5, 6, 7$. The proof of the claim is straightforward and omitted. We let $Q = (x, y, z)$ and $z \notin W_3$. By using the claim, we find 2-PP$[\{(w_0, 5, \emptyset), (w_4, 3, \emptyset)\}|G_1]$ such that the sink of w_0-path is x. Letting P and P' the w_0-path and w_4-path, respectively, we have s_3-path (P, Q) and s_1-path (s_1, P').

Case 2: $k_0 = 2$ and $k_1 = 1$.
We let $s_1, s_2 \in V(G_0)$, $s_3 \in V(G_1)$, and assume $l_1 \geq l_2$. Observe $l_2 \leq 7$.

Subcase 2.1: $l_2 \leq 6$.
If there exists 2-PP$[\{(s_1, 8 - l_2, \{\bar{s_3}\}), (s_2, l_2, W_2^0)\}|G_0]$, and furthermore, letting t_1 be the sink of s_1-path, if there exists 2-PP$[\{(\bar{t_1}, 8 - l_3, W_1^1), (s_3, l_3, W_3^1)\}|G_1]$, we can obtain a 2-PP in $G_0 \oplus G_1$ by merging the two 2-PP's in G_0 and G_1. Hereafter in this case, we assume at least one of the 2-PP's in G_0 and G_1 do not exist.

Subcase 2.1.1: The 2-PP in G_0 does not exist.
By Lemma 16, we assume w.l.o.g. that $s_1 = v_0$, $s_2 = v_4$, $l_2 = 4$, $W_2 = \{v_1\}$, and $\bar{s_3} = v_3$. For each l_1 with $5 \leq l_1 \leq 11$, we are to construction a 3-PP in $G_0 \oplus G_1$. When $5 \leq l_1 \leq 7$, we find 2-PP$[\{(s_1, l_1, W_1^0), (s_2, 8 - l_1, \{\bar{s_3}\})\}|G_0]$, and letting t_2 be the sink of s_2-path, find 2-PP$[\{(\bar{t_2}, 8 - l_3, W_2^1), (s_3, l_3, W_3^1)\}|G_1]$. Note that the 2-PP in G_1 exists since $W_2^1 = \emptyset$. When $l_1 = 8$, we let s_1-path $(s_1, H[\bar{s_1}, x|G_1, \{s_3\}])$ for some $x \notin W_1^1$. We have s_2-path (s_2, v_5, v_6, v_7) and s_3-path (s_3, v_3, v_2, v_1) if $v_1 \notin W_3$; otherwise we have s_2-path (s_2, v_5, v_1, v_2) and s_3-path (s_3, v_3, v_7, v_6). When $l_1 = 9$, we let s_1-path be $(s_1, v_1, H[\bar{v_1}, x|G_1, \{s_3\}])$

for some $x \notin W_1^1$. If $v_2 \notin W_3^0$, we have s_2-path (s_2, v_5, v_6, v_7) and s_3-path (s_3, v_3, v_2); otherwise, we have s_2-path (s_2, v_5, v_6, v_2) and s_3-path (s_3, v_3, v_7).

When $l_1 = 10$, if $v_3 \notin W_3^0$, we have s_3-path (s_3, v_3), s_2-path (s_2, v_5, v_6, v_7), and s_1-path $(s_1, v_1, v_2, H[\bar{v_2}, x|G_1, \{s_3\}])$ for some $x \notin W_1^1$. If $v_3 \in W_3^0$, we first find a hamiltonian path in G_1 $(\bar{s_2}, z_1, z_2, z_3, z_4, z_5, z_6, s_3)$ joining $\bar{s_2}$ and s_3. Then, we let s_2-path be $(s_2, \bar{s_2}, z_1, z_2)$ and s_3-path be (s_3, z_6). The s_1-path can be constructed as follows: if $z_3, z_5 \neq \bar{s_1}$, assuming w.l.o.g. $z_3 \notin W_1^1$, we have s_1-path $(H[s_1, \bar{z_5}|G_0, \{s_2\}], z_5, z_4, z_3)$; otherwise, assuming $z_3 = \bar{s_1}$, we have s_1-path $(s_1, z_3, z_4, z_5, H[\bar{z_5}, y|G_0, \{s_1, s_2\}])$ for some $y \notin W_1^0$. Note that $G_0 \backslash \{s_1, s_2\}$ has a hamiltonian cycle $(v_1, v_2, v_3, v_7, v_6, v_5)$ and thus the hamiltonian path in $G_0 \backslash \{s_1, s_2\}$ exists.

When $l_1 = 11$, if 3-PP$[\{(\bar{s_1}, 4, \emptyset), (\bar{s_2}, 3, \emptyset), (s_3, 1, \emptyset)\}|G_1]$ exists, letting Q_1 and Q_2 be the $\bar{s_1}$-path and $\bar{s_2}$-path in the 3-PP, respectively, we have s_2-path (s_2, Q_2) and s_1-path $(s_1, Q_1, H[\bar{t_1}, x|G_0, \{s_1, s_2\}])$ for some $x \notin W_1^1$, where t_1 is the sink of Q_1. If no such a 3-PP exists, we can observe without difficulty that the triple $\bar{s_1}$, $\bar{s_2}$, and s_3 are adjacent to a common vertex in G_1. Thus, we can see that, assuming $s_3 = w_0$, $\bar{s_2} \in \{w_2, w_3, w_5, w_6\}$. There are two cases $\bar{s_2} = w_2$ and $\bar{s_2} = w_3$ up to symmetry. For the case $\bar{s_2} = w_2$, letting $P_0 = H[s_1, \bar{w_1}|G_0, \{s_2\}]$, we have s_1-path $(P_0, w_1, w_5, w_6, w_7)$ and s_2-path (s_2, w_2, w_3, w_4) if $w_7 \notin W_1^1$; otherwise, we have s_1-path $(P_0, w_1, w_5, w_4, w_3)$ and s_2-path (s_2, w_2, w_6, w_7). For the case $\bar{s_2} = w_3$, letting $P_0 = H[s_1, \bar{w_4}|G_0, \{s_2\}]$, we have s_1-path $(P_0, w_4, w_5, w_6, w_7)$ and s_2-path (s_2, w_3, w_2, w_1) if $w_7 \notin W_1^1$; otherwise, we have s_1-path $(P_0, w_4, w_5, w_1, w_2)$ and s_2-path (s_2, w_3, w_7, w_6).

Subcase 2.1.2: The 2-PP in G_1 does not exist.
We have $l_3 = 4$. It suffices to consider the case that the 2-PP in G_0 exists. Letting t_1 be the sink of s_1-path in the 2-PP of G_0, we assume w.l.o.g. $s_3 = w_0$, $\bar{t_1} = w_4$, $W_1 = \{w_1\}$, and $W_3 = \{w_3\}$. If there exists another 2-PP in G_0 such that the sink of s_1-path is different from t_1, we can see that the 2-PP in G_1 exists and we are done. It is not difficult to observe that another 2-PP can be found when $l_1 = 6 (= l_2)$ and $W_2^0 = \emptyset$. When $l_1 = 6$ and $W_2^0 \neq \emptyset (W_2^1 = \emptyset)$, the construction will result in a 3-PP if we exchange the roles of s_1 and s_2. Let $l_1 = 7 (l_2 = 5)$. If $\bar{s_1} \neq s_3$, letting s_1-path in the 2-PP be (s_1, y, t_1), we have s_3-path $(s_3, \bar{t_1}, t_1, y)$ and s_1-path $(s_1, H[\bar{s_1}, z|G_1, \{s_3, \bar{t_1}\}])$ for some $z \notin W_1^1$. Remember $G_1 \backslash \{s_3, \bar{t_1}\}$ has a hamiltonian cycle of length six. If $\bar{s_1} = s_3$, there exists 2-PP$[\{(s_3, 4, W_3^1), (\bar{s_2}, 4, W_2^1)\}|G_1]$ since $\bar{s_2} \neq w_4$. The 2-PP is merged with $H[s_1, z|G_0, \{s_2\}]$ for some $z \notin W_1^0$. Now, we assume $l_1 \geq 8$. Let the s_1-path in the 2-PP in G_0 be $(s_1, \ldots, u, x, y, t_1)$. If $l_1 = 8$, then $s_1 = u$. For the case $\bar{x} \neq s_3$, we let s_3-path be $(s_3, \bar{t_1}, t_1, y)$ and s_1-path be $(s_1, \ldots, x, H[\bar{x}, z|G_1, \{s_2, \bar{t_1}\}])$ for some $z \notin W_1^1$. For the case $\bar{x} = s_2$, we let s_2-path be (s_2, x, y, t_1) and s_1-path be $(s_1, \ldots, u, H[\bar{u}, z|G_1, \{s_2\}])$ for some $z \notin W_1^1$.

Subcase 2.2: $l_2 = 7$.
We have $l_1 = 7, 8$. When $\bar{s_1} \neq s_3$, there exist 2-PP$[\{(s_1, 1, \emptyset), (s_2, 7, W_2^0)\}|G_0]$ and 2-PP$[\{(\bar{s_1}, l_1 - 1, W_1^1), (s_3, l_3, W_3^1)\}|G_1]$. The two 2-PP's are merged into a 3-PP in $G_0 \oplus G_1$. The construction for $l_1 = 7$ is completed since we can assume w.l.o.g. that $\bar{s_1} \neq s_3$. It remains to construct a 3-PP only when $l_1 = 8$, $l_2 = 7$,

$l_3 = 1$, and $\bar{s_1} = s_3$. We can construct a 3-PP by a enumerative way. The construction is omitted here.

Case 3: $k_0 = 1$ and $k_1 = 2$.
We let $s_1 \in V(G_0)$ and $s_2, s_3 \in V(G_1)$, and assume $l_2 \geq l_3$.

Subcase 3.1: $l_1 \geq 10$.
We claim with an exceptional case when $(s_2, s_3) = (w_0, w_4)$, $(l_2, l_3) = (2, 2)$, $W_2 = \{w_1\}$, and $W_3 = \{w_3\}$, that (i) there exists 2-PP$[\{(s_2, l_2, W_2^1), (s_3, l_3, W_3^1)\}|G_1]$ and furthermore (ii) the vertices of G_1 not contained in s_2-path or s_3-path of the 2-PP form a path Q. The claim can be proved by a case analysis, and omitted here. The length of Q is at least two. We let $Q = (x, \ldots, y)$. If $s_1 \neq \bar{x}, \bar{y}$, assuming w.l.o.g. $y \notin W_1$, we have s_1-path $(H[s_1, \bar{x}|G_0, \emptyset], x, \ldots, y)$. If $s_1 = \bar{x}$ (or symmetrically $s_1 = \bar{y}$), we have s_1-path $(s_1, x, \ldots, y, H[\bar{y}, z|G_0, \{s_1\}])$ for some $z \notin W_1$. The existence of such z is due to Lemma 13.

Now let us consider the exceptional case when $(s_2, s_3) = (w_0, w_4)$, $(l_2, l_3) = (2, 2)$, $W_2 = \{w_1\}$, and $W_3 = \{w_3\}$. When $\bar{s_1} \neq s_2, s_3$, we let s_2-path and s_3-path be $(s_2, \bar{s_2})$ and $(s_3, \bar{s_3})$, respectively. By a case analysis, we can show that there are two s_1-hamiltonian paths with different sinks in $G_0 \oplus G_1 \backslash \{s_2, \bar{s_2}, s_3, \bar{s_3}\}$. We let an s_1-path be one of the two s_1-hamiltonian paths with sink $t \notin W_1$. When $\bar{s_1} \in \{s_2, s_3\}$, we assume w.l.o.g. $\bar{s_1} = s_2$. We let s_2-path and s_3-path be (s_2, w_7) and $(s_3, \bar{s_3})$, respectively. An s_1-path also can be constructed in $G_0 \oplus G_1 \backslash \{s_2, w_7, s_3, \bar{s_3}\}$.

Subcase 3.2: $l_1 = 9$.
We are to develop a basic procedure which is applicable for almost all instances of this subcase. For some $z \in V(G_1)$ with $z \neq s_2, s_3, \bar{s_1}$ and $z \notin W_1^1$, we first find 3-PP$[\{(s_2, l_2, W_2^1), (s_3, l_3, W_3^1), (z, 1, \emptyset)\}|G_1]$, and then the 3-PP is merged with $H[s_1, \bar{z}|G_0, \emptyset]$ into a 3-PP of $G_0 \oplus G_1$. If such a vertex z exists, we are done.

When $\bar{s_1} \in \{s_2, s_3\}$, we can observe that the procedure constructs a 3-PP with an exception up to symmetry that $s_2 = w_0$, $s_3 = w_1$, $l_2 = 5$, $l_3 = 2$, $W_1 = \{w_4\}$, $W_2 = \{w_6\}$, and $W_3 = \{w_2\}$. For the exceptional case, let s_2-path be (s_2, w_5). We let $C = (w_2, w_3, w_7, w_6)$ be a cycle of length four in G_1. By Lemma 17, there exists 2-PP$[\{(s_1, 5, \{\bar{s_3}, \bar{w_5}\}), (\bar{w_4}, 3, \emptyset)\}|G_0]$. The sink of s_1-path is a vertex y with $\bar{y} \in V(C)$, and thus s_1-path of G_0 and C can be merged into an s_1-path in $G_0 \oplus G_1$. The s_2-path in $G_0 \oplus G_1$ is a concatenation of (s_2, w_4) and s_2-path in G_0. Hereafter in this subcase, we assume $\bar{s_1} \neq s_2, s_3$.

When $W_1^1 = \emptyset$, the basic procedure is applicable except only when $s_2 = w_0$, $s_3 = w_1$, $l_2 = 5$, $l_3 = 2$, $W_2 = \{w_6\}$, $W_3 = \{w_2\}$, and $\bar{s_1} = w_4$. Let s_2-path be (s_3, w_5), and let $P = (s_2, w_4, w_3)$ and $Q = (w_2, w_6, w_7)$. Assume $s_1 = v_0$. If $\bar{w_3} \neq v_4$, there exists 2-PP$[\{(s_1, 6, \emptyset), (\bar{w_3}, 2, \emptyset)\}|G_0]$ and furthermore the sink of s_1-path can be located at five possible distinct vertices. We assume w.l.o.g. the sink of s_1-path is w_2. Then, s_1-path in $G_0 \oplus G_1$ is a concatenation of s_1-path in G_0 and Q, and s_2-path is a concatenation of P and $\bar{w_3}$-path of G_0. If $\bar{w_3} = v_4$, the sink of s_1-path in the 2-PP of G_0 can be located at four distinct vertices $X = \{v_2, v_3, v_5, v_6\}$. If at least one of $\bar{w_2}$ and $\bar{w_7}$ are contained in X, we are done. Assume $\{\bar{w_2}, \bar{w_7}\} = \{v_1, v_7\}$. If $v_6 \notin W_1^0$, we have s_1-path $(s_1, v_1, Q, v_7, v_3, v_2, v_6)$

and s_2-path (P, v_4, v_5). Otherwise, we have $v_2 \notin W_1^0$, and we can construct s_1-path and s_2-path in a symmetric way.

When $W_2^1 = \emptyset$, a 3-PP in $G_0 \oplus G_1$ can be constructed by using the basic procedure except only when $s_2 = w_0$, $s_3 = w_3$, $l_2 = 6$, $l_3 = 1$, and $\{\bar{s_1}\} \cup W_1 = \{w_4, w_7\}$. If $\bar{s_1} = w_4$, we can show that at least one 3-PP exist among the four: 3-PP$[\{(s_1, 1, \emptyset), (\bar{s_2}, 5, W_2^0), (\bar{w_7}, 2, \emptyset)\}|G_0]$, 3-PP$[\{(s_1, 1, \emptyset), (\bar{w_2}, 2, W_2^0), (\bar{w_1}, 5, \emptyset)\}|G_0]$, 3-PP$[\{(s_1, 1, \emptyset), (\bar{w_7}, 4, W_2^0), (\bar{w_1}, 3, \emptyset)\}|G_0]$, and 3-PP$[\{(s_1, 1, \emptyset), (\bar{w_7}, 4, W_2^0), (\bar{w_6}, 3, \emptyset)\}|G_0]$. If the first 3-PP exists, then s_1-path is a concatenation of $(s_1, w_4, w_5, w_1, w_2, w_6, w_7)$ and $\bar{w_7}$-path, and s_2-path is a concatenation of (s_2) and $\bar{s_2}$-path. For the other 3-PP, we can construct s_1-path and s_2-path in $G_0 \oplus G_1$ similarly. If $\bar{s_1} = w_7$, the construction of a 3-PP is very similar to the previous case. There exist at least one 3-PP among the four: 3-PP$[\{(s_1, 1, \emptyset), (\bar{s_2}, 5, W_2^0), (\bar{w_4}, 2, \emptyset)\}|G_0]$, 3-PP$[\{(s_1, 1, \emptyset), (\bar{w_4}, 2, W_2^0), (\bar{w_2}, 5, \emptyset)\}|G_0]$, 3-PP$[\{(s_1, 1, \emptyset), (\bar{w_1}, 2, W_2^0), (\bar{w_2}, 5, \emptyset)\}|G_0]$, and 3-PP$[\{(s_1, 1, \emptyset), (\bar{w_2}, 3, W_2^0), (\bar{w_4}, 4, \emptyset)\}|G_0]$. By using each 3-PP, s_1-path and s_2-path in $G_0 \oplus G_1$ can be obtained.

When $W_3^1 = \emptyset$, the basic procedure constructs 3-PP except two cases when (i) $s_2 = w_0$, $s_3 = w_2$, $l_2 = 6$, $l_3 = 1$, $W_2 = \{w_3\}$, and $\{\bar{s_1}\} \cup W_1 = \{w_1, w_6\}$, or (ii) $s_2 = w_0$, $s_3 = w_3$, $l_2 = 6$, $l_3 = 1$, $W_2^1 \neq \emptyset$, and $\{\bar{s_1}\} \cup W_1 = \{w_4, w_7\}$. For the first exceptional case, assuming w.l.o.g. $\bar{s_1} = w_1$, 3-PP$[\{(s_1, 1, \emptyset), (\bar{s_2}, 5, \emptyset), (\bar{w_6}, 2, \emptyset)\}|G_0]$ exists Lemma 5. Then, s_1-path is a concatenation of $(s_1, w_1, w_5, w_4, w_3, w_7, w_6)$ and $\bar{w_6}$-path, and s_2-path is a concatenation of (s_2) and $\bar{s_2}$-path. For the second exceptional case, assuming $\bar{s_1} = w_4$, we can also construct s_1-path and s_2-path from 3-PP$[\{(s_1, 1, \emptyset), (\bar{s_2}, 5, \emptyset), (\bar{w_7}, 2, \emptyset)\}|G_0]$ and $(w_4, w_5, w_1, w_2, w_6, w_7)$.

For the remaining case when $W_1^1, W_2^1, W_3^1 \neq \emptyset$, we assume $l_3 \geq 2$. We are going to develop another procedure to construct a 3-PP for this subcase. If $(l_2, l_3) = (4, 3)$ and 3-PP$[\{(\bar{s_1}, 3, \emptyset), (s_2, 2, \emptyset), (s_3, 3, W_3^1)\}|G_1]$ exists, then letting x and y be the sinks of $\bar{s_1}$-path and s_2-path in the 3-PP, respectively, we find 3-PP$[\{(s_1, 1, \emptyset), (\bar{x}, 5, \emptyset), (\bar{y}, 2, \emptyset)\}|G_0]$. The two 3-PP's are merged into a 3-PP in $G_0 \oplus G_1$. If $(l_2, l_3) = (5, 2)$ and 3-PP$[\{(\bar{s_1}, 3, \emptyset), (s_2, 3, \emptyset), (s_3, 2, W_3^1)\}|G_1]$ exists, then letting x and y be the sinks of $\bar{s_1}$-path and s_2-path, respectively, we find 3-PP$[\{(s_1, 1, \emptyset), (\bar{x}, 5, \emptyset), (\bar{y}, 2, \emptyset)\}|G_0]$ and merge the two 3-PP's into a 3-PP in $G_0 \oplus G_1$. There is a unique case up to symmetry that both procedures fails to construct a 3-PP. The exceptional case is that $s_2 = w_0$, $s_3 = w_1$, $l_2 = 5$, $l_3 = 2$, $W_1 = \{w_4\}$, $W_2 = \{w_6\}$, $W_3 = \{w_2\}$, and $\bar{s_1} \neq w_4$. Let s_3-path be (s_3, w_5) and $P = (s_2, w_4)$. The vertices of G_1 contained in neither s_2-path nor P form a cycle $C = (w_2, w_3, w_7, w_6)$ of length four. There exists 2-PP$[\{(s_1, 5, \{\bar{s_2}, \bar{s_3}, \bar{w_5}\}), (\bar{w_4}, 3, \emptyset)\}|G_0]$ by Lemma 17. The sink of s_1-path is a vertex y with $\bar{y} \in V(C)$. An s_2-path is a concatenation of P and $\bar{w_4}$-path in the 2-PP, and s_1-path is obtained from s_1-path in the 2-PP and the cycle C.

Case 4: $k_0 = 3$ and $k_1 = 0$.
We have three sources s_1, s_2, and s_3 in G_0, and assume $l_1 \geq l_2 \geq l_3$. If $l_2 = l_3 = 1$, an s_1-path can be obtained from a hamiltonian cycle in $G_0 \oplus G_1 \backslash \{s_2, s_3\}$. Thus, we let $l_2 \geq 2$. We can observe by a careful case analysis, with a exception up to symmetry when $(s_1, s_2, s_3) = (v_0, v_1, v_3)$, $(l_1, l_2, l_3) = (13, 2, 1)$, and $W_2 = \{v_2\}$, that there exists l_i' and l_i'', $i = 1, 2, 3$, satisfying all the six conditions:

(A1) for each i, $l'_i + l''_i = l_i$, $1 \le l'_i \le l_i$, (A2) $l'_1 + l'_2 + l'_3 = 8$, (A3) there exists 3-PP$[\{(s_1, l'_1, W'_1), (s_2, l'_2, W'_2), (s_3, l'_3, W'_3)\}|G_0]$, where $W'_i = W^0_i$ if $l_i = l'_i$; otherwise $W'_i = \emptyset$, (B1) for some i, $l'_i = l_i$, (B2) if $l''_i = 1$, then $W^1_i = \emptyset$, and (B3) if there are two l''_is with $l''_i \ge 1$, say l''_a and l''_b, then $l''_a, l''_b \ne 4$.

If there exist two l''_is with $l''_i \ge 1$, say l''_a and l''_b, then letting t_a and t_b be the sinks of s_a-path and s_b-path in the 3-PP of G_0, respectively, we have 2-PP$[\{(\bar{t_a}, l''_a, W^1_a), (\bar{t_b}, l''_b, W^1_b)\}|G_1]$ by Lemma 16. The 3-PP in G_0 and 2-PP in G_1 are merged into a 3-PP in $G_0 \oplus G_1$. If there exists a single l''_i with $l''_i \ge 1$, say l''_a, then $l''_a = 8$. Letting t_a be the sink of s_a-path in the 3-PP of G_0, we find a hamiltonian path $H[t_a, z|G_1, \emptyset]$ for some $z \notin W_a$. A 3-PP in $G_0 \oplus G_1$ can be obtained from the 3-PP in G_0 and the hamiltonian path in G_1. For the exceptional case when $(s_1, s_2, s_3) = (v_0, v_1, v_3)$, $(l_1, l_2, l_3) = (13, 2, 1)$, and $W_2 = \{v_2\}$, we let s_2-path be (s_2, v_5). $G_0 \oplus G_1 \backslash \{s_2, v_5, s_3\}$ has a hamiltonian cycle $(s_1, v_4, H[\bar{v_4}, \bar{v_2}|G_1, \emptyset], v_2, v_6, v_7)$, and thus an s_1-path can be constructed from the hamiltonian cycle. □

The proof of Lemma 6 is by a case analysis. Therefore, Lemma 6 is also checked by a computer program for each $G(8,4) \oplus G(8,4)$ in RHL_4, k, and (s_i, l_i, W_i) with $1 \le i \le k$, $k = 2, 3$.

B Proofs of Lemmas 7, 8, and 9

Proof (of Lemma 7). For the case $v_f \in V(G_0)$ and $v'_f \in V(G_1)$, we assume w.l.o.g. $s \in V(G_0)$. There exists an s-hamiltonian path $P_0 = (s, u_1, u_2, u_3, u_4, u_5, u_6)$ in G_0 with $\bar{u_6} \ne v'_f$ due to Lemma 13. Let P_1 and P'_1 be two $\bar{u_6}$-hamiltonian paths in G_1 with distinct sinks (endvertices different from $\bar{u_6}$). Then, we have two s-hamiltonian paths (P_0, P_1) and (P_0, P'_1) in $G_0 \oplus G_1$ whose sinks are in G_1. The third s-hamiltonian path can be constructed by removing an edge incident to s on a hamiltonian cycle $C = (s, x, \ldots, y)$ in $G_0 \oplus G_1$. Note that at least one of x and y are in G_0. Assuming $y \in V(G_0)$, $C \backslash (s, y)$ is the third one.

For the case $v_f, v'_f \in V(G_0)$, we let $P_0 = (u_0, u_1, u_2, u_3, u_4, u_5)$ be a hamiltonian path in G_0. For the subcase $s \in V(G_0)$, we let $s = u_i$ and assume $i \le 2$. When $i = 0$, for any vertex $w \ne \bar{u_5}$ in G_1, $(P_0, H[\bar{u_5}, w|G_1, \emptyset])$ is an s-hamiltonian path in $G_0 \oplus G_1$. When $i = 1, 2$, we have three s-hamiltonian paths $(s, \ldots, u_0, H[\bar{u_0}, \bar{u_{i+1}}|G_1, \emptyset], u_{i+1}, \ldots, u_5)$, $(s, \ldots, u_0, H[\bar{u_0}, \bar{u_5}|G_1, \emptyset], u_5, \ldots, u_{i+1})$, and $(s, \ldots, u_5, H[\bar{u_5}, \bar{u_0}|G_1, \emptyset], u_0, \ldots, u_{i-1})$. Let us consider the last subcase $s \in V(G_1)$. When $s = \bar{u_0}$, for at least two $\bar{u_5}$-hamiltonian paths P_1 and P'_1 in $G_1 \backslash s$ with distinct sinks, we have s-hamiltonian paths (s, P_0, P_1) and (s, P_0, P'_1). The third one is $(H[s, \bar{u_5}|G_1, \emptyset], P^R_0)$. When $s \ne \bar{u_0}, \bar{u_5}$, we have two s-hamiltonian paths $(H[s, \bar{u_0}|G_1, \emptyset], P_0)$ and $(H[s, \bar{u_5}|G_1, \emptyset], P^R_0)$, whose sinks are in G_0. The third one can be obtained from a hamiltonian cycle in $G_0 \oplus G_1$, whose sink is in G_1. □

Proof (of Lemma 8). We first show that when (v_f, v'_f) is an edge of $G(8,4) \oplus G(8,4)$, there exist four such vertices x_i, $i = 1, \ldots, 4$. For the case $v_f, v'_f \in V(G_0)$, there are two cases up to symmetry that $(v_f, v'_f) = (v_0, v_1)$, (v_0, v_4). We let $(x_1, x_2, x_3, x_4) = (v_2, v_4, v_5, v_7)$ if $(v_f, v'_f) = (v_0, v_1)$, and let $(x_1, x_2, x_3, x_4) = (v_1, v_3, v_5, v_7)$ if $(v_f, v'_f) = (v_0, v_4)$. We claim that for each i, there exists a hypohamiltonian path Q_i in G_0 which does not pass through x_i (as well as the faulty vertices). When $(v_f, v'_f) = (v_0, v_1)$, we have $Q_1 = (v_3, v_4, v_5, v_6, v_7)$, $Q_2 = (v_2, v_3, v_7, v_6, v_5)$, $Q_3 = (v_4, v_3, v_2, v_6, v_7)$, and $Q_4 = (v_2, v_3, v_4, v_5, v_6)$. When $(v_f, v'_f) = (v_0, v_4)$, we have $Q_1 = (v_2, v_3, v_7, v_6, v_5)$, $Q_2 = (v_2, v_1, v_5, v_6, v_7)$, $Q_3 = (v_1, v_2, v_3, v_7, v_6)$, and $Q_4 = (v_3, v_2, v_1, v_5, v_6)$. For each x_i, letting Q_i be a p-q path, we have a hamiltonian cycle $(Q_i, H[\bar{q}, \bar{p}|G_1, \emptyset])$ of $G(8,4) \oplus G(8,4) \backslash \{v_f, v'_f, x_i\}$.

For the case $v_f \in V(G_0)$ and $v'_f \in V(G_1)$, we let $v_f = v_0$. There is a hamiltonian cycle $C = (z_0, z_1, z_2, z_3, z_4, z_5, z_6)$ in G_1. Assuming $\bar{v_4} = z_0$, we have two hamiltonian cycles $(z_0, z_1, \ldots, z_5, H[\bar{z_5}, v_4|G_0, \{v_f\}])$ of $G(8,4) \oplus G(8,4) \backslash \{v_f, v'_f, z_6\}$ and $(z_0, z_6, \ldots, z_2, H[\bar{z_2}, v_4|G_0, \{v_f\}])$ of $G(8,4) \oplus G(8,4) \backslash \{v_f, v'_f, z_1\}$. The existence of the hamiltonian path in G_0 joining $\bar{z_5}$ (resp. $\bar{z_2}$) and v_4 is due to Lemma 13. The two vertices z_6 and z_1 are contained in G_1. In a symmetric way, we can also find two such vertices in G_0.

Now, we show that when (v_f, v'_f) is not an edge, there exist two vertices x_1 and x_2 such that $G(8,4) \oplus G(8,4) \backslash \{v_f, v'_f, x_i\}$ has a hamiltonian cycle for each i. For the case $v_f, v'_f \in V(G_0)$, we let $(x_1, x_2) = (v_1, v_4)$, $Q_1 = (v_3, v_4, v_5, v_6, v_7)$, $Q_2 = (v_1, v_5, v_6, v_7, v_3)$ if $(v_f, v'_f) = (v_0, v_2)$; if $(v_f, v'_f) = (v_0, v_3)$, we let $(x_1, x_2) = (v_4, v_7)$, $Q_1 = (v_5, v_1, v_2, v_6, v_7)$, $Q_2 = (v_4, v_5, v_1, v_2, v_6)$. In the same way as the case $v_f, v'_f \in V(G_0)$ and (v_f, v'_f) is an edge, we can construct a hamiltonian cycle in $G(8,4) \oplus G(8,4) \backslash \{v_f, v'_f, x_i\}$ for each i.

For the case $v_f \in V(G_0)$ and $v'_f \in V(G_1)$, we claim that there exist a pair of vertices z_a and z_b in G_0 which satisfy (i) $z_a, z_b \neq v_f$ and $\bar{z_a}, \bar{z_b} \neq v'_f$, (ii) there exists a hamiltonian path Q_0 between z_a and z_b in G_0, and (iii) there exists a hypohamiltonian path Q_1 between $\bar{z_a}$ and $\bar{z_b}$ in G_1. In G_0, by Lemma 13, there are twelve unordered pairs of vertices such that G_0 has a hamiltonian path joining each pair. Among them, at least six pairs also satisfy the condition (i). Due to Lemma 14, there are at least three pairs satisfying the condition (iii). Thus, the claim is proved. Then, letting $x_1 \in V(G_1)$ be the fault-free vertex not contained in Q_1, we have a hamiltonian cycle (Q_0, Q_1) in $G(8,4) \oplus G(8,4) \backslash \{v_f, v'_f, x_1\}$. Symmetrically, we can find such a vertex x_2 in G_0. □

Proof (of Lemma 9). We assume w.l.o.g. $v_f \in V(G_0)$. For the case $s, t \in V(G_0)$, regarding v_f as a virtual fault-free vertex, we find 3-PP$[\{(v_f, 2, \emptyset), (s, 5, \emptyset), (t, 1, \emptyset)\}|G_0]$. The existence of the 3-PP is due to Lemma 5. Letting v_f-path and s-path in the 3-PP be (v_f, x) and (s, z_1, z_2, z_3, z_4), respectively, we have an s-t hamiltonian path $(s, z_1, z_2, z_3, z_4, H[\bar{z_4}, \bar{t}|G_1, \emptyset], t)$ of $G(8,4) \oplus G(8,4) \backslash \{v_f, x\}$. For the case $s \in V(G_0)$ and $t \in V(G_1)$, we find 2-PP$[\{(v_f, 2, \emptyset), (s, 6, \{\bar{t}\})\}|G_0]$. Then, letting v_f-path and s-path be (v_f, x) and $(s, z_1, z_2, z_3, z_4, z_5)$, respectively, we have an s-t hamiltonian path $(s, z_1, z_2, z_3, z_4, z_5, H[\bar{z_5}, t|G_1, \emptyset])$.

For the last case $s, t \in V(G_1)$, by Lemma 15, there exists a vertex $z \in V(G_1)$ such that (i) $\bar{z} \neq v_f$ and (ii) for any vertex y in G_1 with $y \neq s, t, z$, there exist two disjoint covering paths joining $\{s, t\}$ and $\{z, y\}$. We find 2-PP$[\{(v_f, 2, \emptyset), (\bar{z}, 6, \{\bar{s}, \bar{t}\})\}|G_0]$ and let v_f-path in the 2-PP be (v_f, x). The existence of the 2-PP is due to Lemma 17. Then, letting u be the sink of $\bar{z}$-path in the 2-PP, the $\bar{z}$-path and two disjoint covering paths joining $\{s,t\}$ and $\{z,\bar{u}\}$ are merged into an s-t hamiltonian path in $G(8,4)\oplus G(8,4)\backslash\{v_f,x\}$. This completes the proof. □