A Proof of Lemma 18

To prove Lemma 18, we assume that

$$f_0 = f = 2 \text{ and } k_0 + k_1 + k_2 = k \geq 2.$$

Since each G_i is many-to-many $(0, k)$-disjoint path coverable and $2k-1$-fault hamiltonian, we have

$$2k \leq \delta - 1, \text{ and thus } 1 + 2k \leq \Delta - 1,$$

where $\delta = \min_i \delta(G_i)$ and $\Delta = \delta(G_0 \oplus G_1)$. We have two faulty vertices in G_0, and thus we can not employ Lemma 8 which requires $f + 2k \leq \Delta - 1$, to guarantee existence of pairwise disjoint free bridges. However, by Remark 2, we can see that for any set of terminals $W_l = \{w_1, w_2, \ldots, w_l\}$ contained in one component such that $l \leq 2k$, there exist l pairwise disjoint free bridges of w_i's, $1 \leq i \leq l$.

Case 1: $k_1 \geq 1$.

In this case, Procedures DPC-A and DPC-B are utilized with the roles of G_0 and G_1 interchanged.

Lemma A.1 *When $1 \leq k_1 \leq k-1$, Procedure DPC-A($G_1 \oplus G_0, R, F$) constructs a k-DPC.*

Proof G_1 is $2k_2$-fault k_1-disjoint path coverable since $2k_1 + 2k_2 \leq 2k$, and thus k_1-DPC exists in G_1. G_0 is 2-fault many-to-many $k-1$-disjoint path coverable, and $k_0 + k_2 \leq k - 1$, thus $k_0 + k_2$-DPC exists in G_0. □

Lemma A.2 *When $k_1 = k$, Procedure DPC-B($G_1 \oplus G_0, R, F$) constructs a k-DPC for $k \geq 3$ or for $k = 2$ and $|X_1| = 2$.*

Proof By the choice of s_1 in Procedure DPC-B, if $k \geq 3$, then $|X_1| = 2$. There exists a $k-1$-DPC in G_1 since G_1 is 2-fault $k-1$-disjoint path coverable. The existence of a hamiltonian path in G_0 is straightforward. □

Lemma A.3 *When $k_1 = k = 2$ and $|X_1| \geq 3$, a k-DPC can be constructed.*

Proof Obviously, $|X_1| = 3$. Without loss of generality, we assume that $\bar{s_1}$ and $\bar{s_2}$ are the faulty vertices. First, we find a free bridge $B_{s_2} = (s_2, \ldots, s_2')$ of s_2, and find $H[s_1, t_2|G_1, F']$, where $F' = V_1 \cap V(B_{s_2})$. Let the hamiltonian path be $(s_1, \ldots, t_1, z, \ldots, t_2)$. Then, we find a hamiltonian path $H[s_2', \bar{z}|G_0, F_0]$, and merge the two hamiltonian paths with B_{s_2} and $(z, \bar{z})$. Note that $(z, \bar{z})$ is the free bridge of z. Existence of the hamiltonian paths is straightforward. □

Case 2: $k_0 \geq 1$ and $k_2' \geq 2$ ($k_1 = 0$).

Lemma A.4 *When $k_0 \geq 1$ and $k_2' \geq 2$, Procedure DPC-A($G_0 \oplus G_1, R, F$) constructs a k-DPC.*

Proof G_0 is $2+k_2'+2k_2''$-fault k_0-disjoint path coverable since $2k_0+2+k_2'+2k_2'' \leq 2k_0+2k_2'+2k_2'' = 2k$. The existence of a 0-fault k_2-DPC in G_1 is straightforward. □

Case 3: $k_2 = k$ and $k_2' \geq 2$ ($k_1 = 0$).

Lemma A.5 *When $k_2 = k$ and $k_2' \geq 2$, Procedure DPC-D($G_0 \oplus G_1, R, F$) constructs a k-DPC unless $k_2' = k = 2$.*

Proof We first show a claim that for a set of terminals $W_k = \{w_1, w_2, \ldots, w_k\}$ such that $w_1 = t_1$ and $\{w_2, \ldots, w_k\} = \{s_2, \ldots, s_k\}$, there exist k pairwise disjoint free bridges of w_j's, $1 \leq j \leq k$, in a similar way to the proof of Lemma 8. We utilize the term 'occupied columns' and notation $c(l)$, $f(l)$, and $t(l)$ given in the proof of Lemma 8. We have four base cases as follows:

(i) when $\bar{t_1}$ is free: $c(1) = 1$, $f(1) = 0$, and $t(1) = 1$;

(ii) when $\bar{t_1} \in F_0$: $c(1) = 2$, $f(1) = 1$, and $t(1) = 1$;

(iii) when $\bar{t_1} = s_1$: $c(1) = 2$, $f(1) = 0$, and $t(1) = 2$;

(iv) when $\bar{t_1} = s_i$ for some $i \geq 2$: By the choice of s_1, $\bar{s_1} = t_j$ for some j. We assume w.l.o.g. that $\bar{s_1} = t_2$. Let $w_2 = s_i$. We show that there exist pairwise disjoint free bridges of t_1 and s_i. We first find a free bridge of t_1. The free bridge occupies two columns. Let the column containing (s_1, t_2) be occupied. Then, three columns are occupied and there are three terminals t_1, s_1, and t_2 (excluding s_i) in the occupied columns. A free bridge of s_i exists since there are at least Δ candidates and at most $2k+1$ blocking elements (3 occupied columns, $2k-1-3$ terminals in unoccupied columns, and 2 faulty vertices). Thus, $c(2) = 4$, $f(2) = 0$, and $t(2) = 4$.

Assuming there exist pairwise disjoint free bridges for $W_{l-1} = W_l \backslash w_l$ satisfying $c(l-1) \leq f(l-1) + t(l-1)$, we can always find pairwise disjoint free bridges for W_l satisfying $c(l) \leq f(l) + t(l)$. Thus, we have the claim. To prove existence of the hamiltonian path in G_0, we will show that $2 + |F'| \leq 2k - 2$. When $k_2'' \geq 1$, $2+|F'| = 2+2(k_2''-1)+k_2' \leq 2k-2$. When $k_2'' = 0$, we have $2+|F'| = 2+k_2'-1 \leq 2k-2$ unless $k = 2$. The existence of a $k_2 - 1$-DPC is straightforward. □

Lemma A.6 *When $k_2' = k = 2$, a k-DPC can be constructed.*

Proof We first find a free bridge $B_{s_2} = (s_2, \bar{s_2})$ of s_2, and find a hamiltonian cycle $C_0 = (s_1, x, \ldots, y)$ in $G_0 \backslash F_0 \cup F'$, where $F' = \{s_2\}$. When $\bar{x}$ is a free vertex, we find 2-DPC$[\{(\bar{x}, t_1), (\bar{s_2}, t_2)\}|G_1, \emptyset]$, and then merge the 2-DPC with the hamiltonian cycle. When $\bar{y}$ is a free vertex, a symmetric construction will do. When both $\bar{x}$ and $\bar{y}$ are not free vertices, we assume w.l.o.g. that $\bar{x} = t_1$ and $\bar{y} = t_2$. We find $H[\bar{s_2}, t_2|G_1, F'']$, where $F'' = \{t_1\}$. Finally we merge the hamiltonian path and the hamiltonian cycle C_0 with the free bridges of s_2 and the edge (x, t_1). □

Case 4: $k'_2 = 0$ $(k_1 = 0)$.

Observe that for each $j \in I_2$, $\bar{s_j} = t_i$ for some $i \in I_2$, and thus for any other fault-free vertex v in G_0, $(v, \bar{v})$ is the free bridge of v.

Lemma A.7 *When $k'_2 = 0$ and $k_0 \geq 2$, a k-DPC can be constructed.*

Proof We first find pairwise disjoint free bridges $B_{s_j} = (s_j, \bar{s_j})$, $B_{t_j} = (t_j, \bar{t_j})$ for all $j \in I_0 \backslash \{1, 2\}$, and $B_{s_j} = (s_j, \ldots, s'_j)$ for all $j \in I_2$. And then we find $H[s_2, t_2|G_0, F_0 \cup F']$, where $F' = V_0 \cap [\bigcup_{j \in I_0 \backslash \{1,2\}}(V(B_{s_j}) \cup V(B_{t_j})) \cup \bigcup_{j \in I_2} V(B_{s_j})]$. Let the hamiltonian path be $(s_2, \ldots, x, s_1, \ldots, t_1, y, \ldots, t_2)$. The hamiltonian path exists since $2 + |F'| = 2 + 2(k-2) = 2k - 2$. Finally, we find $k-1$-DPC$[\{(\bar{x}, \bar{y})\} \cup \{(\bar{s_j}, \bar{t_j})|j \in I_0 \backslash \{1, 2\}\} \cup \{(s'_j, t_j)|j \in I_2\}|G_1, \emptyset]$, and merge the hamiltonian path and the $k-1$-DPC. □

Lemma A.8 *When $k'_2 = 0$ and $k_2 \geq 2$, a k-DPC can be constructed.*

Proof We first find pairwise disjoint free bridges $B_{s_j} = (s_j, \bar{s_j})$, $B_{t_j} = (t_j, \bar{t_j})$ for all $j \in I_0$, and $B_{s_j} = (s_j, \ldots, s'_j)$ for all $j \in I_2 \backslash \{k-1, k\}$, and then find a hamiltonian path $P_0 = H[s_{k-1}, s_k|G_0, F_0 \cup F']$, where $F' = V_0 \cap [\bigcup_{j \in I_0}(V(B_{s_j}) \cup V(B_{t_j})) \cup \bigcup_{j \in I_2 \backslash \{k-1,k\}} V(B_{s_j})]$. Let $P_0 = (s_{k-1}, \ldots, x, y, \ldots, s_k)$, where x and y are free vertices. The existence of P_0 is due to that $2 + |F'| = 2k - 2$, and the existence of x and y is due to the fact that the number of vertices on P_0 is at least four since $2k \leq \delta(G_0) - 1 \leq n - 2$. Finally, we find k-DPC$[\{(\bar{x}, t_{k-1}), (\bar{y}, t_k)\} \cup \{(\bar{s_j}, \bar{t_j})|j \in I_0\} \cup \{(s'_j, t_j)|j \in I_2 \backslash \{k-1, k\}\}|G_1, \emptyset]$, and merge the hamiltonian path and the k-DPC. □

Lemma A.9 *When $k'_2 = 0$, $k_0 \leq 1$, and $k_2 \leq 1$, a k-DPC can be constructed.*

Proof It follows that $k_0 = k_2 = 1$. It holds true that $\bar{s_2} = t_2$ since $k_2 = 1$ and $k'_2 = 0$. It suffices to construct a hamiltonian path joining s_1 and t_1 in $G_0 \oplus G_1 \backslash F_0 \cup \{s_2, t_2\}$. First we find a hamiltonian cycle $C_0 = (s_1, \ldots, x, t_1, \ldots, y)$ in $G_0 \backslash F_0 \cup F'$, where $F' = \{s_2\}$. And then we find $H[\bar{x}, \bar{y}|G_1, F'']$, where $F'' = \{t_2\}$. Obviously, $\bar{x}$ and $\bar{y}$ are free vertices. Finally, we merge the hamiltonian cycle and the hamiltonian path with edges $(x, \bar{x})$ and $(y, \bar{y})$. □

Case 5: $k'_2 = 1$ ($k_1 = 0$).

There exists t_j, $j \in I_2$, such that $\bar{t_j} \neq s_i$ for all $i \in I_2$. Such a t_j is unique, and let $\alpha = \bar{t_j}$. We denote by P^R the reverse of a path P, that is, $P^R = (v_l, v_{l-1}, \ldots, v_1)$ for $P = (v_1, v_2, \ldots, v_l)$.

Lemma A.10 *When (a) $k'_2 = 1$, (b) $k''_2 \geq 1$, and either (c1) $k_0 = 0$ or (c2) $k_0 \geq 1$ and $s_i, t_i \neq \alpha$ for all $i \in I_0$, a k-DPC can be constructed.*

Proof We assume w.l.o.g. that $\bar{s_k}$ is a sink in G_1. We first find pairwise disjoint free bridges $B_{s_j} = (s_j, \bar{s_j})$, $B_{t_j} = (t_j, \bar{t_j})$ for all $j \in I_0$, and $B_{s_j} = (s_j, \ldots, s'_j)$ for all $j \in I_2 \backslash k$, and then find a hamiltonian cycle $C_0 = (s_k, x, \ldots, y)$ in $G_0 \backslash F_0 \cup F'$, where $F' = V_0 \cap [\bigcup_{j \in I_0}(V(B_{s_j}) \cup V(B_{t_j})) \cup \bigcup_{j \in I_2 \backslash k} V(B_{s_j})]$. We assume w.l.o.g. $x \neq \alpha$. The hamiltonian cycle exists since $2 + |F'| = 2 + 2(k-1) - 1 = 2k - 1$. Finally, we find k-DPC$[\{(\bar{x}, t_k)\} \cup \{(\bar{s_j}, \bar{t_j}) | j \in I_0\} \cup \{(s'_j, t_j) | j \in I_2 \backslash k\} | G_1, \emptyset]$, and merge the hamiltonian cycle and the k-DPC. □

Lemma A.11 *When (a) $k'_2 = 1$ and either (b1) $k''_2 = 0$ or (b2) $k''_2 \geq 1$, $k_0 \geq 1$, and there exists $i \in I_0$ such that $s_i = \alpha$ or $t_i = \alpha$, a k-DPC can be constructed.*

Proof It follows that $k_0 \geq 1$. If $\alpha \in \{s_i, t_i | i \in I_0\}$, we let $s_1 = \alpha$; else if there exists $i \in I_0$ such that $(s_i, \alpha) \notin E(G_0)$ or $(t_i, \alpha) \notin E(G_0)$, we let s_1 be such a terminal, that is, $(s_1, \alpha) \notin E(G_0)$; else, we let s_1 be an arbitrary source such that $t_1 \in V_0$. We first find pairwise disjoint free bridges $B_{s_j} = (s_j, \bar{s_j})$, $B_{t_j} = (t_j, \bar{t_j})$ for all $j \in I_0 \backslash 1$, and $B_{s_j} = (s_j, \ldots, s'_j)$ for all $j \in I_2$, and then we find a hamiltonian cycle $C_0 = (s_1, x_1, \ldots, y_2, t_1, y_1, \ldots, x_2)$ in $G_0 \backslash F_0 \cup F'$, where $F' = V_0 \cap [\bigcup_{j \in I_0 \backslash 1}(V(B_{s_j}) \cup V(B_{t_j})) \cup \bigcup_{j \in I_2} V(B_{s_j})]$. The hamiltonian cycle exists since $2 + |F'| = 2 + 2(k-1) - 1 = 2k - 1$. When $x_2, y_2 \neq \alpha$, we can construct a k-DPC by finding k-DPC$[\{(\bar{x_2}, \bar{y_2})\} \cup \{(\bar{s_j}, \bar{t_j}) | j \in I_0 \backslash 1\} \cup \{(s'_j, t_j) | j \in I_2\} | G_1, \emptyset]$ and merging the hamiltonian cycle C_0 and the k-DPC in G_1. Similarly, when $x_1, y_1 \neq \alpha$, we can also construct a k-DPC. Observe that such a pair x_i and y_i exist if $\alpha \in F_0$, if $\alpha = s_1$, or if $(s_1, \alpha) \notin E(G_0)$. This completes the proof for the case that the conditions (a) and (b2) are satisfied. Hereafter, we assume that (b1) $k''_2 = 0$. Suppose that no such a pair x_i and y_i exist. Then, (s_1, α) and (t_1, α) must be edges on C_0, that is, $C_0 = (s_1, \alpha, t_1, Q)$. Note that by the choice of s_1, α is a free vertex such that $(s_i, \alpha), (t_i, \alpha) \in E(G_0)$ for all $i \in I_0$. When $k_0 \geq 2$ ($k \geq 3$), we let an s_1-t_1 path be (s_1, Q^R, t_1), and let an s_2-t_2 path be (s_2, α, t_2), and construct all the other paths by utilizing $k-2$-DPC$[\{(\bar{s_j}, \bar{t_j}) | j \in I_0 \backslash \{1, 2\}\} \cup \{(s'_j, t_j) | j \in I_2\} | G_1, \emptyset]$. When $k_0 = 1$ ($k = 2$), we find $H[s_1, t_1 | G_0, F_0]$ and let the hamiltonian path be $(s_1, \ldots, x, s_2, \ldots, \alpha, y, \ldots, t_1)$. Merging the hamiltonian path and

$H[\bar{x}, \bar{y}|G_1, F'']$, where $F'' = \{t_2\}$, with the edges $(x, \bar{x})$, $(y, \bar{y})$, and (α, t_2) results in a 2-DPC. This completes the proof. $\square$

Remark Lemma 18 can be extended to Lemma A.12 as follows. The proof can be derived from the proof of Lemma 18 and is omitted in this paper.

Lemma A.12 *For any $k \geq 2$ and $f \geq 0$, if G_i is many-to-many (f, k)-disjoint path coverable and $f + 2k - 1$-fault hamiltonian for each $i = 0, 1$, then $G_0 \oplus G_1$ with $|F| \leq f + 2$ and $f_0 \geq 2$ is many-to-many k-disjoint path coverable.*