Torus-Like Graphs and Their Paired Many-to-Many Disjoint Path Covers

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Abstract

Given two disjoint vertex-sets, $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ in a graph, a *paired many-to-many k-disjoint path cover* joining S and T is a set of pairwise vertex-disjoint paths $\{P_1, \ldots, P_k\}$ that altogether cover every vertex of the graph, in which each path P_i runs from s_i to t_i. In this paper, we propose a family of graphs, called *torus-like graphs*, that include torus networks, and reveal that a torus-like graph, if built from lower dimensional torus-like graphs that have good Hamiltonian and disjoint-path-cover properties, retain such good properties. As a result, every m-dimensional nonbipartite torus, $m \geq 2$, with at most f vertex and/or edge faults has a paired many-to-many k-disjoint path cover joining arbitrary disjoint sets S and T of size k each, subject to $k \geq 2$ and $f + 2k \leq 2m$. The bound $2m$ on $f + 2k$ is nearly optimal.

Keywords: Disjoint path, path cover, path partition, cylindrical grid, torus.

1. Introduction

Given the internal processor and memory structures in each node, a distributed-memory architecture is characterized primarily by the network used to interconnect the nodes [22]. An interconnection network is frequently modeled as a graph, in which the vertices and edges represent nodes and links, respectively. One of the central issues in the study of interconnection networks is finding parallel paths, which is naturally related to routing among nodes and fault tolerance of the network [11, 20]. Parallel paths correspond to disjoint paths of the underlying graph. Throughout the paper, the vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively.

A number of interconnection networks have been proposed and studied in the literature. Among them, torus is one of the widely recognized interconnection networks. An *m-dimensional torus* is defined as a Cartesian product of m cycles, $C_{d_1} \times \cdots \times C_{d_m}$, where each C_{d_j}, $j \in \{1, \ldots, m\}$, is a cycle of length $d_j \geq 3$. Also, the m-dimensional torus can be constructed recursively from d_m copies

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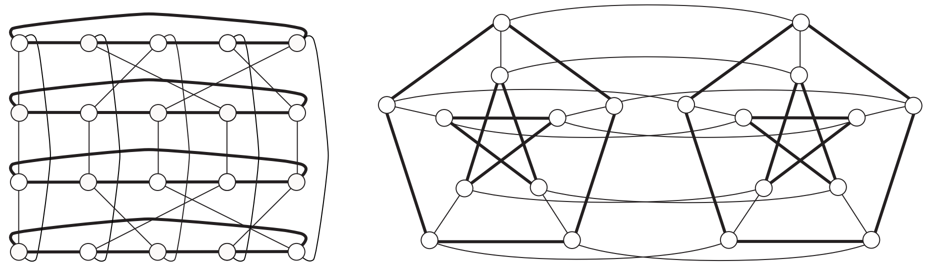


(a) Torus-like graph (b) Product of the Petersen graph and K_2

Fig. 1: Examples of 2-dimensional torus-like graphs, where an intra-component edge is indicated by a thick edge. The graphs (a) and (b) are, in fact, isomorphic.

$G_0, \dots, G_{d_m-1}$ of an $(m-1)$-dimensional torus $C_{d_1} \times \cdots \times C_{d_{m-1}}$ by adding edges, through an identify mapping from $V(G_i)$ to $V(G_{i'})$, to each pair G_i and $G_{i'}$, where $i' = (i+1) \bmod d_m$, for $i \in \{0, \dots, d_m - 1\}$. The edges between G_i and $G_{i'}$ form a perfect matching of the subgraph induced by $V(G_i) \cup V(G_{i'})$.

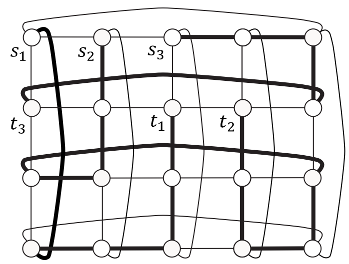
Given two graphs, G_0 and G_1, of the same order and a bijection ϕ from $V(G_0)$ to $V(G_1)$, we denote by $G_0 \oplus_\phi G_1$ the graph whose vertex set is $V(G_0) \cup V(G_1)$ and edge set is $E(G_0) \cup E(G_1) \cup \{(v, \phi(v)) : v \in V(G_0)\}$. To simplify the notation, we often omit the bijection ϕ from $\oplus_\phi$. Given d graphs $G_0, \dots, G_{d-1}$ of the same order n, if we apply the graph constructor $\oplus$ to each pair G_i and $G_{(i+1) \bmod d}$ for $i \in \{0, \dots, d-1\}$, then we obtain a graph with nd vertices. This graph is said to be obtained through the *cycle-based recursive construction*. The torus-like graphs are a class of graphs which are defined recursively as follows:

Definition 1 (Torus-like graphs)**.** An *m-dimensional torus-like graph*, $m \geq 1$, is a graph obtained through the cycle-based recursive construction from $(m-1)$-dimensional torus-like graphs $G_0, \dots, G_{d-1}$, $d \geq 3$, of the same order, where the 0-dimensional torus-like graph is a one-vertex graph K_1.

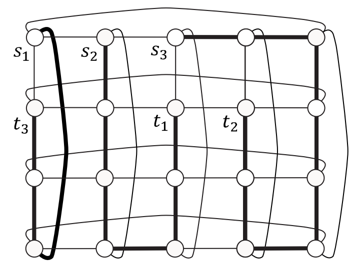
Here, the graphs $G_0, \dots, G_{d-1}$ are called the *components* of the torus-like graph. Refer to Fig. 1 for examples of torus-like graphs. Each vertex v in component G_i has two neighbors outside G_i: one in $G_{(i+1) \bmod d}$, denoted by v^+, and the other in $G_{(i-1) \bmod d}$, denoted by v^-. Contracting the components of the torus-like graph into single vertices results in a cycle C_d of length d.

On the other hand, a *path cover* of a graph G is a set of paths in G such that every vertex of G is contained in at least one path. A *disjoint path cover* (DPC for short) of G is a set of vertex-disjoint paths that altogether cover every vertex of G. This paper is concerned with a DPC in which each path runs from a prescribed source to a prescribed sink.

Definition 2 (Many-to-many disjoint path covers)**.** Given disjoint subsets $S = \{s_1, \dots, s_k\}$ and $T = \{t_1, \dots, t_k\}$ of $V(G)$ for a positive integer k, a *many-to-many k-disjoint path cover* is a DPC composed of k paths that collectively join S and T.



(a) Paired 3-DPC



(b) Unpaired 3-DPC

Fig. 2: Examples of many-to-many disjoint path covers in a 2-dimensional torus.

If each source $s_i \in S$ must be joined to a specific sink $t_i \in T$, the many-to-many k-DPC is called *paired*, and it is *unpaired* if no such constraint is imposed. Refer to Fig. 2 for examples. There are two other DPC types: A *one-to-many k-disjoint path cover* for $S = \{s\}$ and $T = \{t_1, \ldots, t_k\}$ is a DPC made of k paths, each of which joins a pair of source s and sink t_i, $i \in \{1, \ldots, k\}$; when $S = \{s\}$ and $T = \{t\}$, a DPC composed of k paths, each of which joins s and t, is named a *one-to-one k-disjoint path cover*. As intuitively clear, we will call the vertices in S and in T *sources* and *sinks*, respectively, which together form a set of *terminals*.

Definition 3. (See [29].) A graph G is called *f-fault paired* (resp. *unpaired*) *k-disjoint path coverable* if $f + 2k \leq |V(G)|$ and G has a paired (resp. unpaired) k-DPC joining arbitrary disjoint set S of k sources and set T of k sinks in $G - F$ for any fault set $F \subseteq V(G) \cup E(G)$ with $|F| \leq f$.

The disjoint path cover problems are applicable in many areas such as software testing, database design, and code optimization [2, 21]. In addition, the problem is concerned with applications where full utilization of network nodes is important [28]. The disjoint path cover problems have been studied for various classes of graphs, such as hypercubes [6, 7, 12], interval graphs [1, 18, 25], dense graphs [3, 19], torus networks [5, 15, 17, 24], k-ary n-cubes [4, 30, 32], DCell networks [31], etc. Among them, the results on torus networks can be summarized briefly as follows:

Lemma 1 (Kronenthal et al. [15] and Park [23])**.** *A 2-dimensional nonbipartite torus is paired 2-disjoint path coverable.*

Lemma 2. *Let G be an m-dimensional nonbipartite torus $C_{d_1} \times \cdots \times C_{d_m}$, where $m \geq 2$.*
(a) G is f-fault paired k-disjoint path coverable for any f and $k \geq 1$ subject to $f + 2k \leq 2m - 1$ (Kim et al. [13])*.*
(b) If $d_1 = \cdots = d_m \geq 5$, then G is $(2m-4)$-fault paired 2-disjoint path coverable (Chen [4])*.*
(c) If at most one d_j is even, then G is $(2m - 4)$-edge-fault paired 2-disjoint

path coverable, i.e., the graph $G-F$, where $F \subseteq E(G)$ and $|F| \leq 2m-4$, has a paired 2-DPC joining S and T for any terminal sets S and T with $|S|=|T|=2$ (Chen [5]).
(d) Moreover, if $m \geq 3$ and at most one d_j is even, G is $(2m-3)$-edge-fault paired 2-disjoint path coverable (Li et al. [17]).

Note that an m-dimensional torus $C_{d_1} \times \cdots \times C_{d_m}$ is nonbipartite if and only if not every d_j is even. Also, there is a result established by Li et al. [16] that an m-dimensional nonbipartite torus is one-to-one k-disjoint path coverable for every $1 \leq k \leq 2m$. In addition, some work on the DPC problems of a bipartite torus can be found in [5, 15, 24, 26].

In this paper, we reveal that even in the presence of faults, an m-dimensional torus-like graph built from d components $G_0, \ldots, G_{d-1}$ has good Hamiltonian and disjoint-path-cover properties, provided each G_i has such good Hamiltonian and disjoint-path-cover properties. As a result, we obtain that *every m-dimensional nonbipartite torus, $m \geq 2$, is f-fault paired k-disjoint path coverable for any f and $k \geq 2$ subject to $f+2k \leq 2m$.* The result is an improvement of the work shown in Lemma 1 and Lemma 2 (a) through (c). Furthermore, the bound $2m$ on $f+2k$ is nearly optimal, specifically, one less than the bound, $2m+1$, of the necessary condition, given in Lemma 3 below, for a general graph to be f-fault paired k-disjoint path coverable. Note that the connectivity of an m-dimensional torus is $2m$.

Lemma 3. *(See [28].) If a graph G is f-fault paired k-disjoint path coverable, then $f+2k \leq \kappa(G)+1$, where $\kappa(G)$ is the connectivity of G.*

2. Preliminaries

Important parameters of an interconnection network include node degree and network diameter (the longest of the shortest paths between all pairs of nodes) [22]. Also, the connectivity (the minimum number of vertices whose removal results in a disconnected graph or a trivial graph) of the underlying graph has been a primary measure of fault tolerance [11]. We begin with a small lemma concerned with the topological properties of a torus-like graph.

Lemma 4. *Let G be an m-dimensional torus-like graph composed of d components $G_0, \ldots, G_{d-1}$, where each G_i is an $(m-1)$-dimensional torus-like graph.*
(a) G is a regular graph of degree $2m$, which has at least 3^m vertices.
(b) The connectivity of G is $2m$.
(c) The diameter of G is no more than $\lfloor \frac{d}{2} \rfloor$ plus the maximum diameter over all components.
(d) G has no triangle (cycle of length three) if $d \geq 4$ and every G_i has no triangle.
(e) There are at most three common neighbors for any pair of vertices in G. Moreover, if $d \geq 4$ and any pair of vertices in each component have at most two common neighbors, then any pair of vertices in G have at most two common neighbors.

Proof. The proof for (a) is obvious by definition. We can prove the assertion (b) easily by induction on m. Also, the proofs for (c) and (d) are straightforward. The proof for (e) can be completed without difficulty by induction on m. Note that there may exist a pair of vertices that have three common neighbors. (If $d = 3$, the vertices $u \in V(G_0)$ and $v \in V(G_1)$ may have three common neighbors, one in each component.) ■

Lemma 4(c) leads to that compared with an m-dimensional torus $C_{d_1} \times \cdots \times C_{d_m}$, every m-dimensional torus-like graph (of the same order) in which each (sub)component that is j-dimensional is made of d_j (sub)subcomponents that are $(j-1)$-dimensional for all $j \in \{1, \ldots, m\}$ has diameter no more than $\sum_{i=1}^{m} \lfloor \frac{d_i}{2} \rfloor$, the diameter of the torus. This means that there is room for a good interconnection graph with a smaller diameter in the family of torus-like graphs. The torus-like graph shown in Fig. 1, for example, is of diameter 3, while every 2-dimensional torus with 16 or more vertices is of diameter strictly greater than 3.

Another topic we discuss in this section is the Hamiltonian properties of a graph, to which the disjoint path cover problems are closely related. A path that visits each vertex exactly once is a *Hamiltonian path*; a cycle that visits each vertex exactly once is a *Hamiltonian cycle*. A graph G is said to be *f-fault Hamiltonian-connected* (resp. *Hamiltonian*) if any pair of vertices are joined by a Hamiltonian path (resp. there exists a Hamiltonian cycle) in $G - F$ for any fault set $F \subseteq V(G) \cup E(G)$ with $|F| \leq f$. For more details, we refer to the related literature including [8, 9, 10, 27]. It is worth noting that a graph G is f-fault paired 1-disjoint path coverable if and only if G is f-fault Hamiltonian-connected. The Hamiltonian properties of a two-dimensional nonbipartite torus shown below will be utilized in deriving our result on torus networks.

Lemma 5 (Kim et al. [14])**.** *A 2-dimensional nonbipartite torus is 1-fault Hamiltonian-connected and 2-fault Hamiltonian.*

Let G be a finite, simple undirected graph. A *path* from $v \in V(G)$ to $w \in V(G)$, referred to as a v–w path, is a sequence $\langle u_1, \ldots, u_l \rangle$ of distinct vertices of G such that $u_1 = v$, $u_l = w$, and $(u_i, u_{i+1}) \in E(G)$ for all $i \in \{1, \ldots, l-1\}$. If $l \geq 3$ and $(u_l, u_1) \in E(G)$, the sequence is called a *cycle*. Hereafter in this paper, let $\delta(G)$ denote the minimum degree of a graph G. In addition, a vertex v is said to be *free* if it is neither a fault nor a terminal, i.e., $v \notin F$ and $v \notin S \cup T$. An edge (u, v) is said to be *free* if it is nonfaulty and both u and v are free. A paired k-DPC joining $\{(x_1, y_1), \ldots, (x_k, y_k)\}$ refers to a DPC composed of x_1–y_1, …, x_k–y_k paths. For example, Fig. 2(a) shows a paired 3-DPC joining $\{(s_1, t_1), (s_2, t_2), (s_3, t_3)\}$.

3. Chain of torus-like graphs

Let G be an m-dimensional torus-like graph constructed from $(m-1)$-dimensional torus-like graphs $G_0, G_1, \ldots, G_{d-1}$ of the same order. The subgraph of G induced by $V(G_0) \cup \cdots \cup V(G_r)$, $0 \leq r \leq d-2$, forms a *chain of torus-like* *graphs* and will be denoted by $G_0 \oplus \cdots \oplus G_r$ or simply by $G_{0,r}$. In this section, we reveal that in the presence of faults, the chain $H = G_0 \oplus \cdots \oplus G_r$, $r \geq 1$, has good Hamiltonian and disjoint-path-cover properties if every G_i has. Throughout the paper, F will denote a set of faults (vertices and/or edges). We let F_i denote the set of faults contained in G_i, i.e., $F_i = F \cap (V(G_i) \cup E(G_i))$, and let $f_i = |F_i|$. Also, let $F_{i,j}$ denote the fault set of $G_{i,j}$ and let $f_{i,j} = |F_{i,j}|$, so $F_{i,i} = F_i$ and $f_{i,i} = f_i$. Then, the fault set F of H will be $F = F_{0,r-1} \cup F_r \cup F'_{r-1,r}$, where $F'_{i,i+1}$ denotes the set of faulty edges bridging G_i and G_{i+1}.

Theorem 1. *Let G_i, $i \in \{0, \ldots, r\}$, be an $(m-1)$-dimensional torus-like graph of the same order, $m \geq 3$, such that G_i is $(2m-5)$-fault Hamiltonian-connected, $(2m-4)$-fault Hamiltonian, and unpaired 2-disjoint path coverable. Then, the graph H defined as $G_0 \oplus \cdots \oplus G_r$, $r \geq 1$, is (a) $(2m-4)$-fault Hamiltonian-connected and (b) $(2m-3)$-fault Hamiltonian.*

Proof. The proof will proceed by induction on r. Suppose $r = 1$, i.e., $H = G_0 \oplus G_1$ for the base step. For the proof of (a), it suffices to build a Hamiltonian s–t path in $H - F$ for distinct vertices $s, t \in V(G_1)$ and $F = \{s^-, t^-\}$ from Theorem 1 (a) and (b) of [27]. The required Hamiltonian path can be built in three steps as follows: (i) Pick up an edge (x, y) on a Hamiltonian cycle C of $G_0 - F_0$, so $\{x^+, y^+\} \cap \{s, t\} = \emptyset$; (ii) build an unpaired 2-DPC joining $\{s, t\}$ and $\{x^+, y^+\}$ in G_1; (iii) finally, combine the two paths in the DPC with an x–y path $C - (x, y)$ of G_0 through edges (x, x^+) and (y, y^+). The proof of (b) directly follows from Theorem 2(a) of [27].

Suppose $r \geq 2$ for the inductive step, and let $H' = G_0 \oplus \cdots \oplus G_{r-1}$. To prove (a), assume w.l.o.g. $|F| = 2m - 4$ and $f_r + f'_{r-1,r} \leq f_0 + f'_{0,1}$ where $f'_{i,i+1} = |F'_{i,i+1}|$, leading to $f_r + f'_{r-1,r} \leq |F|/2 = m - 2 \leq 2m - 5$, so $G_r - F_r$ is Hamiltonian-connected. Also, $H' - F_{0,r-1}$ is Hamiltonian-connected by the induction hypothesis. A Hamiltonian s–t path will be built in $H - F$ for distinct vertices $s, t \in V(H) \setminus F$. Firstly, let $s, t \in V(H')$. There exists a Hamiltonian s–t path P in $H' - F_{0,r-1}$.

Claim 1. On the path P, there is an edge (x, y) of G_{r-1} such that $\{x^+, (x, x^+), y^+, (y, y^+)\} \cap F = \emptyset$.

Proof. The path P passes through $|V(G_{r-1}) \setminus F_{r-1}| \geq 3^{m-1} - f_{r-1}$ vertices of G_{r-1}. Among them, at least $3^{m-1} - f_{r-1} - 2$ vertices are intermediate vertices of P (excluding s and t). So, P passes through at least $\frac{1}{2}(3^{m-1} - f_{r-1} - 2)$ edges of G_{r-1}, which form candidate edges for (x, y). On the other hand, each fault in $F_r \cup F'_{r-1,r}$ may block at most two candidate edges. Thus, there remain at least $\frac{1}{2}(3^{m-1} - f_{r-1} - 2) - 2(f_r + f'_{r-1,r}) \geq \frac{1}{2}(3^{m-1} - |F| - 3(f_r + f'_{r-1,r}) - 2) \geq \frac{1}{2}(3^{m-1} - (2m-4) - 3(m-2) - 2) \geq 1$ edges for $m \geq 3$, proving the claim. □

It suffices to combine the two subpaths obtained by deleting the edge (x, y) from the path P with a Hamiltonian x^+–y^+ path of G_r through edges (x, x^+) and (y, y^+). Analogously, we can build a Hamiltonian s–t path for $s, t \in V(G_r)$, because there are $|V(G_r) \setminus F_r| - 1 \geq 3^{m-1} - f_r - 1$ candidate edges and at most $f_{r-1} + f'_{r-1,r}$ blocking elements, for which $(3^{m-1} - f_r - 1) - 2(f_{r-1} + f'_{r-1,r}) \geq 3^{m-1} - 2|F| - 1 = 3^{m-1} - 2(2m-4) - 1 \geq 4$. If $s \in V(H')$ and $t \in V(G_r)$

finally, then for a free edge (x, x^+) with $x \in V(G_{r-1})$, combining a Hamiltonian s–x path of H' with a Hamiltonian x^+–t path of G_r results in a required s–t path, completing the proof of (a).

For the proof of (b), we also assume w.l.o.g. $|F| = 2m-3$ and $f_r + f'_{r-1,r} \le f_0 + f'_{0,1}$. So, $f_r + f'_{r-1,r} \le \lfloor |F|/2 \rfloor = m-2 \le 2m-5$ and thus $G_r - F_r$ is Hamiltonian-connected. In addition, $H' - F_{0,r-1}$ is Hamiltonian by the induction hypothesis. A Hamiltonian cycle of $H - F$ can be built from a Hamiltonian cycle C of $H' - F_{0,r-1}$ and a Hamiltonian x^+–y^+ path of $G_r - F_r$ for an edge $(x, y) \in E(G_{r-1})$ on C such that $\{x^+, (x, x^+), y^+, (y, y^+)\} \cap F = \emptyset$. Such edge (x, y) exists because $\frac{1}{2}(3^{m-1} - f_{r-1}) - 2(f_r + f'_{r-1,r}) \ge \frac{1}{2}(3^{m-1} - |F| - 3(f_r + f'_{r-1,r})) \ge \frac{1}{2}(3^{m-1} - (2m-3) - 3(m-2)) \ge 1$ for $m \ge 3$. This completes the proof. ■

Let $S = \{s_1, \dots, s_k\}$ and $T = \{t_1, \dots, t_k\}$ be the source and sink sets given in $H = G_0 \oplus \cdots \oplus G_r$, $r \ge 1$, such that $S \cap T = \emptyset$. We denote by k_i and $k_{i,j}$ the numbers of source-sink pairs in G_i and in $G_i \oplus \cdots \oplus G_j$, respectively, so $k_{i,i} = k_i$. We assume w.l.o.g. that

$$k_0 > k_r, \text{ or } k_0 = k_r \text{ and } f_0 \ge f_r. \tag{1}$$

In addition, let $k'_{r-1,r}$ denote the number of source-sink pairs between H' and G_r, where $H' = G_0 \oplus \cdots \oplus G_{r-1}$, so that $k = k_{0,r-1} + k_r + k'_{r-1,r}$. We let $I_0 = \{1, 2, \dots, k_{0,r-1}\}$, $I_2 = \{k_{0,r-1}+1, \dots, k_{0,r-1}+k'_{r-1,r}\}$, and $I_1 = \{k_{0,r-1} + k'_{r-1,r} + 1, \dots, k\}$. We assume that $\{s_j, t_j : j \in I_0\} \cup \{s_j : j \in I_2\} \subseteq V(H')$ and $\{s_j, t_j : j \in I_1\} \cup \{t_j : j \in I_2\} \subseteq V(G_r)$.

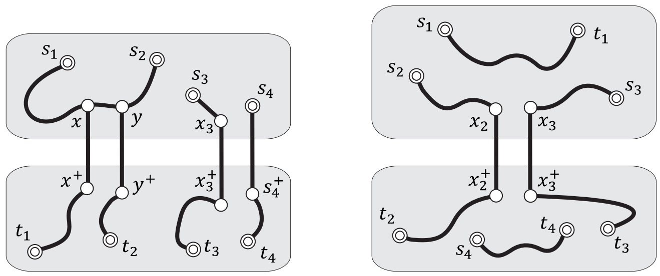
Theorem 2. *Let G_i, $i \in \{0, \dots, r\}$, be an $(m-1)$-dimensional torus-like graph of the same order, $m \ge 3$, such that G_i is f-fault paired k-disjoint path coverable for any f and $k \ge 2$ subject to $f + 2k \le 2m-2$ and moreover, G_i is $(2m-5)$-fault Hamiltonian-connected and $(2m-4)$-fault Hamiltonian. Then, the graph H defined as $G_0 \oplus \cdots \oplus G_r$, $r \ge 1$, is f-fault paired k-disjoint path coverable for any f and $k \ge 2$ subject to $f + 2k \le 2m-1$.*

Proof. Given F, S, and T in H such that $|F| = f$, $|S| = |T| = k \ge 2$, and $f + 2k \le 2m-1$, we will build a paired k-DPC joining S and T in $H - F$. We can assume w.l.o.g.

$$f + 2k = 2m - 1, \tag{2}$$

because for a set F' of arbitrary $(2m-1) - (f+2k)$ fault-free edges, a paired k-DPC joining S and T in $H - (F \cup F')$ is also a paired k-DPC joining them in $H - F$. As a result, we have $k \le m-1$ (i.e., $2k \le \delta(G_i)$) and $f \ge 1$. The proof is by induction on r. Suppose $r = 1$ for the base step, i.e., $H = G_0 \oplus G_1$, where $H' = G_0$ and $k_{0,r-1} = k_0$. There are four cases to consider according to the distribution of faults and terminals:

- $k_1 \ge 1$ or $f_0 \le f - 1$;
- $k_1 = 0$, $f_0 = f$, $k_0 \ge 1$, and for some $a \in I_2$, s_a^+ is not a terminal ($k'_{0,1} \ge 1$);



(a) Find-PDPC-A (b) Find-PDPC-B

Fig. 3: Building a paired k-DPC in the chain of torus-like graphs.

- $k_1 = 0$, $f_0 = f$, $k_0 \geq 1$, and for every $j \in I_2$, s_j^+ is a terminal;
- $k'_{0,1} = k$ and $f_0 = f$.

The proofs for the first three cases are very similar to those for paired disjoint path coverability of RHL graphs given in [29]. This phenomenon occurs because RHL graphs and torus-like graphs both have good Hamiltonian properties in the presence of faults and their numbers of vertices are sufficiently large compared to their degrees. (An m-dimensional RHL graph has 2^m vertices of degree m.) Moreover, the fact $2k \leq \delta(G_i) = 2m - 2$ (or $f \geq 1$) leads to a simpler construction, so some procedures devised in [29] do not need to be used. For the proofs of the first three cases except for the last, we refer to Section 3.1 of [29] (Procedures PairedDPC-A through PairedDPC-E).

Now, let us concentrate on the remaining case, in which $k'_{0,1} = k$ and $f_0 = f \geq 1$. Let I'_2 be the set of indices $j \in I_2$ such that s_j^+ is a sink, i.e., $I'_2 = \{j \in I_2 : s_j^+ \in T\}$, and let $k' = |I'_2|$. Assume w.l.o.g. that $I'_2 = \{1, \ldots, k'\}$. The procedure below builds a required k-DPC.

Procedure Find-PDPC-A(S, T, F, $G_0 \oplus G_1$) // $k'_{0,1} = k$ and $f_0 = f \geq 1$. See Fig. 3(a).

1: For each $j \in I'_2 \setminus \{1, 2\}$, pick up a free edge (x_j, x_j^+) with $x_j \in V(G_0)$ such that (s_j, x_j) is an edge and fault-free.
2: Build a Hamiltonian s_1–s_2 path P in $G_0 - (F_0 \cup F' \cup F'')$, where $F' = \{s_j, x_j : j \in I'_2 \setminus \{1, 2\}\}$ and $F'' = \{s_j : j \in I_2 \setminus I'_2, j \geq 3\}$. Let (x, y) be an edge on the path $P = (s_1, \ldots, x, y, \ldots, s_2)$ such that $x^+, y^+ \notin T$.
3: Build a paired k-DPC in G_1 joining $\{(x^+, t_1), (y^+, t_2)\} \cup \{(x_j^+, t_j) : j \in I'_2 \setminus \{1, 2\}\} \cup \{(s_j^+, t_j) : j \in I_2 \setminus I'_2,\ j \geq 3\}$.
4: Combine the two paths $P - (x, y)$ of G_0 with the k-DPC of G_1 through edges (x, x^+), (y, y^+), (s_j, x_j), (x_j, x_j^+) for $j \in I'_2 \setminus \{1, 2\}$, and (s_j, s_j^+) for $j \in I_2 \setminus I'_2$ with $j \geq 3$.

Claim 2. When $k'_{0,1} = k$ and $f_0 = f \geq 1$, Procedure Find-PDPC-A builds a paired k-DPC in $G_0 \oplus G_1 - F$.

Proof. To prove the existence of the free edges in Step 1, assume $k' \geq 3$ temporarily; suppose otherwise, there is nothing to prove. For each $j \in I_2' \setminus \{1, 2\}$, we can pick up a free edge (x_j, x_j^+) one by one. This is because there are $\delta(G_0)$ candidate edges whereas at most $f + k' + (2k - 2k') + (k' - 2)$ of them could be blocked (f faults, $2k'$ terminals a pair of which could block one candidate edge, $2k - 2k'$ remaining terminals, and $k' - 2$ free edges selected), for which $\delta(G_0) - (f + k' + (2k - 2k') + (k' - 2)) = (2m - 2) - (f + 2k - 2) = 1$. The Hamiltonian path P of Step 2 exists because $|F_0| + |F'| + |F''| \leq f_0 + 2(k - 2) = f + 2k - 4 = 2m - 5$. Also, the paired k-DPC of Step 3 exists because $2k = (2m - 1) - f \leq 2m - 2$. Thus, the procedure builds a required DPC. □

Suppose $r \geq 2$ for the inductive step, and let $H' = G_0 \oplus \cdots \oplus G_{r-1}$. H' always contains a terminal by the assumption (1), whereas G_r may not contain a terminal. Firstly, let $S \cup T \subseteq V(H')$ and $(S \cup T) \cap V(G_r) = \emptyset$. Then, there exists a paired k-DPC joining S and T in $H' - F_{0,r-1}$ by the induction hypothesis.

Claim 3. There is an edge $(x, y) \in E(G_{r-1})$ on a path P_i in the k-DPC such that $\{x^+, (x, x^+), y^+, (y, y^+)\} \cap F = \emptyset$.

Proof. The proof is similar to that of Claim 1. The paths in the k-DPC collectively pass through at least $|V(G_{r-1}) \setminus F_{r-1}| - 2k \geq 3^{m-1} - f_{r-1} - 2k$ vertices of G_{r-1} as intermediate vertices (excluding the terminals). So, there are at least $\lceil \frac{1}{2}(3^{m-1} - f_{r-1} - 2k) \rceil$ candidate edges for (x, y), whereas at most $2(f_r + f'_{r-1,r})$ of them could be blocked (two for each fault in $F_r \cup F'_{r-1,r}$), for which $\lceil \frac{1}{2}(3^{m-1} - f_{r-1} - 2k) \rceil - 2(f_r + f'_{r-1,r}) \geq \lceil \frac{1}{2}(3^{m-1} - (f + 2k) - 3f) \rceil \geq \lceil \frac{1}{2}(3^{m-1} - (2m - 1) - 3(2m - 5)) \rceil \geq 1$ for $m \geq 3$. Note that $f = 2m - 1 - 2k \leq 2m - 5$. Thus, the claim is proven. □

Combining the two subpaths obtained by deleting the edge (x, y) from P_i with a Hamiltonian x^+–y^+ path of $G_r - F_r$ through edges (x, x^+) and (y, y^+) results in a new s_i–t_i path P_i' of H. It suffices to replace P_i in the k-DPC with the new path P_i'.

Now, let G_r, as well as H', have a terminal. The assumption (1) leads to that $k_0 \geq 1$ (meaning $k_r + k'_{r-1,r} \leq k - 1$) or $f_r \leq f - 1$. The procedure below builds a required k-DPC in $H - F$.

Procedure FIND-PDPC-B(S, T, F, $G_0 \oplus \cdots \oplus G_r$) // $r \geq 2$ and G_r contains a terminal. See Fig. 3(b).

1: Pick up $k'_{r-1,r}$ free edges (x_j, x_j^+) with $x_j \in V(G_{r-1})$ for $j \in I_2$.
2: Build a paired $(k_{0,r-1} + k'_{r-1,r})$-DPC in $H' - F_{0,r-1}$ joining $\{(s_j, t_j) : j \in I_0\} \cup \{(s_j, x_j) : j \in I_2\}$.
3: Build a paired $(k_r + k'_{r-1,r})$-DPC in $G_r - F_r$ joining $\{(s_j, t_j) : j \in I_1\} \cup \{(x_j^+, t_j) : j \in I_2\}$.
4: Merge the two DPCs with the free edges selected in Step 1 into a paired k-DPC.

Claim 4. When $r \geq 2$ and G_r contains a terminal, Procedure FIND-PDPC-B builds a paired k-DPC in $H - F$.

Proof. The $k'_{r-1,r}$ free edges of Step 1 exist, because there are $|V(G_{r-1})|$ candidate edges whereas at most $f+2k$ of them could be blocked (by f faults and $2k$ terminals), for which $|V(G_{r-1})|-(f+2k) \geq 3^{m-1}-(2m-1) > (m-1) \geq k \geq k'_{r-1,r}$ for $m \geq 3$. The paired $(k_{0,r-1}+k'_{r-1,r})$-DPC of Step 2 exists by the induction hypothesis if $k_{0,r-1}+k'_{r-1,r} \geq 2$; otherwise, the DPC made of a single Hamiltonian path exists because $f_{0,r-1} \leq f = (2m-1)-2k \leq 2m-5$. Also, the paired $(k_r+k'_{r-1,r})$-DPC of Step 3 exists because $f_r+2(k_r+k'_{r-1,r}) \leq \max\{f+2(k-1),(f-1)+2k\} = f+2k-1 = 2m-2$ if $k_r+k'_{r-1,r} \geq 2$; if $k_r+k'_{r-1,r} = 1$, the DPC exists because $f_r \leq f \leq 2m-5$. Thus, the claim is proven. □

This completes the entire proof of Theorem 2. ■

4. Disjoint path covers in torus-like graphs

Let G be an m-dimensional torus-like graph, $m \geq 3$, composed of d components $G_0,\ldots,G_{d-1}$, where each component G_i is an $(m-1)$-dimensional torus-like graph. In this section, we show that the torus-like graph G has good Hamiltonian and disjoint-path-cover properties if every component G_i has such good properties. Specifically, if each G_i has Hamiltonian property P1 and disjoint-path-cover property P2 below, then G also has both properties:

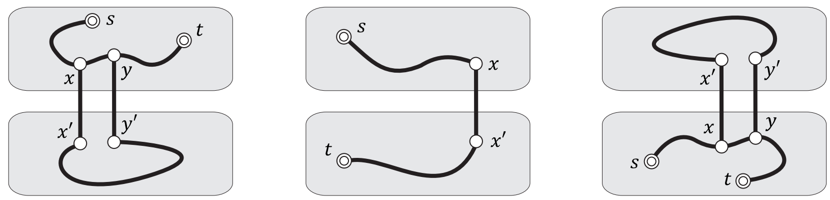
- P1: G_i is $(\delta(G_i)-3)$-fault Hamiltonian-connected and $(\delta(G_i)-2)$-fault Hamiltonian;
- P2: G_i is f-fault paired k-disjoint path coverable for any f and $k \geq 2$ subject to $f+2k \leq \delta(G_i)$.

The following Sections 4.1 and 4.2, respectively, deal with the Hamiltonian and disjoint-path-cover properties of the torus-like graphs.

4.1. Hamiltonian properties of torus-like graphs

Theorem 3. *Let G be an m-dimensional torus-like graph, $m \geq 3$, composed of d components $G_0,\ldots,G_{d-1}$ each of which is $(2m-5)$-fault Hamiltonian-connected, $(2m-4)$-fault Hamiltonian, and f-fault paired k-disjoint path coverable for any f and $k \geq 2$ subject to $f+2k \leq 2m-2$. Then, G is (a) $(2m-3)$-fault Hamiltonian-connected and (b) $(2m-2)$-fault Hamiltonian.*

Proof. For the proof of (a), assume we are given a fault set F with $|F| = 2m-3$ and two distinct vertices s and t in G. If F contains an *inter-component edge*, an edge between G_i and G_{i+1} for some i, then it suffices to build a Hamiltonian s–t path in the chain $G_{i+1}\oplus\cdots\oplus G_i$ by Theorem 1(a). So, we assume there is no inter-component edge fault, leading to $f = f_0+\cdots+f_{d-1} = 2m-3$. Also, assume $f_0 \geq f_j$ for all $j \in \{0,\ldots,d-1\}$, so $f_0 \geq 1$, and let H denote $G_1\oplus\cdots\oplus G_{d-1}$. If $f_0 \leq f-2$, then G_0-F_0 is Hamiltonian-connected; also, $H-F_{1,d-1}$ is Hamiltonian-connected from Theorem 1(a) because $f_{1,d-1} = f-f_0 \leq f-1$. Thus, we can build a Hamiltonian s–t path in $G-F$, as illustrated in Fig. 4. If $f_0 = f-1$, then $f_1 = 0$ or $f_{d-1} = 0$; so, we assume w.l.o.g. $f_1 = 0$. Then,



(a) $s, t \in V(G_0)$ (b) $s \in V(G_0)$, $t \in V(H)$ (c) $s, t \in V(H)$

Fig. 4: Building a Hamiltonian s–t path, where x' and y' are the neighbors of x and y, respectively.

$G_0 \oplus G_1 - F_{0,1}$ and $G_2 \oplus \cdots \oplus G_{d-1} - F_{2,d-1}$ are both Hamiltonian-connected. (Note that $f_{2,d-1} = 1 \leq 2m-5$, and that $G_2 \oplus \cdots \oplus G_{d-1}$ is made of a single component if $d = 3$.) Analogously, we can also build a Hamiltonian s–t path of $G - F$.

Now, let $f_0 = f$. Then, there is a Hamiltonian path in $G_0 - F_0$, joining some vertices x and y, because G_0 is $(2m-4)$-fault Hamiltonian and $f_0 = 2m-3$. In addition, H is unpaired 2-disjoint path coverable by Theorem 2. (Note that a paired 2-disjoint path coverable graph is, by definition, unpaired 2-disjoint path coverable.) Firstly, suppose $s, t \in V(H)$. There exist neighbors $x', y' \in V(G_1) \cup V(G_{d-1})$ of x, y, respectively, such that $\{x', y'\} \neq \{s, t\}$. If $\{x', y'\} \cap \{s, t\} = \emptyset$, it suffices to build an unpaired 2-DPC joining $\{s, t\}$ and $\{x', y'\}$ in H and combine the two paths in the DPC with the x–y path of G_0 through edges (x, x') and (y, y'); if $\{x', y'\} \cap \{s, t\}$ is of size one, say $x' = s$, it suffices to build a Hamiltonian y'–t path in $H - \{s\}$ and combine the y'–t path and one-vertex path $\langle s \rangle$ with the x–y path of G_0. Secondly, suppose $s \in V(G_0)$ and $t \in V(H)$. Let the Hamiltonian x–y path of G_0 be represented as $\langle x, \ldots, z, s, \ldots, y \rangle$ with $z \neq x, s$. For the neighbors $x', y', z' \in V(G_1) \cup V(G_{d-1})$ of x, y, z such that $x', y', z' \neq t$, it suffices to combine the two paths in an unpaired 2-DPC joining $\{y', t\}$ and $\{x', z'\}$ with $\langle x, \ldots, z \rangle$ and $\langle s, \ldots, y \rangle$ through edges (x, x'), (y, y'), and (z, z'). Finally, suppose $s, t \in V(G_0)$. If $\{s, t\} \cap \{x, y\} = \emptyset$, where the Hamiltonian x–y path of G_0 is represented as $\langle x, \ldots, z, s, \ldots, w, t \ldots, y \rangle$, it suffices to combine the three paths $\langle x, \ldots, z \rangle$, $\langle s, \ldots, w \rangle$, $\langle t \ldots, y \rangle$ with the two paths in an unpaired 2-DPC of H joining $\{w', y'\}$ and $\{x', z'\}$ for the distinct neighbors $x', y', z', w' \in V(H)$ of x, y, z, w, respectively. (If $x = z$, then $\{x', z'\}$ will be $\{x^+, x^-\}$.) If $\{s, t\} \cap \{x, y\} \neq \emptyset$, say $x = s$, where the Hamiltonian x–y path of G_0 is $\langle s, \ldots, w, t \ldots, y \rangle$, it suffices to combine $\langle s, \ldots, w \rangle$ and $\langle t \ldots, y \rangle$ with a Hamiltonian w'–y' path of H for the neighbors $w', y' \in V(H)$ of w, y.

For the proof of (b), assume $|F| = 2m-2$. If there is an inter-component edge fault, then $G - F$ is Hamiltonian by Theorem 1(b). So, we assume that F contains no inter-component edge fault, so that $f = f_0 + \cdots + f_{d-1} = 2m-2$. Also, assume $f_0 = \max_j f_j$. If $f_0 \leq 2m-5$, every $G_i - F_i$ is Hamiltonian-connected. So, we can easily build a Hamiltonian cycle of $G - F$. If $f_0 = 2m-4$, then $G_0 - F_0$ has a Hamiltonian cycle C. Moreover, the graph $H - F_{1,d-1}$, where $H = G_1 \oplus \cdots \oplus G_{d-1}$, is Hamiltonian-connected by Theorem 1(a) because

$f_{1,d-1} = 2 \le 2m-4$. For an edge (x,y) on the Hamiltonian cycle C such that $x^+, y^+ \notin F$, it suffices to combine the Hamiltonian x–y path $C-(x,y)$ of G_0 with a Hamiltonian x^+–y^+ path of $H - F_{1,d-1}$. If $f_0 = 2m-3$, then $G_0 - F_0$ has a Hamiltonian path, joining some vertices x and y, because G_0 is $(2m-4)$-fault Hamiltonian. For the neighbors $x', y' \in V(G_1) \cup V(G_{d-1})$ of x, y such that $x', y' \notin F$, it suffices to combine the Hamiltonian x–y path of $G_0 - F_0$ with a Hamiltonian x'–y' path of $H - F_{1,d-1}$. Finally, if $f_0 = 2m-2$, there exist two disjoint paths, say x–y and u–v paths, that collectively cover the vertices of $G_0 - F_0$. This is also because G_0 is $(2m-4)$-fault Hamiltonian. It suffices to combine the x–y and u–v paths with the two paths in an unpaired 2-DPC of H joining $\{x^+, y^-\}$ and $\{u^+, v^-\}$. (Note that the x–y path, also the u–v path, may be a one-vertex path.) This completes the proof. ■

Remark 1. It can be easily proven by induction on r that the graph $H = G_0 \oplus \cdots \oplus G_r$ is unpaired 2-disjoint path coverable if each G_i is unpaired 2-disjoint path coverable and Hamiltonian-connected. So, the last precondition of Theorem 3 that "each G_i is f-fault paired k-disjoint path coverable for any f and $k \ge 2$ subject to $f + 2k \le 2m-2$" can be simplified to a relaxed one that "each G_i is unpaired 2-disjoint path coverable."

4.2. Disjoint-path-cover properties of torus-like graphs

Theorem 4. *Let G be an m-dimensional torus-like graph, $m \ge 3$, composed of d components $G_0, \ldots, G_{d-1}$ each of which is $(2m-5)$-fault Hamiltonian-connected, $(2m-4)$-fault Hamiltonian, and f-fault paired k-disjoint path coverable for any f and $k \ge 2$ subject to $f + 2k \le 2m-2$. Then, G is f-fault paired k-disjoint path coverable for any f and $k \ge 2$ subject to $f + 2k \le 2m$.*

Proof. For the proof, assume we are given terminal sets $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ in G, along with a fault set F with $|F| = f$ and $f + 2k = 2m$. Let J denote the index set $\{1, \ldots, k\}$, and let $J_i = \{j \in J : s_j, t_j \in V(G_i)\}$ and $J'_i = \{j \in J : |\{s_j, t_j\} \cap V(G_i)| = 1\}$ for $i \in \{0, \ldots, d-1\}$. Analogously, let $J_{p,q} = \{j \in J : s_j, t_j \in V(G_p \oplus \cdots \oplus G_q)\}$ and $J'_{p,q} = \{j \in J : |\{s_j, t_j\} \cap V(G_p \oplus \cdots \oplus G_q)| = 1\}$. In addition, let $k_i = |J_i|$, $k'_i = |J'_i|$, $k_{p,q} = |J_{p,q}|$, and $k'_{p,q} = |J'_{p,q}|$.

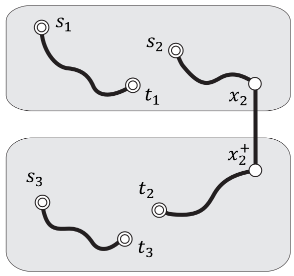
We assume there is no inter-component edge fault by Theorem 2, leading to $f = f_0 + \cdots + f_{d-1} = 2m - 2k$, and also assume for all $i \in \{0, \ldots, d-1\}$,

$$f_0 > f_i, \text{ or } f_0 = f_i \text{ and } k_0 + k'_0 \ge k_i + k'_i. \tag{3}$$

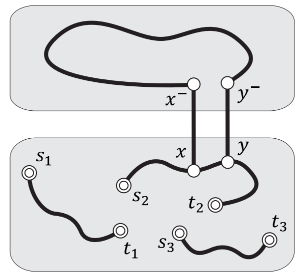
If $k \ge 3$ and $(s_j, t_j) \in E(G) \setminus F$ for some $j \in J$, we can build a paired k-DPC of $G - F$ joining S and T from a paired $(k-1)$-DPC of $G - (F \cup \{s_j, t_j\})$ joining $S \setminus \{s_j\}$ and $T \setminus \{t_j\}$. So, we assume

$$\text{if } k \ge 3, \text{ then } (s_j, t_j) \notin E(G) \setminus F \text{ for all } j \in J. \tag{4}$$

There are four cases according to the distribution of faults and terminals.



(a) Find-PDPC-C



(b) Exceptional case (i)

Fig. 5: Illustrations of Case 1 in the proof of Theorem 4.

Case 1: *$f_0 \leq f-2$, or $k_0+k'_0 < k$ and $f_0+k_0 \geq 1$.* In this case, G_0 is f_0-fault paired $(k_0+k'_0)$-disjoint path coverable if $k_0+k'_0 \geq 2$. Also, we have $f_0+k_0 \geq 1$ because $f_0 \leq f-2$ leads to $f \geq 2$ and $f_0 \geq 1$; so, the subgraph $G_1 \oplus \cdots \oplus G_{d-1}$ is $(f-f_0)$-fault paired $(k-k_0)$-disjoint path coverable if $k-k_0 \geq 2$. Assume w.l.o.g. $s_j \in V(G_0)$ for all $j \in J'_0$. Let $H = G_1 \oplus \cdots \oplus G_{d-1}$.

Procedure Find-PDPC-C(S, T, F, G) // $f_0 \leq f-2$, or $k_0+k'_0 < k$ and $f_0+k_0 \geq 1$. See Fig. 5(a).

1: Pick up a free edge (x_j, x_j^+) with $x_j \in V(G_0)$ for each $j \in J'_0$.
2: Build a paired $(k_0+k'_0)$-DPC in G_0-F_0 joining $\{(s_i,t_i) : i \in J_0\} \cup \{(s_j,x_j) : j \in J'_0\}$.
3: Build a paired $(k-k_0)$-DPC in $H-F_{1,d-1}$ joining $\{(s_i,t_i) : i \in J_{1,d-1}\} \cup \{(x_j^+,t_j) : j \in J'_0\}$.
4: Merge the two DPCs through edges (x_j, x_j^+) for $j \in J'_0$.

Claim 5. When $f_0 \leq f-2$, or $k_0+k'_0 < k$ and $f_0+k_0 \geq 1$, Procedure Find-PDPC-C builds a paired k-DPC in $G-F$ unless (i) $k_0+k'_0 = 0$, (ii) $k_0 = k$, or (iii) $k_0+k'_0 = 1$, $k = 2$, and $f_0 = f$.

Proof. Suppose $k_0+k'_0 \geq 1$. The k'_0 free edges of Step 1 exist because $|V(G_0)| - (f+2k) \geq 3^{m-1} - 2m \geq m \geq k \geq k'_0$. The $(k_0+k'_0)$-DPC of G_0-F_0 exists when $k_0+k'_0 \geq 2$, because $f_0+2(k_0+k'_0) \leq f+2k-2 = 2m-2$; the $(k_0+k'_0)$-DPC when $k_0+k'_0 = 1$, which is made of a Hamiltonian path of G_0-F_0, also exists unless $k=2$ and $f_0 = f$, because $f_0 = 2m-2k-(f-f_0) \leq 2m-5$. Finally, the paired $(k-k_0)$-DPC of $H-F_{1,d-1}$ exists unless $k_0 = k$ by Theorems 1 and 2, because $f_0+k_0 \geq 1$, i.e., $f_{1,d-1} = f-f_0 \leq f-1$ or $k-k_0 \leq k-1$. So, the claim is proven. □

Consider the exceptional case (i) of Claim 5 where $k_0+k'_0 = 0$. If $k \geq 3$ or $f_0 < f$, then $f_0 \leq 2m-5$, hence it suffices to build a paired k-DPC joining S and T in $H-F_{1,d-1}$, which exists by Theorem 2, and then replace an edge $(x,y) \in E(G_1)$ on a path in the DPC such that $x^-, y^- \notin F$ with a Hamiltonian x^-–y^- path of G_0-F_0, as illustrated in Fig. 5(b). Now, let $k=2$ and $f_0 = f$ $(= 2m-4)$. If G_1 or G_{d-1}, say G_1, contains no terminal, we can build a required 2-DPC analogously from a paired 2-DPC of $G_{2,d-1}$ and a Hamiltonian path of $G_{0,1} - F_0$. Note that $G_{0,1} - F_0$ is Hamiltonian-connected by Theorem 1. If G_1 contains a single terminal, say s_1, then for some free edge (x, x^+) with $x \in V(G_1)$, it suffices to build a Hamiltonian s_1–x path in $G_{0,1} - F_0$ and a paired 2-DPC joining $\{(x^+, t_1), (s_2, t_2)\}$ in $G_{2,d-1}$, and then merge them into a required 2-DPC. Finally, if G_1 contains two terminals, say s_1 and s_2, then for some edge $(x, y) \in E(G_1)$ on a Hamiltonian s_1–s_2 path of $G_{0,1} - F_0$ such that both x^+ and y^+ are nonterminals, it suffices to combine a paired 2-DPC of $G_{2,d-1}$ with the s_1–x and s_2–y paths properly.

For the exceptional case (ii) $k_0 = k$ ($f_0 \leq f - 2$), it suffices to build a paired k-DPC in $G_0 - F_0$ and replace an edge (x, y) on a path in the DPC such that $x^+, y^+ \notin F$ with a Hamiltonian x^+–y^+ path of $H - F_{1,d-1}$. Now, consider the exceptional case (iii) $k_0 + k'_0 = 1$, $k = 2$, and $f_0 = f$ ($= 2m - 4$). There exists a Hamiltonian cycle C in $G_0 - F_0$. Firstly, suppose $k_0 = 1$, i.e., $s_1, t_1 \in V(G_0)$. We can extract two disjoint s_1–t_1 and x–y paths from C for some distinct vertices x and y. If there are nonterminal neighbors $x', y' \in V(H)$ of x and y, respectively, we can build an s_2–t_2 path from a paired 2-DPC of H joining $\{(s_2, x'), (y', t_2)\}$ and the x–y path. If no such nonterminal neighbors exist, then assuming w.l.o.g. $x^+ = s_2$ and $x^- = t_2$, concatenating one-vertex path $\langle s_2 \rangle$, the x–y path, and a Hamiltonian y^+–t_2 path of $H - \{s_2\}$ results in an s_2–t_2 path. Secondly, suppose $k'_0 = 1$, i.e., $s_1 \in V(G_0)$. It suffices to extract a Hamiltonian s_1–x path from C for some vertex x that has a nonterminal neighbor x' in H, and combine the path with a paired 2-DPC of H joining $\{(x', t_1), (s_2, t_2)\}$.

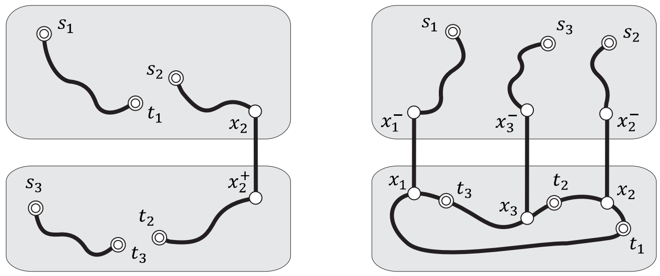
Case 2: $f_0 = f - 1$. We have $f_0 \geq 1$ also in this case because $f \geq 1$. We assume w.l.o.g. $f_1 \leq f_{d-1}$, so $f_1 = 0$ and $f_{0,1} = f - 1$; also, assume $s_j \in V(G_{0,1})$ for all $j \in J'_{0,1}$. Let $H = G_2 \oplus \cdots \oplus G_{d-1}$, possibly made of a single component.

Procedure Find-PDPC-D(S, T, F, G) // $f_0 = f - 1$. See Fig. 6(a).

1: Pick up a free edge (x_j, x_j^+) with $x_j \in V(G_1)$ for each $j \in J'_{0,1}$.
2: Build a paired $(k_{0,1} + k'_{0,1})$-DPC in $G_{0,1} - F_{0,1}$ joining $\{(s_i, t_i) : i \in J_{0,1}\} \cup \{(s_j, x_j) : j \in J'_{0,1}\}$.
3: Build a paired $(k - k_{0,1})$-DPC in $H - F_{2,d-1}$ joining $\{(s_i, t_i) : i \in J_{2,d-1}\} \cup \{(x_j^+, t_j) : j \in J'_{0,1}\}$.
4: Merge the two DPCs through edges (x_i, x_j^+) for $j \in J'_{0,1}$.

Claim 6. When $f_0 = f - 1$, Procedure Find-PDPC-D builds a paired k-DPC in $G - F$ unless (i) $k_{0,1} + k'_{0,1} = 0$, (ii) $k_{0,1} = k$, or (iii) $d = 3$, $f_0 = f_2 = f - 1 = 1$, $S \subset V(G_0)$, and $T \subset V(G_2)$.

Proof. Suppose $k_{0,1} + k'_{0,1} \geq 1$. The existence of $k'_{0,1}$ free edges of Step 1 is obvious (because $3^{m-1} - (f + 2k) \geq m$). The paired DPC of Step 2 exists by Theorems 1 and 2 (because $f_{0,1} = f - 1$). Also, the paired DPC of Step 3 exists unless $k_{0,1} = k$ by Theorems 1 and 2 if $d \geq 4$; by the hypothesis of the theorem if (a) $d = 3$ and (b) $f_2 \leq f - 2$ or $k_{0,1} \geq 1$. That is, the existence is guaranteed unless $d = 3$, $f_2 = f - 1$, and $k_{0,1} = 0$, or equivalently, $d = 3$, $f_0 = f_2 = f - 1 = 1$, $S \subset V(G_0)$, and $T \subset V(G_2)$ from the assumption (3). Thus, the claim is proven. ∎



(a) Find-PDPC-D (b) Exceptional case (iii)

Fig. 6: Illustrations of Case 2 in the proof of Theorem 4.

In this Case 2 where $f_0 = f - 1$, each $G_i - F_i$ is Hamiltonian-connected because $f_i \leq f_0 = f - 1 = 2m - 2k - 1 \leq 2m - 5$. So, a required DPC for the exceptional case (i) can be built from a paired k-DPC of $G_{1,d-1} - F_{1,d-1}$ and a Hamiltonian path of $G_0 - F_0$. Analogously, a required DPC for the exceptional case (ii) can be obtained from a paired k-DPC of $G_{0,1} - F_{0,1}$ and a Hamiltonian path of $H - F_{2,d-1}$. For the exceptional case (iii) finally, we first build a Hamiltonian cycle $\langle x_1, \dots, t_1, \dots, x_k, \dots, t_k \rangle$ in $G_2 - F_2$, from which we extract k disjoint paths, say x_j–t_j paths for $j \in J$, that collectively cover $G_2 - F_2$. (See Fig. 6(b).) It suffices to build a paired k-DPC in $G_{0,1}$ joining $\{(s_j, x_j^-) : j \in J\}$, and then combine the k-DPC with the k disjoint x_j–t_j paths through edges (x_j^-, x_j) for $j \in J$.

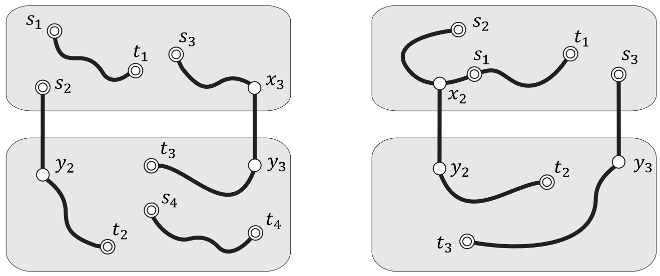
Case 3: $f_0 = f$ *and* $k_0 \geq 1$. We assume w.l.o.g. $s_j \in V(G_0)$ for all $j \in J_0'$, and let $H = G_1 \oplus \cdots \oplus G_{d-1}$. In addition, let $P = \{j \in J_0' :$ both s_j^+ and s_j^- are terminals$\}$, $Q = J_0' \setminus P$, $p = |P|$, and $q = |Q|$, so that $p + q = |J_0'| = k_0'$.

Procedure Find-PDPC-E(S, T, F, G) // $f_0 = f$ and $k_0 \geq 1$. See Fig. 7(a).

1: For each $j \in J_0'$, let an edge (x_j, y_j), where $x_j \in V(G_0)$, be
$$\begin{cases} (s_j, s_j^+) \text{ if } s_j^+ \text{ is not a terminal,} \\ (s_j, s_j^-) \text{ if } s_j^+ \text{ is a terminal but } s_j^- \text{ is not,} \\ \text{a free edge between } G_0 \text{ and } H \text{ if both } s_j^+ \text{ and } s_j^- \text{ are terminals.} \end{cases}$$
2: Build a paired $(k_0 + p)$-DPC in $G_0 - (F_0 \cup F')$ joining $\{(s_i, t_i) : i \in J_0\} \cup \{(s_j, x_j) : j \in P\}$, where $F' = \{s_j : j \in Q\}$.
3: Build a paired $(k - k_0)$-DPC in H joining $\{(s_i, t_i) : i \in J_{1,d-1}\} \cup \{(y_j, t_j) : j \in J_0'\}$.
4: Merge the two DPCs through edges (x_j, y_j) for $j \in J_0'$.

Claim 7. When $f_0 = f$ and $k_0 \geq 1$, Procedure Find-PDPC-E builds a paired k-DPC in $G - F$ unless (i) $k_0 + p \geq 2$, $k_0 + k_0' = k$, and $k_0' \in \{0, 1\}$, or (ii) $k_0 + p = 1$, $k = 3$, and $k_0 + k_0' = k$, or (iii) $k_0 + p = 1$ and $k = 2$.

Proof. The existence of such k_0' edges in Step 1 is straightforward. When $k_0 + p \geq 2$, the $(f_0 + q)$-fault paired $(k_0 + p)$-DPC exists if $k_0 + k_0' < k$ or $k_0' \geq 2$ because $(f_0 + q) + 2(k_0 + p) = f_0 + 2(k_0 + k_0') - q = (f + 2k) - 2(k -$



(a) Find-PDPC-E (b) Exceptional case (ii)

Fig. 7: Illustrations of Case 3 in the proof of Theorem 4.

$(k_0 + k_0')) - q \leq 2m - 2$. Note that when $k_0 + k_0' = k$, the graph H should contain no less than $2p$ terminals, i.e., $k_0' = p + q \geq 2p$, leading to $q \geq p$ and $q \geq \lceil \frac{k_0'}{2} \rceil$; moreover, if $k_0' = 2$, then $q = 2$ by the assumption (4). When $k_0 + p = 1$ ($k_0 = 1$ and $p = 0$), the DPC of Step 2 is, in fact, a Hamiltonian path (joining s_1 and t_1), which exits if $k \geq 4$ or $k = 3$ and $k_0 + k_0' < k$ because $f_0 + q = (f_0 + 2k) - k - (k - q) = 2m - k - (k - (k_0 + k_0') + 1) \leq 2m - 5$. Also, the $(k - k_0)$-DPC of H exists by Theorems 1 and 2 (unless $k_0 = k$, a special subcase of the exceptional case (i)), proving the claim. □

Firstly, we consider the exceptional case (i) $k_0 + p \geq 2$, $k_0 + k_0' = k$, and $k_0' \in \{0, 1\}$. If $k_0' = 0$, i.e., $S \cup T \subseteq V(G_0)$, then from a Hamiltonian cycle of $G_0 - F_0$, we first extract $2k$ disjoint paths, s_j–x_j and t_j–y_j paths for $j \in J$, that collectively cover $G_0 - F_0$. It suffices to combine the $2k$ paths with a paired $(k-1)$-DPC of G_1 joining $\{(x_j^+, y_j^+) : j \in J \setminus \{k\}\}$ and a Hamiltonian x_k^-–y_k^- path of $G_{2,d-1}$. Analogously, we extract $2k-1$ disjoint paths from a Hamiltonian cycle of $G_0 - F_0$ if $k_0' = 1$, i.e., $S \cup (T \setminus \{t_k\}) \subseteq V(G_0)$. The disjoint paths are denoted by s_j–x_j and t_j–y_j paths for $j \in J \setminus \{k\}$, and s_k–x_k path. Assuming w.l.o.g. $t_k \notin V(G_1)$, a required DPC can be built from a paired $(k-1)$-DPC of G_1 joining $\{(x_j^+, y_j^+) : j \in J \setminus \{k\}\}$ and a Hamiltonian x_k^-–t_k path of $G_{2,d-1}$, provided $x_k^- \neq t_k$. If $x_k^- = t_k$, it suffices to combine the s_k–x_k path with $\langle t_k \rangle$ into an s_k–t_k path, and replace an edge $(u, v) \in E(G_1)$ on a path in the DPC such that $u^+, v^+ \neq t_k$ with a Hamiltonian u^+–v^+ path of $G_{2,d-1} - \{t_k\}$.

Now, consider the second exceptional case (ii) $k_0 + p = 1$, $k = 3$, and $k_0 + k_0' = k$, which lead to $k_0 = 1$ and $k_0' = 2$, so $\{s_1, s_2, s_3, t_1\} \subseteq V(G_0)$ and $\{t_2, t_3\} \subseteq V(H)$. There is a Hamiltonian s_2–t_1 path in $G_0 - (F_0 \cup \{s_3\})$, represented as $\langle s_2, \ldots, x_2, s_1, \ldots, t_1 \rangle$, as shown in Fig. 7(b), because $f_0 + 1 = 2m - 2k + 1 = 2m - 5$. Also, there is a nonterminal neighbor $y_3 \in V(H)$ of s_3 because $p = 0$. If there is a nonterminal neighbor $y_2 \in V(H)$ of x_2, it suffices to build a paired 2-DPC in H joining $\{(y_2, t_2), (y_3, t_3)\}$; if the two neighbors in H of x_2 are terminals, t_2 and t_3, then it suffices to combine the s_2–x_2 path with $\langle t_2 \rangle$ into an s_2–t_2 path, and then build a Hamiltonian y_3–t_3 path in $H - \{t_2\}$. Finally, consider the exceptional case (iii) $k_0 + p = 1$ and $k = 2$, leading to

$k_0 = 1$ and $f_0 = 2m - 4$. The subcase where $k'_0 = 0$ is reduced to Case 1, so we assume $k'_0 = 1$, i.e., $\{s_1, t_1, s_2\} \subseteq V(G_0)$ and $t_2 \in V(H)$. Assuming w.l.o.g. $t_2 \notin V(G_1)$, we build a Hamiltonian s_2–t_1 path $\langle s_2, \dots, x_2, s_1, \dots, t_1 \rangle$ in $G_{0,1} - F_0$. If the neighbor $y_2 \in V(G_{2,d-1})$ of x_2 is not equal to t_2, it suffices to build a Hamiltonian y_2–t_2 path in $G_{2,d-1}$; if $y_2 = t_2$, it suffices to merge the s_2–x_2 path and $\langle t_2 \rangle$ into an s_2–t_2 path, and then replace an edge $(u, v) \in E(G_1)$ of the s_1–t_1 or s_2–x_2 path such that $u^+, v^+ \neq t_2$ with a Hamiltonian u^+–v^+ path of $G_{2,d-1} - \{t_2\}$.

Case 4: *$f_0 = f$ and $k_0 = 0$.*

The case where $k_0 + k'_0 < k$ and $f_0 + k_0 \geq 1$ was dealt with in Case 1, so we assume $k_0 + k'_0 = k$ or $f_0 + k_0 = 0$, i.e., $k'_0 = k$ or $f = 0$. Also, we have $k'_0 \geq 1$ from the assumption (3). Assume w.l.o.g. $\{s_1, \dots, s_{k'_0}\} = V(G_0) \cap (S \cup T)$ and moreover, $i \leq j$ if $s_p \in V(G_i)$ and $t_p \in V(G_j)$ for all $p \in \{1, \dots, k\}$.

Case 4.1: *There exist $a, b \in J'_0$ such that $t_a \in V(G_{1,r-1})$ and $t_b \in V(G_{r,d-1})$ for some $r \in \{2, \dots, d-1\}$.* Let $H = G_{1,r-1}$ and $H' = G_{r,d-1}$. Also, let $P = \{j \in J'_0 : t_j \in V(H)$ and s_j^+ is a terminal$\}$ and $Q = \{j \in J'_0 : t_j \in V(H)$ and s_j^+ is not a terminal$\}$. Analogously, let $P' = \{j \in J'_0 : t_j \in V(H')$ and s_j^- is a terminal$\}$ and $Q' = \{j \in J'_0 : t_j \in V(H')$ and s_j^- is not a terminal$\}$. So, $P \cup Q \cup P' \cup Q' = J'_0$. We assume w.l.o.g. $a \in P$ if $P \neq \emptyset$; also, assume $b \in P'$ if $P' \neq \emptyset$.

Procedure FIND-PDPC-F(S, T, F, G) // $t_a \in V(H)$ and $t_b \in V(H')$. See Fig. 8.

1: For each $j \in J'_0 \setminus \{a, b\}$, let an edge (x_j, y_j), where $x_j \in V(G_0)$, be
$$\begin{cases} (s_j, s_j^+) & \text{if } j \in Q, \\ (s_j, s_j^-) & \text{if } j \in Q', \\ \text{a free edge between } G_0 \text{ and } H \text{ with } (s_j, x_j) \in E(G_0) \setminus F_0 & \text{if } j \in P, \\ \text{a free edge between } G_0 \text{ and } H' \text{ with } (s_j, x_j) \in E(G_0) \setminus F_0 & \text{if } j \in P'. \end{cases}$$
2: Build a Hamiltonian s_a–s_b path in $G_0 - (F_0 \cup F')$, where $F' = \{s_i, x_i : i \in (P \cup P') \setminus \{a, b\}\} \cup \{s_j : j \in (Q \cup Q') \setminus \{a, b\}\}$. Through an edge (x_a, x_b) on the Hamiltonian path such that neither x_a^+ nor x_b^- is a terminal, divide the Hamiltonian path into s_a–x_a and s_b–x_b paths. Also, let an s_i–x_i path be $\langle s_i, x_i \rangle$ for $i \in (P \cup P') \setminus \{a, b\}$; let an s_j–x_j path be a one-vertex path $\langle s_j \rangle$ for $j \in (Q \cup Q') \setminus \{a, b\}$.
3: For each $j \in J'_{1,r-1} \cap J'_{r,d-1}$, pick up a free edge (x_j, y_j) with $x_j \in V(G_{r-1})$ and $y_j \in V(G_r)$.
4: Build a paired $(k_{1,r-1} + k'_{1,r-1})$-DPC in H joining $\{(s_i, t_i) : i \in J_{1,r-1}\} \cup \{(y_j, t_j) : j \in P \cup Q\} \cup \{(s_j, x_j) : j \in J'_{1,r-1} \cap J'_{r,d-1}\}$, where $y_a = x_a^+$.
5: Build a paired $(k_{r,d-1} + k'_{r,d-1})$-DPC in H' joining $\{(s_i, t_i) : i \in J_{r,d-1}\} \cup \{(y_j, t_j) : j \in P' \cup Q'\} \cup \{(y_j, t_j) : j \in J'_{1,r-1} \cap J'_{r,d-1}\}$, where $y_b = x_b^-$.
6: Combine the k'_0 paths of $G_0 - F_0$ with the two DPCs through edges (x_j, y_j) for $j \in J'_0 \cup (J'_{1,r-1} \cap J'_{r,d-1})$.

Claim 8. When $t_a \in V(H)$ and $t_b \in V(H')$, Procedure FIND-PDPC-F builds a paired k-DPC in $G - F$ unless $k'_0 = k$ and $(Q \cup Q') \setminus \{a, b\} = \emptyset$.

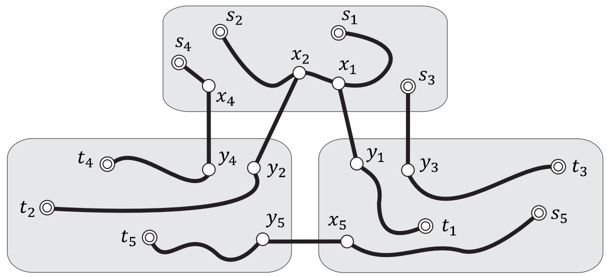


Fig. 8: Illustration of Procedure Find-PDPC-F, where $a = 1$ and $b = 2$.

Proof. The Hamiltonian s_a–s_b path of Step 2 exists unless $k'_0 = k$ and $(Q \cup Q') \setminus \{a, b\} = \emptyset$, because $|F_0 \cup F'| = f_0 + 2 \cdot |(P \cup P') \setminus \{a, b\}| + |(Q \cup Q') \setminus \{a, b\}| \le f_0 + 2(k-2) - 1 \le f + 2k - 5 = 2m - 5$. The paired DPCs of Steps 4 and 5 exist by Theorems 1 and 2, or by the hypothesis of Theorem 4, because $b \notin J_{1,r-1} \cup J'_{1,r-1}$ and $a \notin J_{r,d-1} \cup J'_{r,d-1}$. □

Consider the exceptional case where $k'_0 = k$ and $(Q \cup Q') \setminus \{a, b\} = \emptyset$. Note that $Q = \emptyset$ if $|P \cup Q| \ge 2$, by the assumption that $a \in P$ if $P \ne \emptyset$; symmetrically, $Q' = \emptyset$ if $|P' \cup Q'| \ge 2$. Instead of the s_a–s_b Hamiltonian path in Step 2 of the procedure, we will use a Hamiltonian cycle C of $G_0 - (F_0 \cup F')$, which exists because $f_0 + 2(k - 2) = f + 2k - 4 = 2m - 4$. If $k'_0 \ge 3$, it suffices to extract s_a–x_a and s_b–x_b paths satisfying $x_a^+, x_b^- \notin T$ from the Hamiltonian cycle C. If $k'_0 = 2$, it suffices to extract s_a–x_a and s_b–x_b paths from the Hamiltonian cycle C such that $x_a^+, x_b^- \notin T$, or $x_a^+ \notin T$ and $x_b^- = t_b$, or $x_b^- \notin T$ and $x_a^+ = t_a$, and then combine the two paths with a paired 2-DPC or 1-fault Hamiltonian path of $G_{1,d-1}$.

Case 4.2: *There exists $r \in \{1, \dots, d-1\}$ such that $t_j \in V(G_r)$ for all $j \in J'_0$.* The fact $k'_0 = k$ or $f = 0$ leads to $\{t_1, \dots, t_{k'_0}\} = V(G_r) \cap (S \cup T)$ from the assumption (3). Firstly, suppose $f_0 = f \ge 1$ ($k'_0 = k$). It follows that $S \subseteq V(G_0)$ and $T \subseteq V(G_r)$. Similar to the exceptional case (iii) of Claim 6, we can easily build a paired k-DPC as follows: From a Hamiltonian cycle $\langle s_1, \dots, x_1, \dots, s_k, \dots, x_k \rangle$ of $G_0 - F_0$, we extract k disjoint s_j–x_j paths for $j \in J$ that collectively cover $G_0 - F_0$. Assuming w.l.o.g. $r \ne 1$, it suffices to combine the k disjoint paths with a paired k-DPC of $G_{1,d-1}$ joining $\{(x_j^+, t_j) : j \in J\}$ through edges (x_j, x_j^+) for $j \in J$. Hereafter in this case, we assume $f_0 = f = 0$, and let $H = G_{1,r-1}$ and $H' = G_{r+1,d-1}$.

Secondly, suppose $k'_0 < k$ ($f_0 = 0$ and $k \ge 3$). If $r = 1$ or $r = d - 1$, say $r = 1$, the union of a paired k'_0-DPC of $G_{0,1}$ and a paired $(k - k'_0)$-DPC of $G_{2,d-1}$ forms a required k-DPC. So, let $r \notin \{1, d-1\}$, leading to that H and H' are both nonempty. If $s_p, t_p \in V(H)$ or $s_p, t_p \in V(H')$ for some $p \in J$, say $s_p, t_p \in V(H)$, then for $J' = \{j \in J : s_j \in V(H) \text{ and } t_j \in V(H')\}$, it suffices to pick up free edges (x_j, y_j) between H and $G - H$ for $j \in J'$ and then merge a paired $(k_{1,r-1} + k'_{1,r-1})$-DPC of H and a paired $(k - k_{1,r-1})$-DPC of $G - H$ into a required k-DPC. So, it remains a case that $s_{k'_0+1}, \dots, s_k \in V(H)$ and

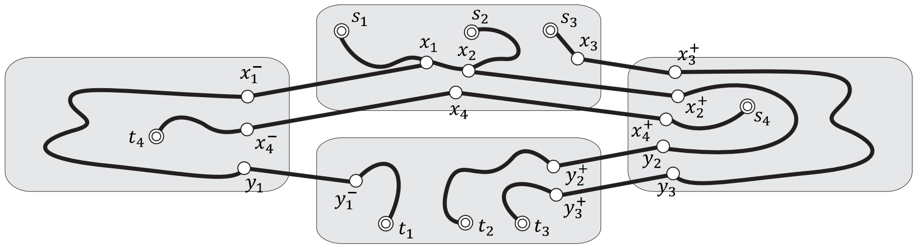


Fig. 9: Illustration of Procedure FIND-PDPC-G.

$t_{k'_0+1}, \ldots, t_k \in V(H')$. If $k'_0 \geq 2$, a required DPC can be constructed as follows:

Procedure FIND-PDPC-G(S, T, F, G) // $2 \leq k'_0 < k$, $\{s_{k'_0+1}, \ldots, s_k\} \subseteq V(H)$, and $\{t_{k'_0+1}, \ldots, t_k\} \subseteq V(H')$. See Fig. 9.

1: For each $j \in J \setminus J'_0$, pick up a vertex $x_j \in V(G_0)$ such that (x_j, x_j^+) and (x_j, x_j^-) are both free.
2: For each $i \in J'_0 \setminus \{1,2\}$, let (x_i, x_i^+) be a free edge between G_0 and G_1 with $(s_i, x_i) \in E(G_0)$.
3: Build a Hamiltonian s_1–s_2 path in $G_0 - F'$, where $F' = \{x_j : j \in J \setminus J'_0\} \cup \{s_i, x_i : i \in J'_0 \setminus \{1,2\}\}$. Through an edge (x_1, x_2) on the Hamiltonian path such that $x_1^-, x_2^+ \notin S \cup T$, divide the Hamiltonian path into s_1–x_1 and s_2–x_2 paths. Also, let an s_i–x_i path be $\langle s_i, x_i \rangle$ for $i \in J'_0 \setminus \{1,2\}$.
4: Pick up a free edge (y_i, y_i^+) with $y_i \in V(G_{r-1})$ for each $i \in J'_0 \setminus \{1\}$, and build a paired $(k-1)$-DPC in H joining $\{(s_j, x_j^+) : j \in J \setminus J'_0\} \cup \{(x_i^+, y_i) : i \in J'_0 \setminus \{1\}\}$.
5: Pick up a free edge (y_1, y_1^-) with $y_1 \in V(G_{r+1})$, and build a paired $(k - k'_0 + 1)$-DPC in H' joining $\{(x_j^-, t_j) : j \in J \setminus J'_0\} \cup \{(x_1^-, y_1)\}$.
6: Build a paired k'_0-DPC in G_r joining $\{(y_i^+, t_i) : i \in J'_0 \setminus \{1\}\} \cup \{(y_1^-, t_1)\}$.
7: Combine the s_i–x_i paths of G_0 with the three DPCs of H, H', and G_r.

Claim 9. When $2 \leq k'_0 < k$, $\{s_{k'_0+1}, \ldots, s_k\} \subseteq V(H)$, and $\{t_{k'_0+1}, \ldots, t_k\} \subseteq V(H')$, Procedure FIND-PDPC-G builds a paired k-DPC in $G - F$.

Proof. The Hamiltonian s_1–s_2 path of $G_0 - F'$ exists because $|F'| \leq 2(k'_0 - 2) + (k - k'_0) = k + k'_0 - 4 \leq 2k - 5 = 2m - 5$. The DPCs of H, H', and G_r exist by Theorem 2, or by the hypothesis of Theorem 4. □

Now, let $k'_0 = 1$. Note that each component contains at most one terminal from the assumption (3). Let $i \in \{1, \ldots, r-1\}$ be the smallest index such that G_i contains a source, say s_2, so that $G_{0,i}$ contains two terminals, s_1 and s_2. We first pick up two free edges (x_1, x_1^+) and (x_2, x_2^-) with $x_1 \in V(G_i)$ and $x_2 \in V(G_0)$, and build a paired 2-DPC in $G_{0,i}$ joining $\{(s_1, x_1), (s_2, x_2)\}$. For $k-2$ free edges (y_j, y_j^+), $j \in J \setminus \{1,2\}$, with $y_j \in V(G_r)$, it suffices to build a paired $(k-1)$-DPC in $G_{i+1,r}$ joining $\{(x_1^+, t_1)\} \cup \{(s_j, y_j) : j \in J \setminus \{1,2\}\}$ and a paired $(k-1)$-DPC in $G_{r+1,d-1}$ joining $\{(x_2^-, t_2)\} \cup \{(y_j^+, t_j) : j \in J \setminus \{1,2\}\}$, and then merge the three DPCs.

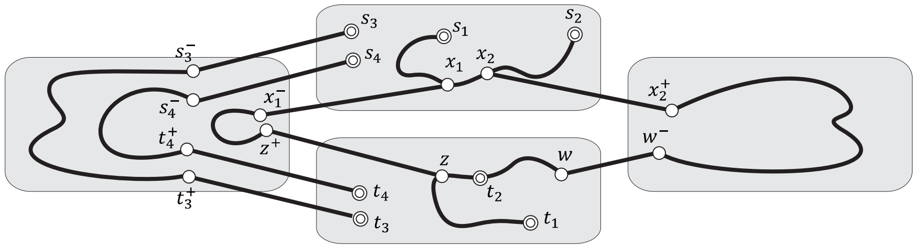


Fig. 10: Illustration of Procedure FIND-PDPC-H.

Finally, suppose $k'_0 = k$ ($f_0 = 0$ and $k \geq 3$), leading to $S \subseteq V(G_0)$ and $T \subseteq V(G_r)$. We assume w.l.o.g. $r \leq \lfloor \frac{d}{2} \rfloor$, so that $G_{r+1,d-1}$ has no fewer components than $G_{1,r-1}$. There are three procedures respectively dealing with the cases (i) $r \leq d-3$, (ii) $r = d-2$ and $k \geq 4$, and (iii) $r = d-2$ and $k = 3$. Note that $r = d-2$ only if $d = 4$ and $r = 2$ or $d = 3$ and $r = 1$.

Procedure FIND-PDPC-H(S, T, F, G) // $k'_0 = k$ and $r \leq d-3$. See Fig. 10.

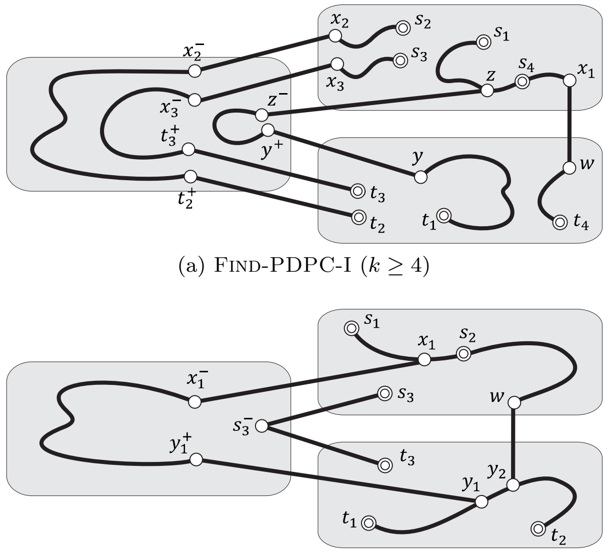
1: Build a Hamiltonian s_1–s_2 path in $G_0 - F'$, where $F' = \{s_j : j \in J \setminus \{1,2\}\}$. Through an edge (x_1, x_2) on the Hamiltonian path such that $x_2^+ \notin T$, divide the Hamiltonian path into s_1–x_1 and s_2–x_2 paths.
2: if $r = 1$, let $w = x_2^+$; otherwise, pick up a free edge (w, w^-) with $w \in V(G_r)$ and $w^- \neq x_2^+$, and then build a Hamiltonian x_2^+–w^- path in $G_{1,r-1}$.
3: Build a Hamiltonian w–t_1 path in $G_r - F''$, where $F'' = \{t_j : j \in J \setminus \{1,2\}\}$, and divide the Hamiltonian path into the w–t_2 and z–t_1 paths for the successor z of t_2.
4: Build a paired $(k-1)$-DPC in $G_{r+1,d-1}$ joining $\{(x_1^-, z^+)\} \cup \{(s_j^-, t_j^+) : j \in J \setminus \{1,2\}\}$.
5: Combine the Hamiltonian paths with the $(k-1)$-DPC.

Claim 10. When $k'_0 = k$ and $r \leq d-3$, Procedure FIND-PDPC-H builds a paired k-DPC in G.

Proof. The proof is straightforward. Note that the Hamiltonian paths in Steps 1 and 3 exist because $|F'| = |F''| = k-2 = m-2 \leq 2m-5$ for $m \geq 3$. $\square$

Procedure FIND-PDPC-I(S, T, F, G) // $k'_0 = k$, $r = d-2$, and $k \geq 4$. See Fig. 11(a).

1: For each $j \in J \setminus \{k\}$, pick up a nonterminal $x_j \in V(G_0)$ such that $x_j^+ \notin T$ and $x_j^- \notin \{t_i^+ : i \in J\}$.
2: Build a paired $(k-1)$-DPC in G_0 joining $\{(s_j, x_j) : j \in J \setminus \{k\}\}$. There is a path, say the s_1–x_1 path, in the DPC that passes through s_k, so the s_1–x_1 path is represented as $\langle s_1, \ldots, z, s_k, \ldots, x_1 \rangle$ where z is the predecessor of s_k.
3: If $r = 1$, let $w = x_1^+$; otherwise, pick up a free edge (w, w^-) with $w \in V(G_r)$ and $w^- \neq x_1^+$, and then build a Hamiltonian x_1^+–w^- path in $G_{1,r-1}$.
4: Let $q \in J \setminus \{k\}$ be an index such that $t_j^+ \neq z^-$ for all $j \in J \setminus \{q, k\}$.



(a) FIND-PDPC-I ($k \geq 4$)

(b) FIND-PDPC-J ($k = 3$)

Fig. 11: Illustrations of Case 4.2 in the proof of Theorem 4, where $k'_0 = k$ and $r = d - 2$.

5: Pick up a free edge (y, y^+) with $y \in V(G_r)$ such that $y \neq w$ and $y^+ \notin \{z^-\} \cup \{x_j^- : j \in J \setminus \{1, k\}\}$.
6: Build a paired 2-DPC in $G_r - F'$ joining $\{(w, t_k), (y, t_q)\}$, where $F' = \{t_j : j \in J \setminus \{q, k\}\}$.
7: If $q = 1$, build a paired $(k-1)$-DPC in G_{d-1} joining $\{(z^-, y^+)\} \cup \{(x_j^-, t_j^+) : j \in J \setminus \{1, k\}\}$; otherwise, build a paired $(k-1)$-DPC joining $\{(z^-, t_1^+), (x_q^-, y^+)\} \cup \{(x_j^-, t_j^+) : j \in J \setminus \{1, q, k\}\}$.
8: Merge the three DPCs into a paired k-DPC of G.

Claim 11. When $k'_0 = k$, $r = d-2$, and $k \geq 4$, Procedure FIND-PDPC-I builds a paired k-DPC in G.

Proof. The proof is similar to those of the previous claims. The $k-1$ nonterminals $x_1, \ldots, x_{k-1}$ exist in G_0 because $|V(G_0)| - 3k \geq 3^{m-1} - 3m \geq m - 1 = k - 1$ for $m = k \geq 4$. The paired 2-DPC of Step 6 exists because $|F'| + 2 \cdot 2 = (k-2) + 4 \leq (k-2) + k = 2m - 2$. □

Procedure FIND-PDPC-J(S, T, F, G) // $k'_0 = k$, $r = d - 2$, and $k = 3$. See Fig. 11(b).

1: Build a Hamiltonian t_1–t_2 path P in $G_r - \{t_3\}$.
2: Case when $s_3^- = t_3^+$: Let an s_3–t_3 path of the DPC be $\langle s_3, s_3^-, t_3 \rangle$.
   a: Pick up an edge (y_1, y_2) on the Hamiltonian path P such that $y_1^+ \notin \{v^- : (v, s_2) \in E(G_0)\}$ and $y_2^- \notin \{s_1, s_3\}$.
   b: If $r = 1$, let $w = y_2^-$; otherwise, pick up a free edge (w, w^+) with $w \in V(G_0)$ and $w^+ \neq y_2^-$, and then build a Hamiltonian w^+–y_2^- path

in $G_{1,r-1}$.

c: Build a Hamiltonian s_1–w path in $G_0 - \{s_3\}$, which is represented as $\langle s_1, \ldots, x_1, s_2, \ldots, w\rangle$ where x_1 is the predecessor of s_2.

d: Build a Hamiltonian x_1^-–y_1^+ path in $G_{d-1} - \{s_3^-\}$, and merge the Hamiltonian paths into s_1–t_1 and s_2–t_2 paths of the DPC.

3: Case when $s_3^- \neq t_3^+$: We will build an s_3–t_3 path that passes through only vertices contained in G_{d-1} as intermediate vertices.

a: Build a Hamiltonian s_1–s_2 path P' in $G_0 - \{s_3\}$.

b: Pick up a pair of vertices x_2 on P' and y_2 on P such that (i) $(x_2, y_2) \in E(G)$ if $r = 1$, (ii) $x_2^+ \neq y_2^-$ if $r \geq 2$, (iii) x_2 and y_2 respectively have predecessors x_1 and y_1 (i.e., $x_2 \neq s_1$ and $y_2 \neq t_1$), and (iv) $x_1^- \neq t_3^+$ and $y_1^+ \neq s_3^-$.

c: If $r \geq 2$, build a Hamiltonian x_2^+–y_2^- path in $G_{1,r-1}$.

d: Build a paired 2-DPC in G_{d-1} joining $\{(x_1^-, y_1^+), (s_3^-, t_3^+)\}$ if $x_1^- \neq y_1^+$; build a Hamiltonian s_3^-–t_3^+ path in $G_{d-1} - \{x_1^-\}$ otherwise.

e: Combine the 2-DPC (or the Hamiltonian s_3^-–t_3^+ path) of G_{d-1} with the Hamiltonian paths P and P' into a paired 3-DPC.

Claim 12. When $k'_0 = k$, $r = d-2$, and $k = 3$, Procedure FIND-PDPC-J builds a paired k-DPC in G.

Proof. Each component of G is a 2-dimensional torus-like graph because $k = m = 3$. The existence of a 1-fault Hamiltonian path or of a paired 2-DPC in a component is due to the hypothesis of this theorem. The edge (y_1, y_2) of Step 2a exists because $(|V(G_r)|-2)-(4+2) \geq (3^2-2)-(4+2) = 1$. Also, the vertex pair x_2 and y_2 of Step 3b exist because $(|V(G_0)|-2)-(2+2) \geq 3^2-2-(2+2) \geq 1$ if $r = 1$, and because $(|V(G_0)|-3)(|V(G_r)|-4) \geq 1$ if $r \geq 2$. □

Therefore, the entire proof of Theorem 4 is complete. ■

Combining Lemmas 1 and 5 with Theorems 3 and 4 leads to that:

Theorem 5. *Every m-dimensional nonbipartite torus, $m \geq 2$, is f-fault paired k-disjoint path coverable for any f and $k \geq 2$ subject to $f + 2k \leq 2m$. Furthermore, the graph is $(2m-3)$-fault Hamiltonian-connected and $(2m-2)$-fault Hamiltonian.*

Proof. Let G be an m-dimensional nonbipartite torus $C_{d_1} \times \cdots \times C_{d_m}$ with d_1 being odd. The proof is by induction on m. If $m = 2$, the theorem holds true from Lemmas 1 and 5. Let $m \geq 3$. Then, by the induction hypothesis, the $(m-1)$-dimensional torus $C_{d_1} \times \cdots \times C_{d_{m-1}}$, denoted by G', is f-fault paired k-disjoint path coverable for any f and $k \geq 2$ subject to $f + 2k \leq 2m - 2$, $(2m-5)$-fault Hamiltonian-connected, and $(2m-4)$-fault Hamiltonian. So, the m-dimensional torus G, which is isomorphic to $G' \times C_{d_m}$, is f-fault paired k-disjoint path coverable for any f and $k \geq 2$ subject to $f+2k \leq 2m$ by Theorem 4; also, the graph G is $(2m-3)$-fault Hamiltonian-connected and $(2m-2)$-fault Hamiltonian by Theorem 3. ■

Finally, we present a conjecture that a three- or higher-dimensional nonbipartite torus admits an optimal construction of a paired many-to-many disjoint path cover, although not every two-dimensional nonbipartite torus is 1-fault paired 2-disjoint path coverable [23].

Conjecture 1. Every m-dimensional nonbipartite torus, $m \geq 3$, is f-fault paired k-disjoint path coverable for any f and $k \geq 2$ subject to $f+2k \leq 2m+1$.

The conjecture is partially proved by Li et al. [17] in the case where $k = 2$, $F \subseteq E(G)$, and at most one d_j is even. We hope that the conjecture could initiate future research.

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References

[1] S. R. Arikati, C. P. Rangan, Linear algorithm for optimal path cover problem on interval graphs, Information Processing Letters 35 (3) (1990) 149–153.

[2] K. Asdre, S. D. Nikolopoulos, The 1-fixed-endpoint path cover problem is polynomial on interval graphs, Algorithmica 58 (3) (2010) 679–710.

[3] H. Cao, B. Zhang, Z. Zhou, One-to-one disjoint path covers in digraphs, Theoretical Computer Science 714 (2018) 27–35.

[4] X.-B. Chen, Paired 2-disjoint path covers of faulty k-ary n-cubes, Theoretical Computer Science 609 (2016) 494–499.

[5] X.-B. Chen, Paired 2-disjoint path covers of multidimensional torus networks with faulty edges, Information Processing Letters 116 (2) (2016) 107–110.

[6] T. Dvořák, P. Gregor, Partitions of faulty hypercubes into paths with prescribed endvertices, SIAM Journal on Discrete Mathematics 22 (4) (2008) 1448–1461.

[7] T. Dvořák, P. Gregor, V. Koubek, Generalized Gray codes with prescribed ends, Theoretical Computer Science 668 (2017) 70–94.

[8] S.-Y. Hsieh, N.-W. Chang, Hamiltonian path embedding and pancyclicity on the Mobius cube with faulty nodes and faulty edges, IEEE Transactions on Computers 55 (7) (2006) 854–863.

[9] S.-Y. Hsieh, G.-H. Chen, C.-W. Ho, Longest fault-free paths in star graphs with edge faults, IEEE Transactions on Computers 50 (9) (2001) 960–971.

[10] S.-Y. Hsieh, G.-H. Chen, C.-W. Ho, Longest fault-free paths in star graphs with vertex faults, Theoretical Computer Science 262 (1-2) (2001) 215–227.

[11] L.-H. Hsu, C.-K. Lin, Graph Theory and Interconnection Networks, CRC press, 2008.

[12] S. Jo, J.-H. Park, K.-Y. Chwa, Paired many-to-many disjoint path covers in faulty hypercubes, Theoretical Computer Science 513 (2013) 1–24.

[13] E. Kim, J.-H. Park, Paired many-to-many disjoint path covers in recursive circulants and tori, Journal of KIISE: Computer Systems and Theory 36 (1) (2009) 40–51.

[14] H.-C. Kim, J.-H. Park, Fault hamiltonicity of two-dimensional torus networks, in: in Proc. of Workshop on Algorithms and Computation WAAC'00, 110–117, 2000.

[15] B. G. Kronenthal, W. H. T. Wong, Paired many-to-many disjoint path covers of hypertori, Discrete Applied Mathematics 218 (2017) 14–20.

[16] J. Li, D. Liu, Y. Yang, J. Yuan, One-to-one disjoint path covers on multi-dimensional tori, International Journal of Computer Mathematics 92 (6) (2015) 1114–1123.

[17] J. Li, G. Wang, L. Chen, Paired 2-disjoint path covers of multi-dimensional torus networks with $2n-3$ faulty edges, Theoretical Computer Science 677 (2017) 1–11.

[18] P. Li, Y. Wu, Spanning connectedness and Hamiltonian thickness of graphs and interval graphs, Discrete Mathematics and Theoretical Computer Science 16 (2) (2015) 125–210.

[19] H.-S. Lim, H.-C. Kim, J.-H. Park, Ore-type degree conditions for disjoint path covers in simple graphs, Discrete Mathematics 339 (2) (2016) 770–779.

[20] J. A. McHugh, Algorithmic Graph Theory, Prentice-Hall, New Jersey, 1990.

[21] S. C. Ntafos, S. L. Hakimi, On path cover problems in digraphs and applications to program testing, IEEE Transactions on Software Engineering 5 (5) (1979) 520–529.

[22] B. Parhami, Introduction to Parallel Processing: Algorithms and Architectures, Springer Science & Business Media, 2006.

[23] J.-H. Park, Many-to-many disjoint path covers in two-dimensional tori, Journal of KIISE: Computer Systems and Theory 38 (1) (2011) 42–48.

[24] J.-H. Park, Paired many-to-many 3-disjoint path covers in bipartite toroidal grids, Journal of Computing Science and Engineering 12 (3) (2018) 115–126.

[25] J.-H. Park, J. Choi, H.-S. Lim, Algorithms for finding disjoint path covers in unit interval graphs, Discrete Applied Mathematics 205 (2016) 132–149.

[26] J.-H. Park, I. Ihm, Many-to-many two-disjoint path covers in cylindrical and toroidal grids, Discrete Applied Mathematics 185 (2015) 168–191.

[27] J.-H. Park, H.-C. Kim, H.-S. Lim, Fault-hamiltonicity of hypercube-like interconnection networks, in: Proc. of IEEE International Parallel and Distributed Processing Symposium IPDPS 2005, IEEE, 2005.

[28] J.-H. Park, H.-C. Kim, H.-S. Lim, Many-to-many disjoint path covers in hypercube-like interconnection networks with faulty elements, IEEE Transactions on Parallel and Distributed Systems 17 (3) (2006) 227–240.

[29] J.-H. Park, H.-C. Kim, H.-S. Lim, Many-to-many disjoint path covers in the presence of faulty elements, IEEE Transactions on Computers 58 (4) (2009) 528–540.

[30] Y.-K. Shih, S.-S. Kao, et al., One-to-one disjoint path covers on k-ary n-cubes, Theoretical Computer Science 412 (35) (2011) 4513–4530.

[31] X. Wang, J. Fan, X. Jia, C.-K. Lin, An efficient algorithm to construct disjoint path covers of DCell networks, Theoretical Computer Science 609 (2016) 197–210.

[32] S. Zhang, S. Wang, Many-to-many disjoint path covers in k-ary n-cubes, Theoretical Computer Science 491 (2013) 103–118.