Unpaired Many-to-Many Disjoint Path Covers in Restricted Hypercube-Like Graphs

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Abstract

For two disjoint vertex-sets, $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ of a graph, an *unpaired many-to-many k-disjoint path cover* joining S and T is a set of pairwise vertex-disjoint paths $\{P_1, \ldots, P_k\}$ that altogether cover every vertex of the graph, in which P_i is a path from s_i to some t_j for $1 \le i, j \le k$. A family of hypercube-like interconnection networks, called *restricted hypercube-like graphs*, includes most nonbipartite hypercube-like networks found in the literature, such as twisted cubes, crossed cubes, Möbius cubes, recursive circulant $G(2^m, 4)$ of odd m, etc. In this paper, we show that every m-dimensional restricted hypercube-like graph, $m \ge 5$, with at most f faulty vertices and/or edges being removed has an unpaired many-to-many k-disjoint path cover joining arbitrary disjoint sets S and T of size k each subject to $k \ge 2$ and $f + k \le m - 1$. The bound $m - 1$ on $f + k$ is the maximum possible.

Keywords: Hypercube-like graph, disjoint path, path cover, path partition, fault tolerance, interconnection network.

1. Introduction

An interconnection network is frequently represented as a graph in which the vertices and edges correspond to nodes and links, respectively. Since node and/or link failure is inevitable in a large network, fault tolerance is essential to the network performance. One of the central issues in the study of interconnection networks is to detect (vertex-)disjoint paths, which is naturally related to routing among nodes and fault tolerance of the network [13, 23]. If each copy of a message is routed along a different path of the disjoint paths, then at least one copy eventually arrives at its sink provided the total number of node and link faults is less than the number of disjoint paths. Furthermore, disjoint path is one of the fundamental notions in graph theory from which many properties of a graph can be deduced [2, 23].

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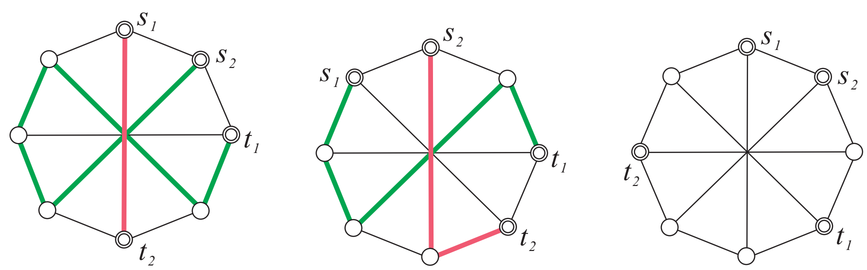
The connectivity of the underlying graph has been a primary measure of fault tolerance [13, 23], and the connectivity of a graph is closely related to the existence of disjoint paths in the graph. Menger's theorem states the connectivity of a graph in terms of the number of disjoint paths (of one-to-one type) between a pair of source and sink, whereas the Fan Lemma states the connectivity of a graph in terms of the number of disjoint paths (of one-to-many type) joining a source to a set of sinks [2]. Moreover, a graph is k-connected if and only if it has k disjoint paths (of many-to-many type), respectively connecting arbitrary k distinct sources and arbitrary k distinct sinks, where, if a source coincides with a sink, then such source itself is regarded as a valid one-vertex path.

Let G be a simple undirected graph whose vertex and edge sets, respectively, are denoted by $V(G)$ and $E(G)$. Given two vertices v and w of G, a *path* P in G from v to w is a sequence $(u_1, \ldots, u_p)$ of distinct vertices of G such that $u_1 = v$, $u_p = w$, and $(u_i, u_{i+1}) \in E(G)$ for all $i \in \{1, \ldots, p-1\}$. A *path cover* of G is a set of paths in G such that every vertex of G is contained in at least one path. A *vertex-disjoint path cover*, or simply a *disjoint path cover*, of G is a special kind of path cover in which every vertex of G is covered by exactly one path. The disjoint path cover problem finds applications in many areas such as software testing, database design, and code optimization [1, 25]. In addition, the problem is concerned with applications where full utilization of network nodes is important [32]. For example, basic communication problems for the dissemination of information, such as broadcasting and information gathering, require visiting every node of the network at least once. Since visiting a node more than once results in unnecessary overhead, a disjoint path cover can be employed to avoid this unsatisfactory situation.

For a positive integer k, let $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ be two disjoint subsets of $V(G)$. Then, a disjoint path cover $\{P_1, \ldots, P_k\}$ of G is said to be an *unpaired many-to-many k-disjoint path cover* (*unpaired k-DPC* for short) joining S and T if for some permutation σ on $\{1, \ldots, k\}$, P_i is a path that runs from s_i to $t_{\sigma(i)}$ for $i \in \{1, \ldots, k\}$ [32]. In addition, the unpaired k-disjoint path cover is regarded as *paired* if σ is constrained to be an identity permutation, so that P_i is a path from s_i to t_i for all $i \in \{1, \ldots, k\}$. Note that a paired k-DPC joining S and T is, by definition, an unpaired k-DPC joining them. Refer to Fig. 1 for examples of unpaired and paired DPCs. Here, the vertices in S and T are often called *sources* and *sinks*, respectively, and *terminals* collectively.

Definition 1. (See [33].) A graph G is called *f-fault unpaired* (resp. *paired*) *k-disjoint path coverable* if $f + 2k \leq |V(G)|$ and G has an unpaired (resp. paired) k-DPC joining arbitrary disjoint set S of k sources and set T of k sinks in $G \setminus F$ for any fault set $F \subseteq V(G) \cup E(G)$ with $|F| \leq f$.

Necessary conditions for a graph G to be f-fault unpaired k-disjoint path coverable have been derived in terms of its connectivity $\kappa(G)$ and its minimum degree $\delta(G)$ of G in [33], as shown below.



(a) Unpaired 2-DPC (b) Paired 2-DPC (c) No unpaired 2-DPC

Fig. 1: Examples of unpaired and paired DPCs. The configuration (a) admits no paired 2-DPC; The configuration (c) admits no unpaired 2-DPC (and no paired 2-DPC).

Lemma 1. *(See [33].) Let G be an f-fault unpaired k-disjoint path coverable graph, where $k \geq 2$. Then, $\kappa(G) \geq f + k$. Furthermore, if G has $f + 2k + 1$ or more vertices, then $\delta(G) \geq f + k + 1$.*

In this paper, we are concerned with the unpaired many-to-many disjoint path coverability for the class of Restricted Hypercube-Like graphs (RHL graphs for short) [31], which are a subset of nonbipartite hypercube-like graphs that have received much attention over the recent decades. The class includes most nonbipartite hypercube-like networks found in the literature, as the following examples: twisted cubes [12], crossed cubes [10], Möbius cubes [7], recursive circulant $G(2^m, 4)$ of odd m [27], multiply twisted cubes [9], Mcubes [35], and generalized twisted cubes [3]. An m-dimensional RHL graph, defined in the next section, has 2^m vertices of degree m. It is an m-regular graph of connectivity m.

Theorem 1. *(See [26].) Every m-dimensional RHL graph, $m \geq 3$, is f-fault unpaired k-disjoint path coverable for any f and $k \geq 1$ subject to $f + k \leq m - 2$.*

The bound, $m - 2$, on $f + k$ in Theorem 1 is one less than the bound, $m - 1$, of the necessary condition in Lemma 1. We will bridge the gap in this paper. Precisely speaking, we will prove our main theorem asserting that *every m-dimensional RHL graph, $m \geq 5$, is f-fault unpaired k-disjoint path coverable for any f and $k \geq 2$ subject to $f + k \leq m - 1$*, achieving the optimal bound $m - 1$ on $f + k$.

The rest of this paper is organized as follows: In the next section, we address previous works and definitions. Sections 3 and 4 are devoted to a proof of our main theorem. Finally, we conclude in Section 5.

2. Previous works and definitions

2.1. Disjoint path covers

Simpler variants of the many-to-many disjoint path covers have also been investigated in the literature. The *one-to-many k-DPC* for $S = \{s\}$ and $T =$

$\{t_1, \ldots, t_k\}$ is a disjoint path cover made of k paths, each joining a pair of source s and sink t_j, $j \in \{1, \ldots, k\}$. The *one-to-one k-DPC* for $S = \{s\}$ and $T = \{t\}$ is a disjoint path cover, each of whose paths joins an identical pair of source s and sink t. The paths in the one-to-many k-DPC or in the one-to-one k-DPC may share a source and/or a sink and thus are pairwise internally disjoint. Readers are recommended to refer to the related literature, such as [16, 20, 28, 32], for more details.

Given disjoint source set S and sink set T in a general graph G, it is NP-complete to determine if there exists a one-to-one, one-to-many, or many-to-many k-DPC joining S and T for any fixed $k \geq 1$ [32, 33]. The disjoint path cover problems have been studied for graphs such as hypercubes [4, 5, 6, 8, 11, 15, 22], recursive circulants [16, 17, 32, 33], hypercube-like graphs [14, 18, 19, 26, 32, 33], cubes of connected graphs [28, 29], k-ary n-cubes [34, 37], alternating group graphs [36], tori [21], and grid graphs [30]. Among others, we mention about the necessity for the existence of an f-fault paired k-disjoint path cover of a general graph and the f-fault paired k-disjoint path coverability of RHL graphs as shown below.

Lemma 2. *(See [32].) If a graph G is f-fault paired k-disjoint path coverable, then $f + 2k \leq \kappa(G) + 1$.*

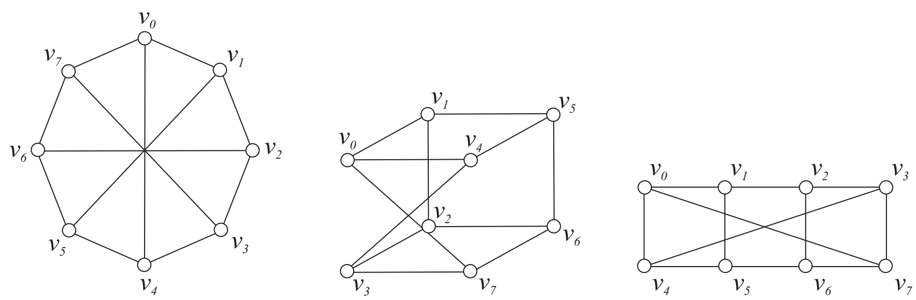
Theorem 2. *(a) Every m-dimensional RHL graph, $m \geq 3$, is f-fault paired k-disjoint path coverable for any f and $k \geq 2$ subject to $f + 2k \leq m$ [33].*
(b) Every m-dimensional RHL graph, $m \geq 5$, is $(m-3)$-fault paired 2-disjoint path coverable [18].

For general $k \geq 2$, there is a gap of size one between the bounds on $f + 2k$ of Theorem 2(a) and of the necessary condition in Lemma 2. For the specific $k = 2$, however, the bound in Theorem 2(b) is optimal. It is still open whether or not the optimal bound $m + 1$ on $f + 2k$ can be achieved for $k \geq 3$.

It is worth noting that there is a class of graphs, called Recursive-Circulant-Like graphs (RCL graphs for short) [16, 17], that allows for the optimal constructions of unpaired and paired DPCs, achieving the optimal bounds of the necessary conditions in Lemmas 1 and 2. That is, it was proven that for $m \geq 5$, every m-dimensional RCL graph, which has 2^m vertices of degree m (as an m-dimensional RHL graph does), is f-fault *unpaired* k-disjoint path coverable for any f and $k \geq 2$ subject to $f + k \leq m - 1$ [16], and is f-fault *paired* k-disjoint path coverable for any f and $k \geq 2$ subject to $f + 2k \leq m + 1$ [17].

2.2. Restricted hypercube-like graphs

A 3-dimensional RHL graph is isomorphic to recursive circulant $G(8, 4)$ whose vertex and edge sets, respectively, are $\{v_i : 0 \leq i \leq 7\}$ and $\{(v_i, v_j) : i + 1 \text{ or } i + 4 \equiv j \pmod 8\}$. The 3-dimensional RHL graph is isomorphic to a 3-dimensional twisted cube TQ_3 and to a Möbius ladder with four spokes [24] as shown in Fig. 2. An m-dimensional RHL graph, $m \geq 4$, is recursively defined with a graph operation $\oplus$. Given two graphs, G_0 and G_1, with the same number of vertices and a bijection ϕ from $V(G_0)$ to $V(G_1)$, we denote



(a) $G(8,4)$ (b) TQ_3 (c) Möbius ladder

Fig. 2: The 3-dimensional RHL graph.

by $G_0 \oplus_\phi G_1$ the graph whose vertex set is $V(G_0) \cup V(G_1)$ and edge set is $E(G_0) \cup E(G_1) \cup \{(v, \phi(v)) : v \in V(G_0)\}$. To simplify the notation, we often omit the bijection ϕ from $\oplus_\phi$.

Definition 2. (See [31].) A graph that belongs to RHL_m is called an *m-dimensional RHL graph*, where

- $RHL_3 = \{G(8,4)\}$, and
- $RHL_m = \{G_0 \oplus_\phi G_1 : G_0, G_1 \in RHL_{m-1}$, ϕ is a bijection from $V(G_0)$ to $V(G_1)\}$ for $m \geq 4$.

Lemma 3. *(See [18].) Let G be an m-dimensional RHL graph, where $m \geq 3$.*
(a) G has 2^m vertices of degree m. Moreover, it is nonbipartite.
(b) G has no triangle (cycle of length three).
(c) There are at most two common neighbors for any pair of vertices in G.

The disjoint path cover of a graph is naturally related to its Hamiltonian properties. For instance, a Hamiltonian path between two distinct vertices in a graph G is in fact a 1-DPC, irrespective of its type, of G joining the vertices. The Hamiltonian properties of RHL graphs were studied in [31] as shown below, where a graph G is said to be *f-fault Hamiltonian-connected* (resp. *Hamiltonian*) if any pair of vertices are joined by a Hamiltonian path (resp. there exists a Hamiltonian cycle) in $G \setminus F$ for any fault set $F \subseteq V(G) \cup E(G)$ with $|F| \leq f$.

Lemma 4. *(See [31].) Every m-dimensional RHL graph, $m \geq 3$, is $(m-3)$-fault Hamiltonian-connected and is $(m-2)$-fault Hamiltonian.*

2.3. Generalized disjoint path covers

The unpaired k-disjoint path cover is defined on necessarily disjoint source set S and sink set T (of size k each) in a graph, so that each path in the DPC connects a pair of distinct source and sink. It is sometimes useful if we define an extension of the unpaired k-disjoint path cover on arbitrary (not necessarily disjoint) sets, S and T, of sources and sinks in such a way that a vertex that belongs to both sets is considered as a valid, one-vertex path. A *generalized k-disjoint path cover* joining S and T in a graph G is defined as a set of k pairwise disjoint paths of G composed of

- f' one-vertex paths for terminals in $S \cap T$, where $f' = |S \cap T|$, and
- $k - f'$ paths that form an unpaired $(k - f')$-DPC joining $S \setminus T$ and $T \setminus S$ in $G \setminus (S \cap T)$.

It follows that given a fault set F in a graph G, the graph $G \setminus F$ has a generalized k-DPC joining S and T if and only if the graph $G \setminus (F \cup F')$ has an unpaired $(k - f')$-DPC joining S' and T', where $F' = S \cap T$, $f' = |F'|$, $S' = S \setminus F'$, and $T' = T \setminus F'$.

Hereafter in this paper, an *s–t path* refers to a path that runs from s to t; and an *s-path* refers to a path starting at vertex s. If $G = H_0 \oplus H_1$, subgraphs H_0 and H_1 are called the *components* of G. For a vertex v in a component H_i, we denote by $\bar{v}$ the neighbor of v contained in the other component H_{1-i}, for $i = 0, 1$. Extended to a vertex set of a component, $\bar{X}$ for $X \subseteq V(H_i)$ is defined as the set $\{\bar{x} : x \in X\}$. In addition, a vertex v is called to be *free* if it is neither a fault nor a terminal. An edge (u, v) is called to be *free* if it is nonfaulty and both u and v are free. Finally, $N_G(v)$, or $N(v)$ if the graph G is clear in the context, represents the open neighborhood of a vertex $v \in V(G)$, i.e., $N_G(v) = \{u \in V(G) : (u, v) \in E(G)\}$.

3. Unpaired many-to-many disjoint path covers

In this section, we will construct f-fault unpaired k-disjoint path covers in m-dimensional RHL graphs with $m \geq 5$ for any f and $k \geq 2$ satisfying the optimal bound $f + k \leq m - 1$ given in Lemma 1. That is, we will establish the following theorem.

Theorem 3. *Every m-dimensional RHL graph, $m \geq 5$, is f-fault unpaired k-disjoint path coverable for any $f \geq 0$ and $k \geq 2$ subject to $f + k \leq m - 1$.*

Corollary 1. *Every m-dimensional RHL graph G, $m \geq 5$, in which a fault set $F \subseteq V(G) \cup E(G)$ with $|F| \leq f$ and k-vertex sets $S, T \subseteq V(G) \setminus F$ are given, has a generalized k-DPC joining S and T if $f + k \leq m - 1$ and $|S \setminus T| = |T \setminus S| \geq 2$ $(k \geq 2)$.*

Proof. The corollary follows from Theorem 3, since for $f' := |S \cap T| \leq k - 2$, G is $(f + f')$-fault unpaired $(k - f')$-disjoint path coverable, where $k - f' \geq 2$ and $(f + f') + (k - f') = f + k \leq m - 1$. □

The proof of Theorem 3 will proceed by induction on m. The base step of $m = 5$, however, will be dealt with in the next Section 4.2. It will be verified by using the various kinds of disjoint path cover properties of the 4-dimensional RHL graphs, which will be developed later in Section 4.1.

For the inductive step, let $m \geq 6$. Recall that an m-dimensional RHL graph G is isomorphic to $H_0 \oplus H_1$ for some $(m-1)$-dimensional RHL graphs H_0 and H_1. We will construct an f-fault unpaired k-DPC for any given set S of k sources and set T of k sinks in G having at most f faults such that $f + k \leq m - 1$. The induction hypothesis states that each component H_i, $i \in \{0, 1\}$, of G has an f'-fault generalized k'-DPC joining $S' \subseteq V(H_i)$ and $T' \subseteq V(H_i)$ if $f' + k' \leq m - 2$ and $|S' \setminus T'| = |T' \setminus S'| \geq 2$. Another useful fact from Lemma 4 is that both H_0 and H_1 are $(m-4)$-fault Hamiltonian-connected and $(m-3)$-fault Hamiltonian.

An unpaired k-DPC with a fault set F is also an unpaired k-DPC with a virtual fault set $F \cup F'$, where F' is a set of arbitrary $m - 1 - k - |F|$ fault-free edges. As a result, it can be assumed that

$$f = |F| \text{ and } f + k = m - 1.$$

Let F_0 and F_1 respectively denote the sets of faults contained in H_0 and H_1, and F_2 denote the set of faulty edges bridging H_0 and H_1. Let $f_i = |F_i|$ for $i \in \{0, 1, 2\}$, so that $f = f_0 + f_1 + f_2$. We denote by S_i and T_i the sets of sources and sinks in H_i for $i \in \{0, 1\}$, respectively. Assume w.l.o.g. that $|S_0| \geq |T_0|$ and $|S_1| \leq |T_1|$. We let $k_0 = |T_0|$, $k_1 = |S_1|$, and $k_2 = k - (k_0 + k_1)$. Then, H_0 has $k_0 + k_2$ sources and k_0 sinks, and H_1 has k_1 sources and $k_1 + k_2$ sinks. So, it is assumed that $S_0 = \{s_i : 1 \leq i \leq k_0 + k_2\}$, $S_1 = \{s_i : k_0 + k_2 < i \leq k\}$, $T_0 = \{t_j : 1 \leq j \leq k_0\}$, and $T_1 = \{t_j : k_0 < j \leq k\}$. Furthermore, we also assume w.l.o.g. that

$$k_0 \geq k_1, \text{ and if } k_0 = k_1, f_0 \geq f_1.$$

(Suppose otherwise, it suffices to switch G_0 and G_1 and then switch S and T.)

We have the following three cases to consider:

1. $k_1 \geq 1$ or $f_0 \leq f - 1$;
2. $k_1 = 0$, $f_0 = f$, and $k_0 + f_0 \geq 1$ ($k_0 \geq 1$ or $f_0 \geq 1$); and
3. $k_2 = k$ and $f = 0$.

The first two cases are relatively easy to deal with, because they can be proven by mainly using the unpaired DPC properties and Hamiltonian properties of components, H_0 and H_1. Furthermore, the proofs for the two cases are exactly the same as those for Cases 1 and 2 of Theorem 3 given in [16], which states that every m-dimensional RCL graph, $m \geq 5$, is f-fault unpaired k-disjoint path coverable for any $f \geq 0$ and $k \geq 2$ subject to $f + k \leq m - 1$. This unexpected phenomenon occurs because the proofs for the two cases given in [16] rely on the following three properties of an m-dimensional RCL graph G':

- P1: G' is isomorphic to $H'_0 \oplus H'_1$ for some $(m-1)$-dimensional RCL graphs H'_0 and H'_1;
- P2: G' has 2^m vertices of degree m; and
- P3: G' is $(m-3)$-fault Hamiltonian-connected and $(m-2)$-fault Hamiltonian.

Note that an m-dimensional RHL graph G also has the following three properties Q1, Q2 and Q3 corresponding to P1, P2 and P3, by Definition 2 and Lemmas 3 and 4:

- Q1: G is isomorphic to $H_0 \oplus H_1$ for some $(m-1)$-dimensional RHL graphs H_0 and H_1;
- Q2: G has 2^m vertices of degree m; and
- Q3: G is $(m-3)$-fault Hamiltonian-connected and $(m-2)$-fault Hamiltonian.

For the proofs of the first two cases, we refer to Cases 1 and 2 in Section 5 of [16].

Now, let us concentrate on the third case where $k_2 = k$ and $f = 0$, which is the hardest one among the three. In this case, all the sources are contained in H_0 whereas all the sinks are contained in H_1. So, an unpaired k-DPC joining S and T, if exists, contains k paths each of which must pass through some edge $(v, \bar{v})$, $v \in V(H_0)$, between H_0 and H_1. There are no faults, and moreover, we have $k_2 = m-1$ by the assumption of $f + k = m - 1$. One of the natural approaches to construct a desired k-DPC would be that we first pick up a k-vertex set $X \subseteq V(H_0)$ that admits two generalized k-DPCs, one joining S and X in H_0 and the other joining $\bar{X}$ and T in H_1, and then we combine the two generalized DPCs into a desired DPC with edges between X and $\bar{X}$.

If there exists a vertex $\alpha \in V(H_0)$ such that $N_{H_0}(\alpha) = S$, then the edge $(\alpha, \bar{\alpha})$ must be passed through by a path in any unpaired k-DPC joining S and T; and thus, α must be included in the k-vertex subset X. Such a vertex α is called a *critical* vertex of H_0, and the edge $(\alpha, \bar{\alpha})$ is called a *critical* edge of $H_0 \oplus H_1$. Note that there exists at most one critical vertex of H_0 by Lemma 3(c). Symmetrically, a vertex $\beta \in V(H_1)$ such that $N_{H_1}(\beta) = T$ is called a *critical* vertex of H_1, and the edge $(\beta, \bar{\beta})$ is called a *critical* edge. Also, $\bar{\beta}$ must be included in the set X. It is possible that critical edges $(\alpha, \bar{\alpha})$ and $(\beta, \bar{\beta})$ coincide, i.e., $\bar{\alpha} = \beta$ and $\bar{\beta} = \alpha$.

In order to select a desirable k-vertex set $X \subseteq V(H_0)$, we will reply on disjoint path cover properties and Hamiltonian properties of components of H_0 and H_1, i.e., *subcomponents* of $H_0 \oplus H_1$. Note that H_0 and H_1 are $(m-1)$-dimensional RHL graphs, so that H_0 is isomorphic to $G_0 \oplus G_1$ for some $(m-2)$-dimensional RHL graphs G_0 and G_1; and also, H_1 is isomorphic to $G_2 \oplus G_3$ for some $(m-2)$-dimensional RHL graphs G_2 and G_3, as illustrated in Fig. 3. Contracting each subcomponent of $H_0 \oplus H_1$ to a single vertex results in a four-vertex graph isomorphic to a complete graph or to a cycle (whereas subcomponent contraction of an RCL graph always results in a four-vertex cycle).

Let $E_{i,j}$ denote the set of edges of $H_0 \oplus H_1$ that connect two subcomponents G_i and G_j for $i, j \in \{0, 1, 2, 3\}$, i.e., $E_{i,j} = \{(u, v) \in E(H_0 \oplus H_1) : u \in V(G_i), v \in V(G_j)\}$. Then, we have $|E_{0,1}| = |E_{2,3}| = 2^{m-2}$ and $|E_{0,3}| + |E_{0,2}| = |E_{1,2}| + |E_{1,3}| = 2^{m-2}$. Moreover, we have $|E_{0,3}| = |E_{1,2}|$ and $|E_{0,2}| = |E_{1,3}|$ since the union $E_{0,3} \cup E_{0,2} \cup E_{1,2} \cup E_{1,3}$ forms a perfect matching of $H_0 \oplus H_1$. It is assumed w.l.o.g. that

$$|E_{0,3}| = |E_{1,2}| \geq |E_{0,2}| = |E_{1,3}|, \text{ so that } |E_{0,3}| = |E_{1,2}| \geq 2^{m-3}.$$

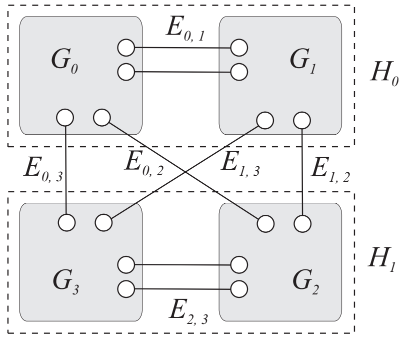


Fig. 3: Recursive structure of $H_0 \oplus H_1$, where there are two components, H_0 and H_1, and four subcomponents, G_0, G_1, G_2, and G_3.

(Suppose otherwise, it suffices to switch G_2 and G_3.) In addition, we assume w.l.o.g. that

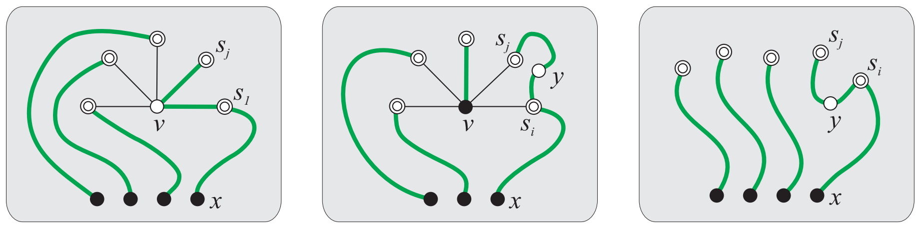
A1: $|V(G_0) \cap S| \geq |V(G_1) \cap S|$;
A2: if $|E_{0,3}| = |E_{1,2}| = |E_{0,2}| = |E_{1,3}| = 2^{m-3}$, then $|V(G_3) \cap T| \geq |V(G_2) \cap T|$;
A3: if there is no critical vertex in $H_0 \oplus H_1$, then $|V(G_0) \cap S| \geq |V(G_2) \cap T|, |V(G_3) \cap T|$; and
A4: if there is a single critical vertex in $H_0 \oplus H_1$, it is included in H_0.

The following procedure will construct an unpaired k-DPC joining S and T in an m-dimensional RHL graph $H_0 \oplus H_1$, where $m \geq 7$ and $k = m - 1$. The inductive step of $m = 6$ is deferred to Section 4.3. This is because for the proof when $m = 6$, we need disjoint path cover properties of a 4-dimensional RHL graph, which will be developed in Section 4.1. Note that the subcomponents of a 6-dimensional RHL graph are 4-dimensional RHL graphs.

Procedure Find-Unpaired-DPC(S, T, $H_0 \oplus H_1$)
/* It is assumed that $m \geq 7$, $k = m - 1$, $S \subseteq V(H_0)$ and $T \subseteq V(H_1)$. */

1: Find a generalized k-DPC between S and some k-vertex set $X \subseteq V(H_0)$ that satisfies the following four conditions simultaneously:
   C1: $|\bar{X} \cap V(G_3)| = k - 2$ and $|\bar{X} \cap V(G_2)| = 2$;
   C2: if there is a critical vertex α in H_0, then $\alpha \in X$;
   C3: if there is a critical vertex β in H_1, then $\bar{\beta} \in X$; and
   C4: $\bar{X} \cap T = \emptyset$, or $\bar{X} \cap T = \{\bar{\alpha}\}$ for the critical vertex α of H_0.
2: Find a generalized k-DPC joining $\bar{X}$ and T in H_1.
3: Merge the two generalized k-DPCs of H_0 and H_1 with k edges between X and $\bar{X}$.

The existence of such k-vertex set X in Step 1 of Procedure Find-Unpaired-DPC will be the most crucial part in the correctness proof. For our purpose, we start with two basic lemmas concerned with DPC properties of subcomponents, in which more than $k - 2$ sources are given.



(a) $N_{G_r}(v) = A$ & $v \notin B$ (b) $N_{G_r}(v) = A$ & $v \in B$ (c) $N_{G_r}(v) \neq A \ \forall v \in V(G_r)$

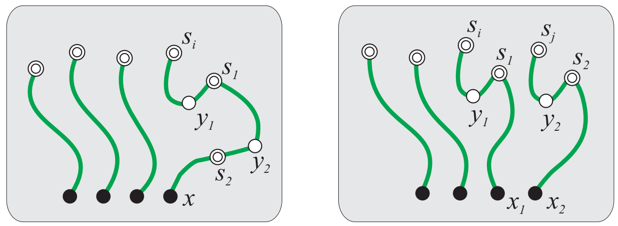
Fig. 4: Illustrations of the proof of Lemma 5.

Lemma 5. *Let G_r be a subcomponent of an m-dimensional RHL graph, $m \geq 7$, in which a source set A and a sink set B with $|A| = k-1$, $|B| = k-2$, and $|B \setminus A| \geq 2$ are given, where $k = m-1$.*
(a) If there is a common neighbor, v, of sources (i.e., $N_{G_r}(v) = A$) such that $v \notin B$, then there exists a generalized $(k-1)$-DPC joining A and $B \cup \{v\}$ in G_r.
(b) If there is a common neighbor, v, of sources such that $v \in B$, or if there is no common neighbor of sources, then for any vertex $w \in V(G_r)$, there exists a generalized $(k-1)$-DPC joining A and $B \cup \{y\}$ in G_r for some vertex $y \in V(G_r)$ with $y \neq w$.

Proof. The common neighbor of sources, if exists, is unique by Lemma 3(c), and moreover, it is not a source. Note that G_r is an $(m-2)$-dimensional RHL graph.

Proof of (a): If we regard an arbitrary source $s_1 \in A$ as a *virtual* non-source vertex, there exists a generalized $(k-2)$-DPC joining $A \setminus s_1$ and B in G_r by the induction hypothesis. Then, there exists a unique path in the generalized $(k-2)$-DPC that passes through s_1 (see Fig. 4(a)), which may be represented as (s_j, v, s_1, P_x, x) for some $s_j \in A \setminus s_1$, $x \in B$, and subpath P_x. Note that since $N_{G_r}(v) = A$, the path starts from some source $s_j \in A \setminus s_1$, next visits v and s_1 in order, and then runs to some vertex $x \in B$. Here, the subpath (s_1, P_x, x) will be a one-vertex path in the case when $s_1 \in B$. Dividing the s_j–x path into two, (s_j, v) and (s_1, P_x, x), results in a generalized $(k-1)$-DPC of G_r joining A and $B \cup \{v\}$, completing the proof of (a).

Proof of (b): Let $s_i \in A$ be a source such that $s_i \notin N_{G_r}(w)$ if w is not a common neighbor of sources (i.e., if there is a common neighbor, $v \in B$, of sources such that $v \neq w$, or if there is no common neighbor of sources); Let s_i be an arbitrary source in A otherwise (i.e., w is a common neighbor of sources such that $w \in B$). If we regard the source s_i as a *virtual* non-source vertex, the subcomponent G_r has a generalized $(k-2)$-DPC joining $A \setminus s_i$ and B by the induction hypothesis. The path in the generalized $(k-2)$-DPC that passes through s_i may be represented as $(s_j, P_y, y, s_i, P_x, x)$ for some $s_j \in A \setminus s_i$, $x \in B$, and subpaths P_y, P_x (see Figs. 4(b) and 4(c)). Here, the vertex y, the



(a) $s_1, s_2 \in V(P_i)$

(b) $s_1 \in V(P_i)$ & $s_2 \in V(P_j)$

Fig. 5: Illustrations of the proof of Lemma 6, where $V(P_i)$ denotes the set of vertices of an s_i-path, P_i, in the generalized k-DPC.

immediate predecessor of s_i in the path, is distinct from s_i (since $s_j \neq s_i$), and moreover $y \neq w$ (suppose otherwise, it would follow that $s_i \in N_{G_r}(w)$ and, by the choice of s_i, w is the common neighbor of sources such that $w \in B$, implying the s_j–x path of the DPC visits two distinct sinks y $(= w)$ and x, which is a contradiction). Dividing the s_j–x path into two, (s_j, P_y, y) and (s_i, P_x, x), results in a generalized $(k-1)$-DPC of G_r joining A and $B \cup \{y\}$, completing the proof of (b). □

Lemma 6. *Let G_r be a subcomponent of an m-dimensional RHL graph, $m \geq 7$, in which a source set A and a sink set B with $|A| = k$, $|B| = k-2$, and $|B \setminus A| \geq 2$ are given, where $k = m-1$. Then, there exists a generalized k-DPC joining A and $B \cup \{y_1, y_2\}$ in G_r for some vertices $y_1, y_2 \in V(G_r)$.*

Proof. If we regard arbitrary two sources s_1, s_2 as *virtual* non-source vertices, the subcomponent G_r has a generalized $(k-2)$-DPC joining $A \setminus \{s_1, s_2\}$ and B by the induction hypothesis. If there is a path in the generalized $(k-2)$-DPC that passes through s_1 and s_2 both (see Fig. 5(a)), which may be represented as $(s_i, P_y^1, y_1, s_1, P_y^2, y_2, s_2, P_x, x)$ or $(s_i, P_y^2, y_2, s_2, P_y^1, y_1, s_1, P_x, x)$ for some $s_i \in A \setminus \{s_1, s_2\}$ and $x \in B$, it suffices to divide the path into three disjoint paths joining $\{s_i, s_1, s_2\}$ and $\{x, y_1, y_2\}$. If some s_i–x_1 path in the generalized $(k-2)$-DPC passes through s_1 and some s_j–x_2 path passes through s_2 (see Fig. 5(b)), where $s_i, s_j \in A \setminus \{s_1, s_2\}$ and $x_1, x_2 \in B$, then it suffices to divide the s_i–x_1 path into s_i–y_1 and s_1–x_1 paths for some $y_1 \in V(G_r)$, and divide the s_j–x_2 path into s_j–y_2 and s_2–x_2 paths for some $y_2 \in V(G_r)$. This completes the proof. □

Now, we are ready to demonstrate the existence of a k-vertex set X in Step 1 of Procedure Find-Unpaired-DPC. For a vertex u in a subcomponent G_j of $H_0 \oplus H_1$, we denote by $\hat{u}$ the neighbor of u that is contained in the same component as u but in the other subcomponent than G_j. (If $u \in V(G_0)$, then $\hat{u} \in V(G_1)$ and vice versa. If $u \in V(G_2)$, then $\hat{u} \in V(G_3)$ and vice versa.) This definition is naturally extended to a vertex subset, Y, of a subcomponent, so that $\hat{Y}$ represents the set $\{\hat{u} : u \in Y\}$.

Lemma 7. *There exists a k-vertex set $X \subseteq V(H_0)$ that admits a generalized k-DPC of H_0 joining S and X and moreover satisfies the four conditions in Step 1 of Procedure* Find-Unpaired-DPC.

Proof. Let $S' = S \cap V(G_0)$ and $S'' = S \cap V(G_1)$, so that $S = S' \cup S''$. Assume that $S' = \{s_1, \ldots, s_p\}$ and $S'' = \{s_{p+1}, \ldots, s_k\}$ for $p = |S'|$, where $k = m - 1$. There are three cases depending on the number of critical edges.

Case 1: *There is no critical edge in $H_0 \oplus H_1$.* Then, there is no critical vertex either. We have three possibilities according to the size of S'.

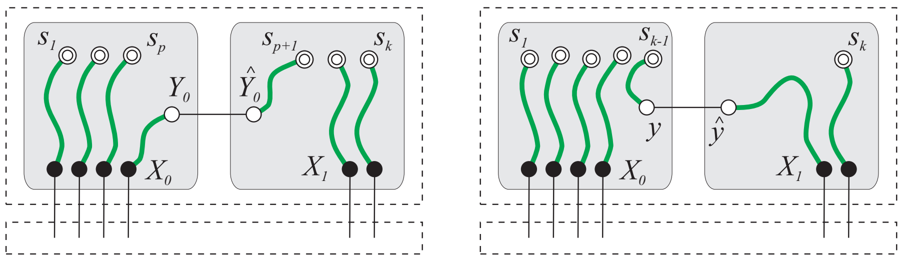
Case 1.1: $|S'| \leq k - 2$. A desired k-vertex set X and a generalized k-DPC joining S and X can be obtained in the following way (see Fig. 6(a)):

1: Pick up $k-2$ free edges from $E_{0,3}$, whose set of end-vertices in G_0 is denoted by X_0.
2: Pick up $(k-2)-p$ free edges from $E_{0,1}$ whose set of end-vertices in G_0, denoted by Y_0, has no intersection with X_0.
3: Find a generalized $(k-2)$-DPC joining $S' \cup Y_0$ and X_0 in G_0.
4: Pick up 2 free edges from $E_{1,2}$ whose set of end-vertices in G_1, denoted by X_1, has no intersection with $\hat{Y_0}$.
5: Find a generalized $(k-p)$-DPC joining S'' and $\hat{Y_0} \cup X_1$ in G_1.
6: Merge the two generalized DPCs of G_0 and G_1 with edges between Y_0 and $\hat{Y_0}$ into a generalized k-DPC of H_0 joining S and X, where $X = X_0 \cup X_1$.

The $k-2$ free edges of Step 1 exist because there are $|E_{0,3}|$ candidate edges whereas at most $2k$ of them could be blocked by terminals, for which $|E_{0,3}|-2k \geq 2^{m-3} - 2k = 2^{m-3} - 2(m-1) \geq m-3 = k-2$ for every $m \geq 7$. Similarly, the free edges of Step 2 also exist since $2^{m-2} - 2k \geq m-3 = k-2$ for all $m \geq 7$. The generalized $(k-2)$-DPC of Step 3 exists due to the induction hypothesis. Also, a similar counting argument leads to the existence of the free edges of Step 4, because $|S''| + |\hat{Y_0}| = (k-p) + (k-2-p) = ((k-p)-p) + (k-2) \leq k-2$. In addition, the generalized $(k-p)$-DPC of Step 5 exists by the induction hypothesis since $k - p = |S''| \leq |S'| \leq k-2$. Thus, there exists a generalized k-DPC joining S and X in H_0; and moreover, it is straightforward to check that the four conditions of Procedure Find-Unpaired-DPC are satisfied.

Case 1.2: $|S'| = k-1$. We claim $N_{G_0}(\hat{s_k}) \neq S'$. Suppose otherwise, $\hat{s_k}$ would be a critical vertex of H_0, i.e., $N_{H_0}(\hat{s_k}) = S$, contradicting the hypothesis of Case 1. So, G_0 has no common neighbor of sources in S', or G_0 has a common neighbor of sources in S' other than $\hat{s_k}$. (The common neighbor of sources is unique, if any, by Lemma 3(c).) Through Lemma 5, we construct a generalized k-DPC of H_0 joining S and some desirable k-vertex set $X \subseteq V(H_0)$ as follows (see Fig. 6(b)):

1: Pick up $k-2$ free edges from $E_{0,3}$, whose set of end-vertices in G_0 is denoted by X_0.
2: Find a generalized $(k-1)$-DPC joining S' and $X_0 \cup \{y\}$ in G_0 for some $y \in V(G_0)$ with $\hat{y} \neq s_k$.
3: Pick up 2 free edges from $E_{1,2}$ whose set of end-vertices in G_1, denoted by X_1, does not contain $\hat{y}$.



(a) $|S'| \leq k-2$ (b) $|S'| = k-1$

Fig. 6: Illustrations of Case 1 in the proof of Lemma 7.

4: Find a generalized 2-DPC joining $\{s_k, \hat{y}\}$ and X_1 in G_1.
5: Merge the two generalized DPCs of G_0 and G_1 with edge $(y, \hat{y})$ into a generalized k-DPC of H_0 joining S and X, where $X = X_0 \cup X_1$.

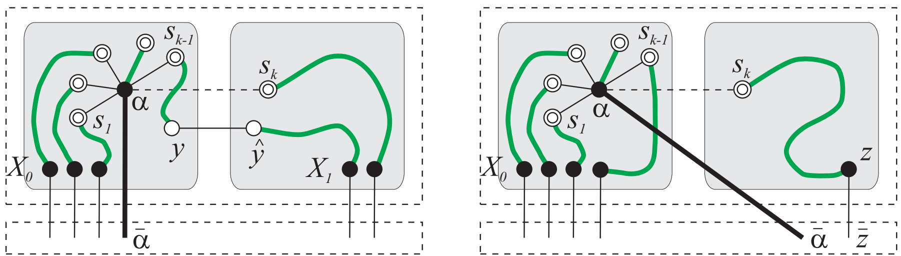
The same counting arguments as those of Case 1.1 lead to the existence of the free edges in Steps 1 and 3. The generalized $(k-1)$-DPC of Step 2 exists by Lemma 5(a) if there is a common neighbor, v, of sources of S' (where $v \neq \hat{s_k}$ as claimed) and $v \notin X_0$, and by Lemma 5(b) if otherwise (i.e., there is a common neighbor, v, of sources of S' and $v \in X_0$, or there is no common neighbor of sources). Also, the generalized DPC of Step 4 exists by the induction hypothesis. Then, the proof for this case is completed.

Case 1.3: *$|S'| = k$, i.e., $S' = S$ and $S'' = \emptyset$.* A generalized k-DPC of H_0 joining S and some desirable k-vertex set X can be constructed in a way similar to that of Case 1.2.

1: Pick up $k-2$ free edges from $E_{0,3}$, whose set of end-vertices in G_0 is denoted by X_0.
2: Find a generalized k-DPC joining S' and $X_0 \cup \{y_1, y_2\}$ in G_0 for some $y_1, y_2 \in V(G_0)$.
3: Pick up 2 free edges from $E_{1,2}$ whose set of end-vertices in G_1, denoted by X_1, does not contain $\hat{y_1}$, $\hat{y_2}$.
4: Find a generalized 2-DPC joining $\{\hat{y_1}, \hat{y_2}\}$ and X_1 in G_1.
5: Merge the two generalized DPCs of G_0 and G_1 with edges $(y_1, \hat{y_1})$ and $(y_2, \hat{y_2})$ into a generalized k-DPC of H_0 joining S and X, where $X = X_0 \cup X_1$.

The existence of a generalized k-DPC of Step 2 is due to Lemma 6.

Case 2: *There is a single critical edge in $H_0 \oplus H_1$.* Then, there is a critical vertex, α, in H_0 by our assumption A4. So, the critical edge will be $(\alpha, \bar{\alpha})$. Moreover, the critical vertex α is contained in G_0 again by our assumption A1. So, we have $|S'| = k-1$, $|S''| = 1$, and α is the unique common neighbors of sources in S', i.e., $N_{G_0}(\alpha) = S'$. Possibly, there may exist another critical vertex $\beta \in V(H_1)$ such that $\beta = \bar{\alpha}$. As we did in Case 1.2 where $|S'| = k-1$, we will construct a generalized k-DPC of H_0 joining S and some desirable k-vertex set $X \subseteq V(H_0)$ through Lemma 5. Here, the critical vertex α must be included



(a) $(\alpha, \bar{\alpha}) \in E_{0,3}$ (b) $(\alpha, \bar{\alpha}) \in E_{0,2}$

Fig. 7: Illustrations of Case 2 in the proof of Lemma 7.

in X. There are two possibilities on the location of $\bar{\alpha}$.

Case 2.1: *$\bar{\alpha} \in V(G_3)$, so that $(\alpha, \bar{\alpha}) \in E_{0,3}$.* Refer to Fig. 7(a).

1: Pick up $k-3$ free edges from $E_{0,3} \setminus (\alpha, \bar{\alpha})$, whose set of end-vertices in G_0 is denoted by X_0.
2: Pick up a free edge $(y, \hat{y})$, where $y \in V(G_0)$, from $E_{0,1}$ such that $y \notin X_0 \cup \{\alpha\}$.
3: Find a generalized $(k-1)$-DPC of G_0 joining S' and $X_0 \cup \{\alpha, y\}$.
4: Pick up 2 free edges from $E_{1,2}$ whose set of end-vertices in G_1, denoted by X_1, does not contain $\hat{y}$.
5: Find a generalized 2-DPC joining $\{s_k, \hat{y}\}$ and X_1 in G_1.
6: Merge the two generalized DPCs of G_0 and G_1 with edge $(y, \hat{y})$ into a generalized k-DPC of H_0 joining S and X, where $X = X_0 \cup X_1 \cup \{\alpha\}$.

The existence of free edges of Steps 1, 2, and 4 is straightforward from the counting arguments similar to those of Case 1.1. Also, the existence of generalized DPCs of Steps 3 and 5 is by Lemma 5(a) and by the induction hypothesis, respectively. Thus, the procedure constructs a generalized k-DPC of H_0 joining S and some desirable k-vertex set $X \subseteq V(H_0)$ that includes α, completing the proof for this case.

Case 2.2: *$\bar{\alpha} \in V(G_2)$, so that $(\alpha, \bar{\alpha}) \in E_{0,2}$.* A slight modification of the procedure for Case 2.1 leads to a procedure for this case as shown below (see Fig. 7(b)):

1: Pick up $k-2$ free edges from $E_{0,3}$, whose set of end-vertices in G_0 is denoted by X_0.
2: Find a generalized $(k-1)$-DPC of G_0 joining S' and $X_0 \cup \{\alpha\}$.
3: Pick up a free edge, $(z, \bar{z})$ where $z \in V(G_1)$, from $E_{1,2}$.
4: Find a generalized 1-DPC joining S'' and $\{z\}$ in G_1.
5: The union of two generalized DPCs of G_0 and G_1 forms a generalized k-DPC of H_0 joining S and X, where $X = X_0 \cup \{\alpha, z\}$.

The existence of free edges in Steps 1 and 3 is guaranteed from the counting arguments as before. Notice that $\alpha \notin X_0$, since $(\alpha, \bar{\alpha})$ is an edge of $E_{0,2}$. The generalized $(k-1)$-DPC in Step 2 exists by Lemma 5(a). The 1-DPC of Step 4 is in fact a Hamiltonian s_k–z path of G_1, which exists by Lemma 4. Thus, we conclude that the procedure shown above constructs a generalized k-DPC of H_0 joining S and some desirable k-vertex set $X \subseteq V(H_0)$.

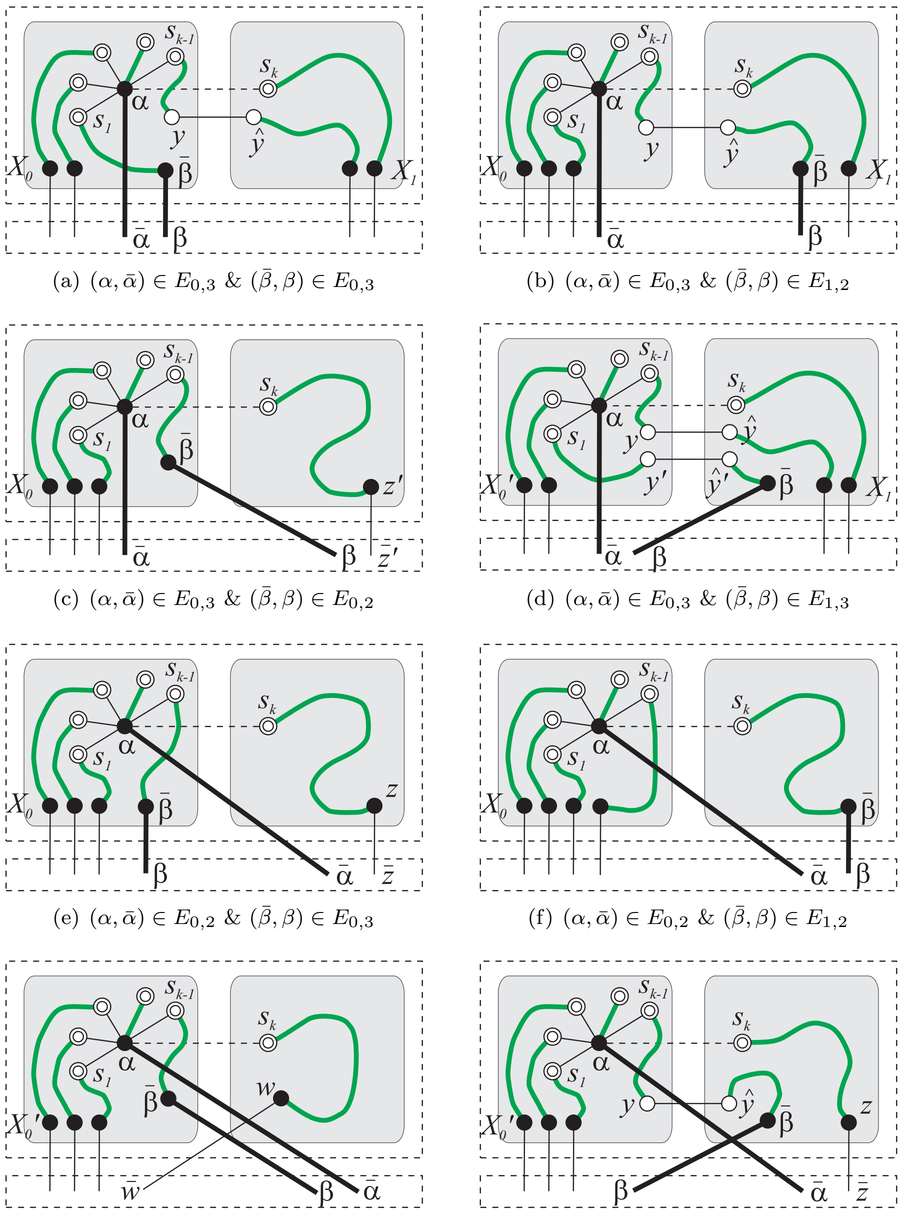
Case 3: *There are two distinct critical edges in* $H_0 \oplus H_1$. Let α and β, respectively, be critical vertices of H_0 and H_1, where $\bar{\alpha} \neq \beta$ and $\bar{\beta} \neq \alpha$. Then, the critical edges will be $(\alpha, \bar{\alpha})$ and $(\beta, \bar{\beta})$. Again α is a vertex of G_0 by our assumption. There are two subcases depending on the position of $\bar{\alpha}$.

Case 3.1: $\bar{\alpha} \in V(G_3)$, *so that* $(\alpha, \bar{\alpha}) \in E_{0,3}$. The procedure for Case 2.1 where $(\alpha, \bar{\alpha}) \in E_{0,3}$ will be recycled with a slight modification for each of the four possibilities on the location of $(\bar{\beta}, \beta)$. Firstly, assume $(\bar{\beta}, \beta) \in E_{0,3}$ (see Fig. 8(a)). It suffices to pick up $k-3$ edges from $E_{0.3} \setminus (\alpha, \bar{\alpha})$ including $k-4$ free edges and the critical edge $(\bar{\beta}, \beta)$ (instead of selecting $k-3$ free edges) in Step 1. Secondly, assume $(\bar{\beta}, \beta) \in E_{1,2}$. It suffices to modify Step 4 to pick up 2 edges from $E_{1,2}$ including one free edge whose end-vertex in G_1 is different from $\hat{y}$ and the critical edge $(\bar{\beta}, \beta)$ (instead of selecting 2 free edges), as shown in Fig. 8(b). The generalized 2-DPC of Step 5 exists by the induction hypothesis if $\{s_k, \hat{y}\} \cap X_1 = \emptyset$, and by Lemma 4 if $\{s_k, \hat{y}\} \cap X_1 \neq \emptyset$ (i.e., $\{s_k, \hat{y}\} \cap X_1 = \{\bar{\beta}\}$).

Thirdly, assume $(\bar{\beta}, \beta) \in E_{0,2}$ (see Fig. 8(c)). It suffices to hand over the role of the two free edges $(y, \hat{y})$ and $(z, \bar{z})$ for some $z \in X_1$ to the critical edge $(\bar{\beta}, \beta)$, so that some s_j-path, $s_j \in S'$, of the generalized k-DPC of H_0 passes through $(\bar{\beta}, \beta)$ (instead of passing through $(y, \hat{y})$ and $(z, \bar{z})$ in order). That is, we first construct a generalized $(k-1)$-DPC of G_0 joining S' and $X_0 \cup \{\alpha, \bar{\beta}\}$ through Lemma 5(a), and then find a generalized 1-DPC of G_1 joining s_k and z' for some free edge $(z', \bar{z'}) \in E_{1,2}$ where $z' \in V(G_1)$. The union of two generalized DPCs results in a desired k-DPC of H_0 joining S and X, where $X = X_0 \cup \{\alpha, \bar{\beta}, z'\}$.

Finally, assume $(\bar{\beta}, \beta) \in E_{1,3}$ (see Fig. 8(d)). It suffices to hand over the role of one free edge $(x, \bar{x})$ for $x \in X_0$ to the two edges, one free edge $(y', \hat{y'})$ for $y' \in V(G_0) \setminus (X_0 \cup \{\alpha\})$ and the critical edge $(\bar{\beta}, \beta)$, so that some s_j-path of the generalized k-DPC of H_0 runs to a vertex of G_3 via $(\bar{\beta}, \beta)$ (instead of $(x, \bar{x})$). That is, we first pick up $k-4$ free edges from $E_{0,3} \setminus (\alpha, \bar{\alpha})$, whose set of end-vertices in G_0 is denoted by X'_0, and pick up two free edge, $(y, \hat{y})$ and $(y', \hat{y'})$ where $y, y' \in V(G_0)$, from $E_{0,1}$ such that $y, y' \notin X'_0 \cup \{\alpha\}$. Then, a generalized $(k-1)$-DPC of G_0 joining S' and $X'_0 \cup \{\alpha, y, y'\}$, which exists by Lemma 5(a), is combined with a generalized 3-DPC of G_1 joining $\{s_k, \hat{y}, \hat{y'}\}$ and $\{\bar{\beta}\} \cup X_1$, where X_1 is the set of end-vertices in G_1 of 2 free edges from $E_{1,2}$ such that $\hat{y}, \hat{y'} \notin X_1$.

Case 3.2: $\bar{\alpha} \in V(G_2)$, *so that* $(\alpha, \bar{\alpha}) \in E_{0,2}$. Similar to Case 3.1, the procedure for Case 2.2 where $(\alpha, \bar{\alpha}) \in E_{0,2}$ will be reused with a slight modification for each of the four possibilities on the location of $(\bar{\beta}, \beta)$. Firstly, assume $(\bar{\beta}, \beta) \in E_{0,3}$ (see Fig. 8(e)). It suffices to pick up $k-2$ edges composed of $k-3$ free edges from $E_{0,3} \setminus (\bar{\beta}, \beta)$ and the critical edge $(\bar{\beta}, \beta)$ in Step 1 (instead of selecting $k-2$ free edges). Secondly, assume $(\bar{\beta}, \beta) \in E_{1,2}$ (see Fig. 8(f)). It suffices to replace the free edge $(z, \bar{z})$ of Step 3 with the critical edge $(\bar{\beta}, \beta)$. In Step 4, if $\bar{\beta} \neq s_k$, we find a Hamiltonian s_k–$\bar{\beta}$ path of G_1. If $\bar{\beta} = s_k$ accidentally, we let s_k–$\bar{\beta}$ path be a one-vertex path; and for an edge (u, v) of a path in



(a) $(\alpha, \bar{\alpha}) \in E_{0,3}$ & $(\bar{\beta}, \beta) \in E_{0,3}$

(b) $(\alpha, \bar{\alpha}) \in E_{0,3}$ & $(\bar{\beta}, \beta) \in E_{1,2}$

(c) $(\alpha, \bar{\alpha}) \in E_{0,3}$ & $(\bar{\beta}, \beta) \in E_{0,2}$

(d) $(\alpha, \bar{\alpha}) \in E_{0,3}$ & $(\bar{\beta}, \beta) \in E_{1,3}$

(e) $(\alpha, \bar{\alpha}) \in E_{0,2}$ & $(\bar{\beta}, \beta) \in E_{0,3}$

(f) $(\alpha, \bar{\alpha}) \in E_{0,2}$ & $(\bar{\beta}, \beta) \in E_{1,2}$

(g) $(\alpha, \bar{\alpha}) \in E_{0,2}$ & $(\bar{\beta}, \beta) \in E_{0,2}$

(h) $(\alpha, \bar{\alpha}) \in E_{0,2}$ & $(\bar{\beta}, \beta) \in E_{1,3}$

Fig. 8: Illustrations of Case 3 in the proof of Lemma 7.

the generalized $(k-1)$-DPC of G_0 such that $u, v \neq \alpha$, replace the edge with a Hamiltonian $\hat{u}$–$\hat{v}$ path of $G_1 \setminus s_k$.

Thirdly, assume $(\bar{\beta}, \beta) \in E_{0,2}$ (see Fig. 8(g)). It suffices to hand over the roles of the two free edges, $(x, \bar{x}) \in E_{0,3}$ for some $x \in X_0$ and $(z, \bar{z}) \in E_{1,2}$, to the other two edges, the critical edge $(\bar{\beta}, \beta) \in E_{0,2}$ and an edge $(w, \bar{w}) \in E_{1,3}$ where $w \in V(G_1)$ such that $\bar{w} \neq t_1$. (There exists such an edge $(w, \bar{w})$ in $E_{1,3}$ since $|E_{1,3}| = |E_{0,2}| \geq 2$.) That is, we first pick up $k-3$ free edges from $E_{0,3}$, whose set of end-vertices in G_0 is denoted by X'_0, in Step 1 (instead of selecting $k-2$ free edges), and construct a generalized $(k-1)$-DPC of G_0 joining S' and $X'_0 \cup \{\alpha, \bar{\beta}\}$ through Lemma 5(a). In addition, we find a Hamiltonian s_k–w path of G_1 if $w \neq s_k$. If $w = s_k$ accidentally, we let s_k–w path be a one-vertex path and replace an edge (u, v) on a path in the generalized $(k-1)$-DPC of G_0 such that $u, v \neq \alpha$ with a Hamiltonian $\hat{u}$–$\hat{v}$ path of $G_1 \setminus s_k$. Then, we have a desired k-DPC of H_0 joining S and X, where $X = X'_0 \cup \{\alpha, \bar{\beta}, w\}$.

Finally, assume $(\bar{\beta}, \beta) \in E_{1,3}$ (see Fig. 8(h)). It suffices to hand over the role of one free edge $(x, \bar{x})$ for $x \in X_0$ to the two edges, one free edge $(y, \hat{y})$ for $y \in V(G_0) \setminus (X_0 \cup \{\alpha\})$ and the critical edge $(\bar{\beta}, \beta)$, so that some s_j-path of the generalized k-DPC of H_0 runs to a vertex of G_3 via $(\bar{\beta}, \beta)$ (instead of $(x, \bar{x})$). That is, we first pick up $k-3$ free edges from $E_{0,3}$, whose set of end-vertices in G_0 is denoted by X'_0, and pick up a free edge, $(y, \hat{y})$ where $y \in V(G_0)$, from $E_{0,1}$ such that $y \notin X'_0$. Then, a generalized $(k-1)$-DPC of G_0 joining S' and $X'_0 \cup \{\alpha, y\}$, which exists by Lemma 5(a), is combined with a generalized 2-DPC of G_1 joining $\{s_k, \hat{y}\}$ and $\{\bar{\beta}, z\}$, where z is the end-vertex in G_1 of a free edge from $E_{1,2}$ such that $z \neq \hat{y}$. This completes the entire proof. □

We now turn our attention to the existence of a generalized k-DPC of H_1 in Step 2 of Procedure Find-Unpaired-DPC.

Lemma 8. *There exists a generalized k-DPC of H_1 joining $\bar{X}$ and T in Step 2 of Procedure* Find-Unpaired-DPC.

Proof. Let $Y = \bar{X}$, $Y' = Y \cap V(G_3)$, and $Y'' = Y \cap V(G_2)$, so that $Y = Y' \cup Y''$. It follows that $|Y'| = k-2$, $|Y''| = 2$, and $|Y \cap T| \leq 1$ by the construction of X in Procedure Find-Unpaired-DPC, where $k = m-1 \geq 6$. In addition, a critical vertex β of H_1, if exists, is contained in Y. Let $T' = T \cap V(G_3)$ and $T'' = T \cap V(G_2)$, so that $T = T' \cup T''$. Assume that $T' = \{t_1, \ldots, t_q\}$ and $T'' = \{t_{q+1}, \ldots, t_k\}$ for $q = |T'|$. There are five cases depending on the size of T'.

Case 1: $2 \leq |T'|, |T''| \leq k-2$. (See Fig. 9(a).) There exist $(k-2)-q$ free edges from $E_{3,2}$ whose set of end-vertices in G_3, denoted by Z, has no intersection with Y' and also, $\hat{Z}$ has no intersection with Y'', because $|E_{3,2}| - 2k = 2^{m-2} - 2(m-1) \geq m-3 = k-2$ for all $m \geq 7$. If we combine a generalized $(k-2)$-DPC of G_3 joining Y' and $T' \cup Z$ with a generalized $(k-q)$-DPC of G_2 joining $\hat{Z} \cup Y''$ and T'', we obtain a generalized k-DPC of H_1 joining Y and T. Note that the generalized DPCs of G_3 and G_2 exist by the induction hypothesis or by Lemma 4.

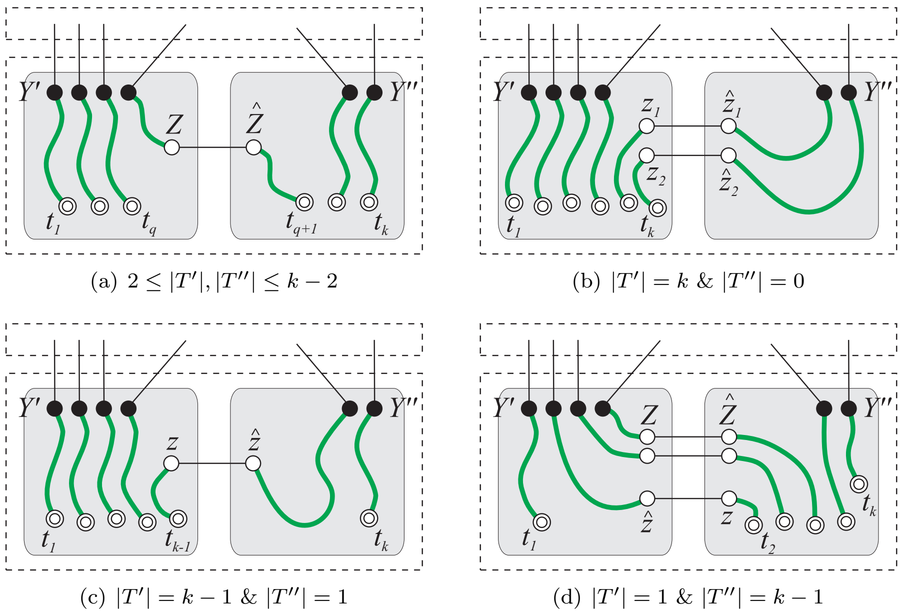


Fig. 9: Illustrations of the proof of Lemma 8.

Case 2: *$|T'| = k$ & $|T''| = 0$.* (See Fig. 9(b).) There exists a generalized k-DPC of G_3 joining $Y' \cup \{z_1, z_2\}$ and T' for some $z_1, z_2 \in V(G_3)$ by Lemma 6. If $\{\hat{z_1}, \hat{z_2}\} \neq Y''$, it suffices to combine the k-DPC of G_3 with a generalized 2-DPC of G_2 joining Y'' and $\{\hat{z_1}, \hat{z_2}\}$. If $\{\hat{z_1}, \hat{z_2}\} = Y''$ accidentally, let the two paths joining Y'' and $\{\hat{z_1}, \hat{z_2}\}$ be one-vertex paths. We only have to combine the k-DPC of G_3 with the two one-vertex paths; and for an edge (u, v) of a path in the k-DPC of G_3 such that $\{u, v\} \cap \{z_1, z_2\} = \emptyset$, replace the edge with a Hamiltonian $\hat{u}$–$\hat{v}$ path of $G_2 \setminus \{\hat{z_1}, \hat{z_2}\}$. The Hamiltonian path exists by Lemma 4.

Case 3: *$|T'| = k-1$ & $|T''| = 1$.* (See Fig. 9(c).) We claim that there exists a generalized $(k-1)$-DPC of G_3 joining $Y' \cup \{z\}$ and T' for some $z \in V(G_3)$ with $\hat{z} \neq t_k$. The claim follows directly from Lemma 5(b) if there is no common neighbor of sinks of T' (i.e., $N_{G_3}(v) \neq T'$ for all $v \in V(G_3)$). Now, assume that there exists a common neighbor, v, of sinks in T'. If $v \in Y'$, the claim follows again by Lemma 5(b). If $v \notin Y'$, meaning v is not a critical vertex of H_1, the claim follows from Lemma 5(a), since $\hat{v} \neq t_k$. Thus, the claim is proven. If $\{t_k, \hat{z}\} \neq Y''$, it suffices to combine the generalized $(k-1)$-DPC of G_3 with a generalized 2-DPC of G_2 joining Y'' and $\{t_k, \hat{z}\}$. If $\{t_k, \hat{z}\} = Y''$ accidentally, we only need to combine the generalized $(k-1)$-DPC of G_3 with the two one-vertex paths joining Y'' and $\{t_k, \hat{z}\}$; and for an edge (u, v) of a path in the generalized $(k-1)$-DPC of G_3 such that $\{\hat{u}, \hat{v}\} \cap \{t_k, \hat{z}\} = \emptyset$, replace the edge with a Hamiltonian $\hat{u}$–$\hat{v}$ path of $G_2 \setminus \{t_k, \hat{z}\}$.

Case 4: $|T'| = 1$ *&* $|T''| = k - 1$. (See Fig. 9(d).) We first pick up $k - 4$ free edges from $E_{3,2}$ whose set of end-vertices in G_3, denoted by Z, has no intersection with Y' and also, $\hat{Z}$ has no intersection with Y''. The existence of $k-4$ free edges is obvious from the same counting argument as Case 1. Similar to the claim of Case 3, there exists a generalized $(k-1)$-DPC of G_2 joining $(\hat{Z} \cup Y'') \cup \{z\}$ and T'' for some $z \in V(G_2)$ with $\hat{z} \neq t_1$. We only have to combine the generalized $(k-1)$-DPC of G_2 with a generalized $(k-2)$-DPC of G_3 joining Y' and $Z \cup \{t_1, \hat{z}\}$.

Case 5: $|T'| = 0$ *&* $|T''| = k$. In the same manner as Case 4, we pick up $k - 4$ free edges from $E_{3,2}$ whose set of end-vertices in G_3, denoted by Z, has no intersection with Y' and also, $\hat{Z}$ has no intersection with Y''. Then, there exists a generalized k-DPC of G_2 joining $(\hat{Z} \cup Y'') \cup \{z_1, z_2\}$ and T'' for some $z_1, z_2 \in V(G_2)$ by Lemma 6. We only need to combine the k-DPC of G_2 with a generalized $(k-2)$-DPC of G_3 joining Y' and $Z \cup \{\hat{z_1}, \hat{z_2}\}$. This completes the proof. □

Lemmas 7 and 8 lead to the following:

Lemma 9. *Procedure* FIND-UNPAIRED-DPC *constructs an unpaired k-DPC joining S and T in an m-dimensional RHL graph $H_0 \oplus H_1$ if $m \geq 7$, $k = m-1$, $S \subseteq V(H_0)$, and $T \subseteq V(H_1)$.*

4. Disjoint path covers of the low-dimensional RHL graphs

In this section, we deal with the base step of $m = 5$ and the inductive step of $m = 6$ for the remaining case where $k = m - 1 = 5$ and $S \subseteq V(H_0)$ & $T \subseteq V(H_1)$, which are not covered in the previous section for the proof of Theorem 3. We rely on the various kinds of DPC properties of the 4-dimensional RHL graphs, with which we begin.

4.1. Disjoint path cover properties of the 4-dimensional RHL graphs

By Theorem 1 or by Theorem 2(a), we can see that every 4-dimensional RHL graph is (0-fault) unpaired 2-disjoint path coverable. Moreover, we have the following:

Lemma 10. *Every 4-dimensional RHL graph G has a generalized 2-DPC joining S and T for any sets $S, T \subseteq V(G)$ such that $|S| = |T| = 2$ and $S \neq T$.*

Proof. The proof is a direct consequence of Theorem 1 and Lemma 4. □

DPC properties of the 4-dimensional RHL graphs, addressed in the following Lemmas 11 through 18, were verified from computer programs that exhaustively search for (generalized) DPCs mostly on the basis of a variation of depth-first search. The source codes may be downloaded from `http://tcs.catholic.ac.kr/~jhpark/papers/DPC_properties.zip`.

The first one is a negative result stating that there exists a 4-dimensional RHL graph, say the Cartesian product $G(8,4) \times K_2$ of $G(8,4)$ and a complete graph K_2, that does not admit an f-fault unpaired k-DPC for some F, S, and T satisfying the conditions of Theorem 3.

Lemma 11. *(a) $G(8,4) \times K_2$ is not unpaired 3-disjoint path coverable.*
(b) $G(8,4) \times K_2$ is not 1-vertex-fault unpaired 2-disjoint path coverable.
(c) $G(8,4) \times K_2$ is not 1-edge-fault unpaired 2-disjoint path coverable.

Lemmas 12, 13, and 14 below are concerning the 4-dimensional RHL graphs in which S and T with $|S| = 3 > |T|$ are given.

Lemma 12. *Given $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2\}$ in a 4-dimensional RHL graph G, there exists a vertex $x \notin T$ such that G has a generalized 3-DPC joining S and $T \cup \{x\}$. Furthermore, the number of such vertices x is at least three.*

Lemma 13. *Let $S = \{s_1, s_2, s_3\}$ and $T = \{t_1\}$ be given in a 4-dimensional RHL graph G. Then, for any 8-vertex set $Y \subseteq V(G) \setminus T$, there exists a 2-vertex set $X \subset Y$ that admits a generalized 3-DPC joining S and $T \cup X$. Furthermore, the number of such sets X is at least two.*

Lemma 14. *Given $S = \{s_1, s_2, s_3\}$ in a 4-dimensional RHL graph G, there exists a vertex γ, called* absolute vertex*, such that G has a generalized 3-DPC joining S and $\{\gamma\} \cup X$ for any 2-vertex set $X \subseteq V(G) \setminus \gamma$. Furthermore, the number of such absolute vertices γ is at least three.*

Remark. No absolute vertex γ is a source, because there exists no generalized 3-DPC between S and $\{\gamma\} \cup X$ for a 2-vertex set X with $\{\gamma\} \cup X = S$.

Lemmas 15 and 16 account for the case where $|S| = 4 > |T|$.

Lemma 15. *Given $S = \{s_1, s_2, s_3, s_4\}$ and $T = \{t_1, t_2\}$ in a 4-dimensional RHL graph G, there exists a 2-vertex set $X \subseteq V(G) \setminus T$ such that G has a generalized 4-DPC joining S and $T \cup X$. Furthermore, the number of such 2-vertex sets X is at least four. In addition, if there exists no common neighbor, v, of the four sources such that $v \neq t_1, t_2$, then for every vertex $z \in V(G) \setminus T$, there exists such 2-vertex set X with $z \notin X$.*

Lemma 16. *Given $S = \{s_1, s_2, s_3, s_4\}$ in a 4-dimensional RHL graph G, there exists an unordered pair $\Gamma = \{\gamma_1, \gamma_2\}$ of vertices, called* absolute pair*, such that G has a generalized 4-DPC joining S and $\Gamma \cup X$ for any 2-vertex set $X \subseteq V(G) \setminus \Gamma$. Furthermore, the number of such absolute pairs Γ is at least four.*

Remark. An absolute pair Γ always includes a common neighbor, v, of the four sources, if exists.

Finally, the case where $|S| = |T|$ are dealt with in Lemmas 17 and 18 below.

Lemma 17. *Given $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ with $|S \cap T| \leq 1$ in a 4-dimensional RHL graph G, there exist two distinct vertices $x \notin S$ and $y \notin T$ such that between $S \cup \{x\}$ and $T \cup \{y\}$, there exists a generalized 4-DPC in which no x–y path is included (i.e., each of the four paths runs from s_i for some $i \in \{1, 2, 3\}$ or runs to t_j for some $j \in \{1, 2, 3\}$). Furthermore, for any 2-vertex set $Z \subset V(G)$, we can pick up such vertices x and y with $x, y \notin Z$.*

Lemma 18. *Given pairwise disjoint $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$ in a 4-dimensional RHL graph G with a single edge fault, there exist two distinct vertices $x \notin S$ and $y \notin T$ such that between $S \cup \{x\}$ and $T \cup \{y\}$, there exists a generalized 3-DPC in which no x–y path is included (i.e., each of the three paths runs from s_i for some $i \in \{1, 2\}$ or runs to t_j for some $j \in \{1, 2\}$). Furthermore, for any 2-vertex set $Z \subset V(G)$, we can pick up such vertices x and y with $x, y \notin Z$.*

4.2. Base step of $m = 5$

We will prove the base step of Theorem 3 asserting that every 5-dimensional RHL graph is f-fault unpaired k-disjoint path coverable for any $f \geq 0$ and $k \geq 2$ subject to $f + k \leq 4$. The case of $k = 2$ is a direct consequence of Theorem 2(b). (By definition, a paired 2-DPC is an unpaired 2-DPC for the same fault, source and sink sets.) We now need to show that every 5-dimensional RHL graph is (i) unpaired 4-disjoint path coverable, (ii) 1-vertex-fault unpaired 3-disjoint path coverable, and (iii) 1-edge-fault unpaired 3-disjoint path coverable. For the first two assertions (i) and (ii), it suffices to prove the following:

Lemma 19. *Every 5-dimensional RHL graph G has a generalized 4-DPC joining S and T for any sets $S, T \subseteq V(G)$ such that $|S| = |T| = 4$ and $|S \cap T| \leq 1$.*

Proof. Let $G = H_0 \oplus H_1$, where H_0 and H_1 are 4-dimensional RHL graphs. As we did in the earlier part of Section 3, we assume w.l.o.g. $k_0 \geq k_1$ and $|S_0| \geq |T_0|$, so that $|S_0| \geq |T_0|, |S_1|, |T_1|$, where $S_0 = S \cap V(H_0) = \{s_1, \ldots, s_{|S_0|}\}$, $T_0 = T \cap V(H_0) = \{t_1, \ldots, t_{|T_0|}\}$, etc. There are nine cases according to the distribution of sources and sinks.

Case 1: *$|S_0| = 2$ & $|T_0| = 2$ ($|S_1| = 2$ & $|T_1| = 2$).* There exists a generalized 2-DPC joining S_0 and T_0 in H_0 by Lemma 10. The union of the 2-DPC of H_0 and a generalized 2-DPC of H_1 joining S_1 and T_1 results in a generalized 4-DPC of $H_0 \oplus H_1$.

Case 2: *$|S_0| = 3$ & $|T_0| = 1$ ($|S_1| = 1$ & $|T_1| = 3$).* There is an absolute vertex, γ, with $\gamma \neq t_1$ in H_0 by Lemma 14. For some vertex $z \in V(H_1)$ with $\bar{z} \neq t_1$, there exists a generalized 3-DPC joining $\{s_4, \bar{\gamma}, z\}$ and T_1 in H_1 by Lemma 12. It suffices to combine the 3-DPC of H_1 with a generalized 3-DPC of H_0 joining S_0 and $\{t_1, \gamma, \bar{z}\}$.

Case 3: *$|S_0| = 3$ & $|T_0| = 2$ ($|S_1| = 1$ & $|T_1| = 2$).* There exists a vertex $z \in V(H_0)$ that admits a generalized 3-DPC joining S_0 and $T_0 \cup \{z\}$ in H_0 by Lemma 12. Moreover, we can assume that $\bar{z} \neq s_4$ if $S_1 \cap T_1 = \emptyset$; $\bar{z}$ is a non-terminal vertex otherwise. It suffices to merge the 3-DPC of H_0 and a generalized 2-DPC of H_1 joining $\{s_4, \bar{z}\}$ and T_1.

Case 4: *$|S_0| = 3$ & $|T_0| = 3$ ($|S_1| = 1$ & $|T_1| = 1$).* By Lemma 17, there exist vertices $x \in V(H_0) \setminus S$ and $y \in V(H_0) \setminus T$ that admit a generalized 4-DPC of H_0 joining $S \cup \{x\}$ and $T \cup \{y\}$. Moreover, we can assume that the 4-DPC does not include an x–y path and also $\bar{x}, \bar{y} \notin \{s_4, t_4\}$. It suffices to combine the 4-DPC of H_0 with a generalized 2-DPC of H_1 joining $\{s_4, \bar{y}\}$ and $\{t_4, \bar{x}\}$.

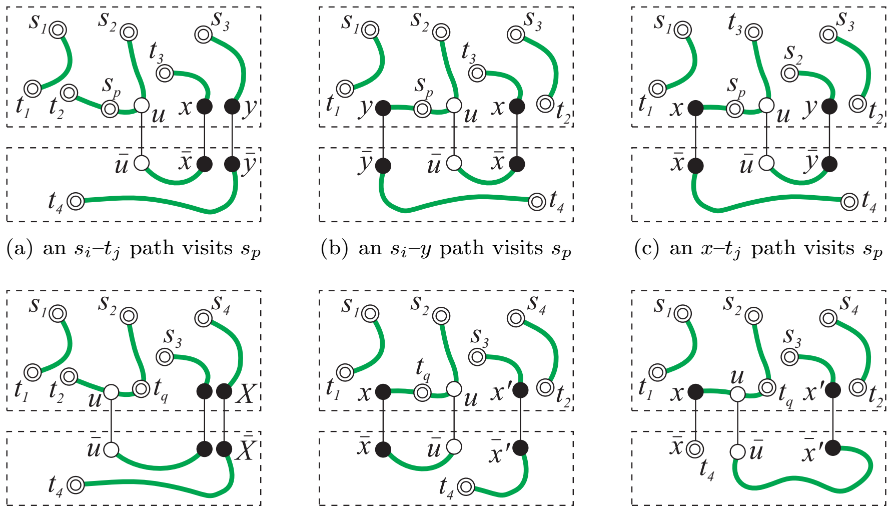
Case 5: $|S_0| = 4$ *&* $|T_0| = 0$ *(*$|S_1| = 0$ *&* $|T_1| = 4$*).* There exists an absolute pair $\{\gamma_1, \gamma_2\}$ in H_0 by Lemma 16. For some $z_1, z_2 \in V(H_1)$, there is a generalized 4-DPC joining $\{\bar{\gamma_1}, \bar{\gamma_2}, z_1, z_2\}$ and T_1 in H_1 by Lemma 15. It suffices to merge the 4-DPC of H_1 and a generalized 4-DPC of H_0 joining S_0 and $\{\gamma_1, \gamma_2, \bar{z_1}, \bar{z_2}\}$.

Case 6: $|S_0| = 4$ *&* $|T_0| = 1$ *(*$|S_1| = 0$ *&* $|T_1| = 3$*).* There exists an absolute vertex, γ, in H_1 with $\bar{\gamma} \neq t_1$ by Lemma 14, so that there is a generalized 3-DPC joining $\{\gamma\} \cup X$ and T_1 in H_1 for any 2-vertex set $X \subseteq V(H_1) \setminus \gamma$. For some $z_1, z_2 \in V(H_0)$, H_0 has a generalized 4-DPC joining S_0 and $\{t_1, \bar{\gamma}, z_1, z_2\}$ by Lemma 15. It suffices to combine the 4-DPC of H_0 with a generalized 3-DPC of H_1 joining $\{\gamma, \bar{z_1}, \bar{z_2}\}$ and T_1.

Case 7: $|S_0| = 4$ *&* $|T_0| = 2$ *(*$|S_1| = 0$ *&* $|T_1| = 2$*).* By Lemma 15, there exist vertices $z_1, z_2 \in V(H_0)$ with $\{\bar{z_1}, \bar{z_2}\} \neq T_1$ that admit a generalized 4-DPC of H_0 joining S_0 and $T_0 \cup \{z_1, z_2\}$. It suffices to combine the 4-DPC of H_0 with a generalized 2-DPC of H_1 joining $\{\bar{z_1}, \bar{z_2}\}$ and T_1.

Case 8: $|S_0| = 4$ *&* $|T_0| = 3$ *(*$|S_1| = 0$ *&* $|T_1| = 1$*).* Firstly, we assume that there exists a source $s_p \in S_0$ such that $s_p \notin N_{H_0}(\bar{t_4})$ and $s_p \notin T$. If we regard s_p as a *virtual* non-source vertex and apply Lemma 17, we obtain a generalized 4-DPC of H_0 joining $(S_0 \setminus s_p) \cup \{x\}$ and $T_0 \cup \{y\}$ for some $x, y \in V(H_0)$, in which no x–y path is included and $x, y \neq \bar{t_4}$. If an s_i–t_j path in the 4-DPC passes through s_p (see Fig. 10(a)), represented as $(s_i, P', u, s_p, P'', t_j)$ where u is the immediate predecessor of s_p in the path, then it suffices to divide the s_i–t_j path into two, (s_i, P', u) and (s_p, P'', t_j) paths, and combine the resultant 5-DPC of H_0 joining $S_0 \cup \{x\}$ and $T_0 \cup \{y, u\}$ with a generalized 2-DPC of H_1 joining $\{\bar{y}, \bar{u}\}$ and $\{t_4, \bar{x}\}$. (Note that the four vertices $\bar{x}$, $\bar{y}$, $\bar{u}$, and t_4 are distinct.) If an s_i–y path visits s_p (see Fig. 10(b)), we divide the path into s_i–u and s_p–y paths for the immediate predecessor u of s_p, resulting in a 5-DPC joining $S_0 \cup \{x\}$ and $T_0 \cup \{y, u\}$. It suffices to combine the 5-DPC with a generalized 2-DPC of H_1 joining $\{\bar{y}, \bar{u}\}$ and $\{t_4, \bar{x}\}$. (Also, $\bar{x}$, $\bar{y}$, $\bar{u}$, and t_4 are distinct.) If an x–t_j path visits s_p (see Fig. 10(c)), we divide the path into x–s_p and u–t_j paths for the immediate successor u of s_p, resulting in a 5-DPC joining $S_0 \cup \{u\}$ and $T_0 \cup \{x, y\}$. It suffices to combine the 5-DPC with a generalized 2-DPC of H_1 joining $\{\bar{x}, \bar{y}\}$ and $\{t_4, \bar{u}\}$. (Again, $\bar{x}$, $\bar{y}$, $\bar{u}$, and t_4 are distinct.)

We now assume that no such source s_p exists. It follows that at least three sources that are non-sink vertices are neighbors of $\bar{t_4}$ in H_0, which implies that there exists a sink $t_q \in T_0 \setminus S_0$ such that $t_q \notin N_{H_0}(\bar{t_4})$. If we regard t_q as a *virtual* non-sink vertex, then by Lemma 15, there exists a generalized 4-DPC of H_0 joining S_0 and $(T_0 \setminus t_q) \cup X$ for some 2-vertex set $X \subset V(H_0)$. If some s_i–t_j path in the 4-DPC passes through t_q (see Fig. 10(d)), we divide the path into s_i–t_q and u–t_j paths for the immediate successor u of t_q, resulting in a 5-DPC of H_0 joining $S_0 \cup \{u\}$ and $T_0 \cup X$. It suffices to combine the 5-DPC with a generalized 2-DPC of H_1 joining $\bar{X}$ and $\{t_4, \bar{u}\}$. (Note that $\bar{u} \neq t_4$ by the choice of t_q, and $\bar{u} \notin \bar{X}$.) Now, let some s_i–x path for $x \in X$ pass through t_q. If $\bar{x} \neq t_4$ (see Fig. 10(e)), we divide the path into s_i–u and t_q–x paths for the immediate predecessor u of t_q, resulting in a 5-DPC joining $S_0 \cup \{x\}$ and $T_0 \cup \{x', u\}$ where



(a) an s_i–t_j path visits s_p (b) an s_i–y path visits s_p (c) an x–t_j path visits s_p

(d) an s_i–t_j path visits t_q (e) an s_i–x path visits t_q, where $\bar{x} \neq t_4$ (f) an s_i–x path visits t_q, where $\bar{x} = t_4$ and $x \neq t_q$

Fig. 10: Illustrations of Case 8 in the proof of Lemma 19.

x' is the vertex in X other than x. It suffices to combine the 5-DPC with a generalized 2-DPC of H_1 joining $\{\bar{x}', \bar{u}\}$ and $\{t_4, \bar{x}\}$. (Note that $\bar{u} \neq t_4$ by the choice of t_q, and $\bar{u} \neq \bar{x}, \bar{x}'$.) If $\bar{x} = t_4$ and $x \neq t_q$ (see Fig. 10(f)), it suffices to divide the s_i–x path into s_i–t_q and u–x paths for the immediate successor u of t_q, and merge the s_j–x' path, a Hamiltonian $\bar{x}'$–$\bar{u}$ path of $H_1 \setminus t_4$, the u–x path, and one-vertex path (t_4). (Note that $u \neq x$ by the choice of t_q.) Finally, if $\bar{x} = t_4$ and $x = t_q$, it suffices to combine the s_j–x' path with a Hamiltonian $\bar{x}'$–t_4 path of H_1.

Case 9: *$|S_0| = 4$ & $|T_0| = 4$ ($|S_1| = 0$ & $|T_1| = 0$).* Let s_p be a source in $S_0 \setminus T_0$, and t_q be a sink in $T_0 \setminus S_0$. If we regard s_p and t_q as *virtual* free vertices, then by Lemma 17, there exists a generalized 4-DPC of H_0 joining $(S_0 \setminus s_p) \cup \{x\}$ and $(T_0 \setminus t_q) \cup \{y\}$ for some $x, y \in V(H_0)$, in which no x–y path is included. Similar to the proof of Case 8, if we divide the path(s) of the 4-DPC that pass(es) through s_p and/or t_q, we can obtain a generalized 6-DPC of H_0. Combining the 6-DPC with some generalized 2-DPC of H_1 appropriately results in a generalized 4-DPC of $H_0 \oplus H_1$ joining S and T. This completes the entire proof. □

It remains to verify the assertion (iii) for the proof of the base step.

Lemma 20. *Every 5-dimensional RHL graph G is 1-edge-fault unpaired 3-disjoint path coverable.*

Proof. Let $G = H_0 \oplus H_1$ for some 4-dimensional RHL graphs H_0 and H_1. Given an edge-fault set F with $|F| = 1$, and source and sink sets S and T in G such that $|S| = |T| = 3$ and $S \cap T = \emptyset$, we will construct an unpaired 3-DPC of $G \setminus F$ joining S and T. As in the proof of Lemma 19, we assume w.l.o.g. $|S_0| \geq |T_0|, |S_1|, |T_1|$. There are six cases.

Case 1: *$|S_0| = 2$ & $|T_0| = 1$ ($|S_1| = 1$ & $|T_1| = 2$).* If $f_0 + f_1 = 0$, it suffices to pick up a free edge $(x, \bar{x})$ where $x \in V(H_0)$ and then merge two generalized DPCs with $(x, \bar{x})$, a generalized 2-DPC of H_0 joining S_0 and $T_0 \cup \{x\}$ and a generalized 2-DPC of H_1 joining $S_1 \cup \{\bar{x}\}$ and T_1. We now assume w.l.o.g. $f_0 = 1$. There exists a Hamiltonian s_1–s_2 path in $H_0 \setminus F_0$ by Lemma 4. From the Hamiltonian path, we can extract a generalized 2-DPC joining S_0 and $T_0 \cup \{u\}$ where u is the immediate predecessor or successor of t_1 such that $\bar{u} \neq s_3$. It suffices to combine the 2-DPC of H_0 with a generalized 2-DPC of H_1 joining $S_1 \cup \{\bar{u}\}$ and T_1.

Case 2: *$|S_0| = 2$ & $|T_0| = 2$ ($|S_1| = 1$ & $|T_1| = 1$).* If $f_0 = 0$, the union of a generalized 2-DPC of H_0 joining S_0 and T_0 and a Hamiltonian s_3–t_3 path of $H_1 \setminus F_1$ will be a desired DPC. Now, assume $f_0 = 1$. Then, by Lemma 18, there exists a generalized 3-DPC of $H_0 \setminus F_0$ joining $S_0 \cup \{x\}$ and $T_0 \cup \{y\}$ for some $x, y \in V(H_0)$, in which no x–y path is included and moreover $\{\bar{x}, \bar{y}\} \cap \{s_3, t_3\} = \emptyset$. It suffices to combine the 3-DPC with a generalized 2-DPC of H_1 joining $\{s_3, \bar{y}\}$ and $\{t_3, \bar{x}\}$.

Case 3: *$|S_0| = 3$ & $|T_0| = 0$ ($|S_1| = 0$ & $|T_1| = 3$).* Firstly, assume $f_0 + f_1 = 0$. By Lemma 14, there exists an absolute vertex γ in H_0 such that $(\gamma, \bar{\gamma}) \notin F_2$; and also, there exists an absolute vertex γ' in H_1 such that $(\gamma', \bar{\gamma'}) \notin F_2$ and $\gamma' \neq \bar{\gamma}$. For some free edge $(x, \bar{x})$ where $x \in V(H_0)$ such that $(x, \bar{x}) \neq (\gamma, \bar{\gamma}), (\gamma', \bar{\gamma'})$, it suffices to merge two DPCs, a generalized 3-DPC of H_0 joining S_0 and $\{\gamma, \bar{\gamma'}, x\}$ and a generalized 3-DPC of H_1 joining $\{\bar{\gamma}, \gamma', \bar{x}\}$ and T_1. Now, assume w.l.o.g. $f_0 = 1$. For an absolute vertex γ in H_1, we find a Hamiltonian $\bar{\gamma}$–s_1 path in $H_0 \setminus F_0$. From the Hamiltonian path, we extract a generalized 3-DPC $H_0 \setminus F_0$ joining S_0 and $\{\bar{\gamma}, u, v\}$ where u, v are the immediate successors of s_2, s_3 in the Hamiltonian path. It suffices to combine the 3-DPC of H_0 with a generalized 3-DPC of H_1 joining $\{\gamma, \bar{u}, \bar{v}\}$ and T_1.

Case 4: *$|S_0| = 3$ & $|T_0| = 1$ ($|S_1| = 0$ & $|T_1| = 2$).* Firstly, assume $f_0 = 1$. There exists a Hamiltonian s_1–s_2 path, P, in $H_0 \setminus F_0$. We claim that from P, we can extract a generalized 3-DPC of $H_0 \setminus F_0$ joining S_0 and $T_0 \cup X$ for some 2-vertex set X with $\bar{X} \neq T_1$. If we encounter t_1 and then encounter s_3 when we traverse the path P from s_1, which can be represented as $(s_1, P_u, u, t_1, v, P_v, s_3, P', s_2)$ where u and v respectively are the immediate predecessor and successor of t_1, it suffices to divide the path into $\{(s_1, P_u, u, t_1), (v, P_v, s_3), (P', s_2)\}$ or $\{(s_1, P_u, u), (t_1, v, P_v, s_3), (P', s_2)\}$. The claim can be proven symmetrically for the other possibility where we encounter s_3 and t_1 in order. If we combine the 3-DPC with a generalized 2-DPC of H_1 joining $\bar{X}$ and T_1, we obtain a desired DPC. Secondly, assume $f_2 = 1$. There exists an absolute vertex γ such that $\gamma \neq t_1$ and $(\gamma, \bar{\gamma}) \notin F_2$ by Lemma 14. For some free edge $(x, \bar{x})$, where $x \in V(H_0)$, such that $x \neq \gamma$, it suffices to combine a generalized 3-DPC of H_0 joining S_0 and $T_0 \cup \{\gamma, x\}$ with a generalized 2-DPC of H_1 joining $\{\bar{\gamma}, \bar{x}\}$ and T_1. Thirdly, assume $f_1 = 1$. There exists a

Hamiltonian t_2–t_3 path, P, in $H_1 \setminus F_1$, from which we can extract a generalized 2-DPC joining $\{u, v\}$ and T_1 if (u, v) is an edge of P. There are at least two absolute vertices of H_0 other than t_1 by Lemma 14, so one of them, say γ, has an immediate predecessor or successor, u, in the path P such that $\bar{u} \neq t_1$ (i.e., $(\bar{\gamma}, u)$ is an edge of P and $\bar{u} \neq t_1$). It suffices to combine a generalized 3-DPC of H_0 joining S_0 and $T_0 \cup \{\gamma, \bar{u}\}$ with a generalized 2-DPC of $H_1 \setminus F_1$ joining $\{\bar{\gamma}, u\}$ and T_1.

Case 5: *$|S_0| = 3$ & $|T_0| = 2$ ($|S_1| = 0$ & $|T_1| = 1$).* If $f_0 = 0$, then by Lemma 12, there exists a generalized 3-DPC of H_0 joining S_0 and $T_0 \cup \{x\}$ for some $x \in V(H_0) \setminus T_0$ such that $\bar{x} \neq t_3$ and $(x, \bar{x}) \notin F$. It suffices to combine the 3-DPC of H_0 with a Hamiltonian $\bar{x}$–t_3 path of $H_1 \setminus F_1$. Now, assume $f_0 = 1$. There exists a Hamiltonian t_1–t_2 path P in $H_0 \setminus F_0$, represented as $(t_1, \ldots, u_1, s_{i_1}, v_1, \ldots, u_2, s_{i_2}, v_2, \ldots, u_3, s_{i_3}, v_3, \ldots, t_2)$ for $\{i_1, i_2, i_3\} = \{1, 2, 3\}$. If $\bar{v_3} \neq t_3$, it suffices to divide the path into a generalized 4-DPC of $H_0 \setminus F_0$ composed of $(t_1, \ldots, u_1, s_{i_1})$, $(v_1, \ldots, u_2, s_{i_2})$, $(v_2, \ldots, u_3, s_{i_3})$, $(v_3, \ldots, t_2)\}$ paths, and combine the 4-DPC with a generalized 2-DPC of H_1 joining $\{\bar{v_1}, \bar{v_2}\}$ and $\{t_3, \bar{v_3}\}$. Symmetrically, we can also construct a desired DPC in the remaining case where $\bar{v_3} = t_3$ (so, $\bar{u_1} \neq t_3$), through a generalized 4-DPC made of $(t_1, \ldots, u_1)$, $(s_{i_1}, v_1, \ldots, u_2)$, $(s_{i_2}, v_2, \ldots, u_3)$, and $(s_{i_3}, v_3, \ldots, t_2)$ paths.

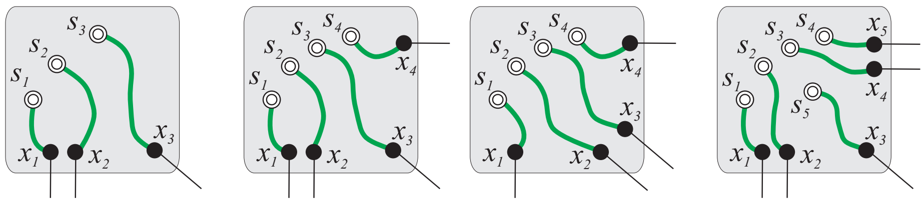
Case 6: *$|S_0| = 3$ & $|T_0| = 3$ ($|S_1| = 0$ & $|T_1| = 0$).* If $f_0 = 0$, then by Lemma 17, there exists a generalized 4-DPC of H_0 joining $S_0 \cup \{x\}$ and $T_0 \cup \{y\}$ for some $x \in V(H_0) \setminus S_0$ and $y \in V(H_0) \setminus T_0$, in which no x–y path is included. Moreover, we can assume $(x, \bar{x}), (y, \bar{y}) \notin F$. We only need to combine the 4-DPC with a Hamiltonian $\bar{y}$–$\bar{x}$ path of $H_1 \setminus F_1$. Now, assume $f_0 = 1$. There exists a Hamiltonian s_1–s_2 path in $H_0 \setminus F_0$, which can be represented as the following example: $(s_1, \ldots, u_0, s_3, v_0, \ldots, u_1, t_{j_1}, v_1, \ldots, u_2, t_{j_2}, v_2, \ldots, u_3, t_{j_3}, v_3, \ldots, s_2)$ for $\{j_1, j_2, j_3\} = \{1, 2, 3\}$. In a similar way as Case 5, we can always extract from the Hamiltonian path, a generalized 5-DPC of $H_0 \setminus F_0$, composed of $(s_1, \ldots, u_0)$, $(s_3, v_0, \ldots, u_1, t_{j_1})$, $(v_1, \ldots, u_2, t_{j_2})$, $(v_2, \ldots, u_3, t_{j_3})$, and $(v_3, \ldots, s_2)$ paths for example, in which one path runs from a source in S_0 to a sink in T_0, and each of the other four paths runs from a source or runs to a sink but not both. A desired DPC is obtained through some generalized 2-DPC of H_1 as before. $\square$

4.3. Inductive step of $m = 6$

We will prove the inductive step of $m = 6$ for the case where $k = m - 1 = 5$ and $S \subseteq V(H_0)$ & $T \subseteq V(H_1)$, which remains uncovered in Section 3 for the proof of Theorem 3. Let $G = H_0 \oplus H_1$ be a 6-dimensional RHL graph. Recall the recursive structure of G illustrated in Fig. 3 and the four assumptions A1, A2, A3, and A4 made in the previous section, so that $|S \cap V(G_0)| \geq |S \cap V(G_1)|$; and moreover, if there is a single critical vertex in $H_0 \oplus H_1$, it is included in H_0, especially in G_0. We further assume

$$E_{0,2} \neq \emptyset, \text{ so that } E_{1,3} \neq \emptyset.$$

Suppose otherwise, the graph G can be represented as $H'_0 \oplus H'_1$, where H'_0 and H'_1 are 5-dimensional RHL graphs defined as $H'_0 = G_0 \oplus G_3$ and $H'_1 = G_1 \oplus G_2$.



(a) $p = 3$ (b) $p = 4$ (c) $p = 5$

Fig. 11: Illustrations of the p-vertex sets X of Lemmas 21, 22, and 23.

So, the case where $S \nsubseteq V(G_0)$ or $T \nsubseteq V(G_2)$ is reduced to the cases where $S \nsubseteq V(H'_0)$ or $T \nsubseteq V(H'_1)$, and moreover, the procedure DPC-G in [16] is applicable for the remaining case where $S \subseteq V(G_0)$ and $T \subseteq V(G_2)$. (The procedure DPC-G relies only on the three properties P1, P2, and P3 shown in the earlier part of Section 3, where an m-dimensional RHL graph also has the three properties Q1, Q2 and Q3 corresponding to P1, P2 and P3.)

First of all, we will pick up some vertex set $X = \{x_1, \ldots, x_p\}$ in G_0 that admits a generalized p-DPC of G_0 joining S' and X, where $S' = S \cap V(G_0)$ and $p = |S'| \geq 3$. Then, through the p edges of $\{(x_i, \bar{x_i}) : i = 1, \ldots, 3\} \cup \{(x_j, \hat{x_j}) : j = 4, \ldots, p\}$, an unpaired 5-DPC of G joining S and T will be constructed. The following Lemmas 21, 22, and 23 are concerning the constructions of such sets X that satisfy some additional conditions for $p = 3, 4, 5$, respectively. The p-vertex sets X of the three lemmas are illustrated in Fig. 11. Note that there is a critical vertex (or a critical edge) in $H_0 \oplus H_1$ only if $p = 4$. We denote by $Y_0 \subseteq V(G_0)$ the set of end-vertices in G_0 of edges of $E_{0,3}$, where $|Y_0| = |E_{0,3}| \geq 8$ by our assumption.

Lemma 21. *Given a source set $S' = \{s_1, s_2, s_3\}$ in G_0, there exists a 3-vertex set $X = \{x_1, x_2, x_3\} \subseteq V(G_0)$ that admits a generalized 3-DPC of G_0 joining S' and X, where $\bar{x_1}, \bar{x_2} \in V(G_3)$ and $\bar{x_3} \in V(G_2)$. Furthermore, there exists another 3-vertex set $X' = \{x'_1, x'_2, x_3\}$ that satisfies the above conditions, where $\{x'_1, x'_2\} \neq \{x_1, x_2\}$ and $x_3 \in X' \cap X$.*

Proof. By Lemma 14, there exists an absolute vertex, γ, in G_0 that admits a generalized 3-DPC joining S and $\{\gamma\} \cup Z$ for any 2-vertex set $Z \subseteq V(G_0)$. If $(\gamma, \bar{\gamma}) \in E_{0,3}$, it suffices to let $x_1 = x'_1 = \gamma$ and pick up three vertices $x_2, x'_2, x_3 \in V(G_0)$ such that $x_2, x'_2 \in Y_0 \setminus \gamma$ and $\bar{x_3} \in V(G_2)$. If $(\gamma, \bar{\gamma}) \in E_{0,2}$, it suffices to let $x_3 = \gamma$ and pick up three vertices $x_1, x'_1, x_2 (= x'_2) \in Y_0$, completing the proof. □

Lemma 22. *Given a source set $S' = \{s_1, s_2, s_3, s_4\}$ in G_0 and a source s_5 in G_1, there exists a 4-vertex set $X = \{x_1, x_2, x_3, x_4\} \subseteq V(G_0)$ that admits a generalized 4-DPC of G_0 joining S' and X such that:*

- *The set $\{(x_i, \bar{x_i}) : i = 1, \ldots, 3\}$ includes all critical edges from $E_{0,3} \cup E_{0,2}$.*

- $\bar{x_1} \in V(G_3)$, $\bar{x_3} \in V(G_2)$, *and* $\hat{x_4} \neq s_5$. *If there are two distinct critical edges from* $E_{0,2}$, *then* $\bar{x_2} \in V(G_2)$; *if otherwise,* $\bar{x_2} \in V(G_3)$.

Furthermore, if there is at most one critical edge from $E_{0,3} \cup E_{0,2}$, *there exists another* 4*-vertex set* $X' = \{x_1', x_2', x_3, x_4\}$ *that satisfies the above conditions, where* $\{x_1', x_2'\} \neq \{x_1, x_2\}$ *and* $\{x_3, x_4\} \subseteq X' \cap X$.

Proof. There are at least four absolute pairs in G_0 by Lemma 16. We first claim that there exists an absolute pair $\Gamma = \{\gamma_1, \gamma_2\}$ such that (i) if there is a common neighbor, v, of sources, then $\gamma_1 = v$, and (ii) $\hat{\gamma_2} \neq s_5$. If G_0 has no common neighbor of sources, it suffices to pick up an arbitrary absolute pair and appropriately name its members as γ_1 and γ_2. Otherwise, there are at least three absolute pairs satisfying the conditions, since every absolute pair contains the common neighbor, which is proving the claim. Keep in mind that if there is a critical vertex, α, of H_0, then $\alpha \in V(G_0)$ is the common neighbor of the four sources in S'. To include the critical edge $(\alpha, \bar{\alpha})$, if any, in the set $\{(x_i, \bar{x_i}) : i = 1, \ldots, 3\}$, we will always set $x_1 = \gamma_1$ if $(\gamma_1, \bar{\gamma_1}) \in E_{0,3}$ and set $x_3 = \gamma_1$ if $(\gamma_1, \bar{\gamma_1}) \in E_{0,2}$. Also, we will include γ_2 in X to guarantee the existence of a generalized 4-DPC joining S' and X.

Case 1: $\bar{\gamma_1}, \bar{\gamma_2} \in V(G_3)$. If there is at most one critical edge from $E_{0,3} \cup E_{0,2}$ (i.e., no critical vertex, or one critical vertex $\alpha \in V(G_0)$, or two critical vertices $\alpha \in V(G_0)$ and $\beta \in V(H_1)$ such that either $\bar{\beta} = \alpha$ or $\bar{\beta} \in V(G_1)$), it suffices to let $x_1 = x_1' = \gamma_1$, $x_4 = \gamma_2$, and pick up three vertices $x_2, x_2', x_3 \in V(G_0)$ such that $x_2, x_2' \in Y_0 \setminus \Gamma$ and $\bar{x_3} \in V(G_2)$. Now, assume that there are two distinct critical edges, $(\alpha, \bar{\alpha})$ and $(\bar{\beta}, \beta)$, where $\alpha, \bar{\beta} \in V(G_0)$. It follows that $\gamma_1 = \alpha$ and $\alpha \neq \bar{\beta}$. If $(\bar{\beta}, \beta) \in E_{0,3}$ and $\bar{\beta} \neq \gamma_2$, it suffices to let $x_1 = \gamma_1$, $x_2 = \bar{\beta}$, $x_4 = \gamma_2$, and pick up a vertex $x_3 \in V(G_0)$ such that $\bar{x_3} \in V(G_2)$. If $(\bar{\beta}, \beta) \in E_{0,3}$ and $\bar{\beta} = \gamma_2$, it suffices to let $x_1 = \gamma_1$, $x_2 = \gamma_2$, and pick up vertices $x_3, x_4 \in V(G_0)$ such that $\bar{x_3} \in V(G_2)$ and $x_4 \notin \{x_1, x_2, x_3, \hat{s_5}\}$. Finally, if $(\bar{\beta}, \beta) \in E_{0,2}$ $(\bar{\beta} \neq \gamma_2)$, it suffices to let $x_1 = \gamma_1$, $x_4 = \gamma_2$, $x_3 = \bar{\beta}$, and pick up a vertex $x_2 \in Y_0 \setminus \Gamma$.

Case 2: $\bar{\gamma_1}, \bar{\gamma_2} \in V(G_2)$. If there is at most one critical edge from $E_{0,3} \cup E_{0,2}$, it suffices to let $x_3 = \gamma_1$, $x_4 = \gamma_2$, and pick up three vertices $x_1, x_1', x_2 (= x_2') \in Y_0$. Now, assume that there are two distinct critical edges, $(\alpha, \bar{\alpha})$ and $(\bar{\beta}, \beta)$, where $\alpha, \bar{\beta} \in V(G_0)$. Then, we have $\gamma_1 = \alpha$ and $\alpha \neq \bar{\beta}$. If $(\bar{\beta}, \beta) \in E_{0,3}$ $(\bar{\beta} \neq \gamma_2)$, it suffices to let $x_1 = \bar{\beta}$, $x_3 = \gamma_1$, $x_4 = \gamma_2$, and pick up a vertex $x_2 \in Y_0 \setminus \bar{\beta}$. If $(\bar{\beta}, \beta) \in E_{0,2}$ and $\bar{\beta} \neq \gamma_2$, it suffices to let $x_2 = \bar{\beta}$, $x_3 = \gamma_1$, $x_4 = \gamma_2$, and pick up a vertex $x_1 \in Y_0$. Finally, if $(\bar{\beta}, \beta) \in E_{0,2}$ and $\bar{\beta} = \gamma_2$, it suffices to let $x_2 = \bar{\beta}$, $x_3 = \gamma_1$, and pick up vertices $x_1, x_4 \in V(G_0)$ such that $x_1 \in Y_0$ and $x_4 \notin \{x_1, x_2, x_3, \hat{s_5}\}$.

Case 3: $\bar{\gamma_1} \in V(G_3)$ *and* $\bar{\gamma_2} \in V(G_2)$. If there is at most one critical edge from $E_{0,3} \cup E_{0,2}$, it suffices to let $x_1 = x_1' = \gamma_1$, $x_3 = \gamma_2$, and pick up three vertices $x_2, x_2', x_4 \in V(G_0)$ such that $x_2, x_2' \in Y_0 \setminus \gamma_1$ and $x_4 \notin \{x_1, x_2, x_2', x_3, \hat{s_5}\}$. Now, let $(\alpha, \bar{\alpha})$ and $(\bar{\beta}, \beta)$ be two distinct critical edges, where $\alpha, \bar{\beta} \in V(G_0)$. Then, $\gamma_1 = \alpha$ and $\alpha \neq \bar{\beta}$. If $(\bar{\beta}, \beta) \in E_{0,3}$, it suffices to let $x_1 = \gamma_1$, $x_2 = \bar{\beta}$, $x_3 = \gamma_2$, and pick up a vertex $x_4 \in V(G_0) \setminus \{x_1, x_2, x_3, \hat{s_5}\}$. If $(\bar{\beta}, \beta) \in E_{0,2}$

and $\bar{\beta} \neq \gamma_2$, it suffices to let $x_1 = \gamma_1$, $x_3 = \bar{\beta}$, $x_4 = \gamma_2$, and pick up a vertex $x_2 \in Y_0 \setminus \gamma_1$. Finally, if $(\bar{\beta}, \beta) \in E_{0,2}$ and $\bar{\beta} = \gamma_2$, it suffices to let $x_1 = \gamma_1$, $x_3 = \gamma_2$, and pick up vertices $x_2, x_4 \in V(G_0)$ such that $x_2 \in Y_0 \setminus \gamma_1$ and $x_4 \notin \{x_1, x_2, x_3, \hat{s_5}\}$.

Case 4: $\bar{\gamma_1} \in V(G_2)$ *and* $\bar{\gamma_2} \in V(G_3)$. If there is at most one critical edge from $E_{0,3} \cup E_{0,2}$, it suffices to let $x_3 = \gamma_1$, $x_4 = \gamma_2$, and pick three vertices $x_1, x_1', x_2 (= x_2') \in Y_0 \setminus \gamma_2$. Assume now $(\alpha, \bar{\alpha})$ and $(\bar{\beta}, \beta)$ are two distinct critical edges, where $\alpha, \bar{\beta} \in V(G_0)$. Then, $\gamma_1 = \alpha$ and $\alpha \neq \bar{\beta}$. If $(\bar{\beta}, \beta) \in E_{0,3}$ and $\bar{\beta} \neq \gamma_2$, it suffices to let $x_1 = \gamma_2$, $x_2 = \bar{\beta}$, $x_3 = \gamma_1$, and pick up a vertex $x_4 \in V(G_0) \setminus \{x_1, x_2, x_3, \hat{s_5}\}$. If $(\bar{\beta}, \beta) \in E_{0,3}$ and $\bar{\beta} = \gamma_2$, it suffices to let $x_1 = \gamma_2$, $x_3 = \gamma_1$, and pick up vertices $x_2, x_4 \in V(G_0)$ such that $x_2 \in Y_0 \setminus \gamma_2$ and $x_4 \notin \{x_1, x_2, x_3, \hat{s_5}\}$. Finally, if $(\bar{\beta}, \beta) \in E_{0,2}$, it suffices to let $x_2 = \bar{\beta}$, $x_3 = \gamma_1$, $x_4 = \gamma_2$, and pick up a vertex $x_1 \in Y_0 \setminus \gamma_2$. This completes the proof. □

Lemma 23. *Given a source set* $S = \{s_1, s_2, s_3, s_4, s_5\}$ *in* G_0*, there exists a* 5*-vertex set* $X = \{x_1, x_2, x_3, x_4, x_5\} \subseteq V(G_0)$ *that admits a generalized* 5*-DPC of* G_0 *joining* S *and* X*, where* $\bar{x_1}, \bar{x_2} \in V(G_3)$ *and* $\bar{x_3} \in V(G_2)$*. Furthermore, there exists another* 5*-vertex set* $X' = \{x_1', x_2', x_3, x_4, x_5'\}$ *that satisfies the above conditions, where* $\{x_1', x_2'\} \neq \{x_1, x_2\}$ *and* $\{x_3, x_4\} \subseteq X' \cap X$*.*

Proof. If we regard s_5 as a *virtual* non-source vertex and set $s_5 = \hat{s_1}$ temporarily, then by Lemma 22, there exists a set $Z = \{x_1, x_2, x_3, x_4\} \subseteq V(G_0)$ that admits a generalized 4-DPC joining $\{s_1, s_2, s_3, s_4\}$ and Z, where $\bar{x_1}, \bar{x_2} \in V(G_3)$ and $\bar{x_3} \in V(G_2)$. Note that there is no critical vertex (and no critical edge) in this configuration, so that $\bar{x_2} \in V(G_3)$. If we divide the path in the DPC, say s_i–x_j path, that passes through s_5 into the two paths of s_i–u and s_5–x_j where u is the immediate predecessor of s_5, there remains a generalized 5-DPC joining S and $Z \cup \{u\}$. We only need to let $X = Z \cup \{u\}$. To obtain another 5-vertex set X', it suffices to repeat the above process to a set $Z' = \{x_1', x_2', x_3, x_4\} \subseteq V(G_0)$ of Lemma 22 other than Z, and let $X' = Z' \cup \{u'\}$ for some $u' \in V(G_0)$. Thus, the lemma is proven. □

Now, we construct an unpaired 5-DPC joining S and T in $H_0 \oplus H_1$, in which each s_i-path for $s_i \in S'$ passes through exactly one of the p edges in $\{(x_i, \bar{x}_i) : i = 1, \ldots, 3\} \cup \{(x_j, \hat{x_j}) : j = 4, \ldots, p\}$, where $p = |S'| \geq 3$. Note that if there is a critical vertex, β, of H_1 such that $\bar{\beta} \in V(G_1)$, the critical edge $(\bar{\beta}, \beta)$ is not included in the p-edge set, whereas all the other types of critical edges, if any, is included in the edge set by Lemma 22. Let $S'' = S \cap V(G_1)$, $T' = T \cap V(G_3)$, and $T'' = T \cap V(G_2)$, so that $S = S' \cup S''$ and $T = T' \cup T''$.

Lemma 24. *Given* $S \subseteq V(H_0)$ *and* $T \subseteq V(H_1)$ *with* $|S| = |T| = 5$*, there exists an unpaired* 5*-DPC joining* S *and* T *in* $H_0 \oplus H_1$*.*

Proof. By Lemmas 21, 22, and 23, there exists a p-vertex set $X = \{x_1, \ldots, x_p\} \subseteq V(G_0)$ that admits a generalized p-DPC of G_0 joining S' and X, where $\bar{x_1} \in V(G_3)$ and $\bar{x_3} \in V(G_2)$. Moreover, we have $\hat{x_4} \neq s_5$ if $p = 4$. Let W be a

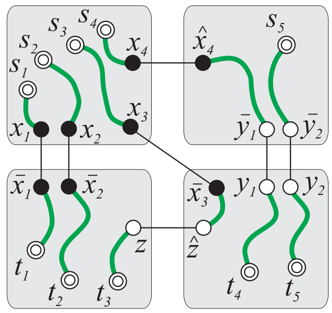
2-vertex set of G_1 defined as $W = \{s_4, s_5\}$ if $p = 3$; $W = \{\hat{x_4}, s_5\}$ if $p = 4$; $W = \{\hat{x_4}, \hat{x_5}\}$ if $p = 5$. The set W will always serve as a source set when we construct a generalized DPC of G_1. Also, let $Y_2 \subseteq V(G_2)$ be the end-vertices in G_2 of edges in $E_{1,2}$ (i.e., $E_{1,2} = \{(y, \bar{y}) : y \in Y_2\}$), so that $|Y_2| = |E_{1,2}| \geq 8$. There are several cases according to the size of T'.

Case 1: $|T'| = 3$ *or* 2. In this case, there is no critical vertex of H_1. So, we have $\bar{x_2} \in V(G_3)$ and $p \geq 3$. We can assume $\{\bar{x_1}, \bar{x_2}\} \neq \{t_1, t_2\}$ by Lemmas 21, 22, and 23. Firstly, assume that $|T'| = 3$ and $|T''| = 2$ (see Fig. 12(a)). If we find the following three generalized DPCs of subcomponents other than G_0 and merge them into one, we obtain a desired 5-DPC: (i) a generalized 3-DPC of G_3 joining $\{\bar{x_1}, \bar{x_2}, z\}$ and T' for some $z \in V(G_3)$ with $\hat{z} \notin T''$, which exists by Lemma 12; (ii) a generalized 3-DPC of G_2 joining $\{\bar{x_3}, y_1, y_2\}$ and $T'' \cup \{\hat{z}\}$ for some $y_1, y_2 \in Y_2$ with $\{\bar{y_1}, \bar{y_2}\} \neq W$ by Lemma 13; and (iii) a generalized 2-DPC of G_1 joining W and $\{\bar{y_1}, \bar{y_2}\}$ by Lemma 10. Note that $\bar{x_3} \notin Y_2$. Secondly, assume that $|T'| = 2$ and $|T''| = 3$. From the DPCs of subcomponents below, we obtain a desired DPC: (i) a generalized 2-DPC of G_3 joining $\{\bar{x_1}, \bar{x_2}\}$ and T'; (ii) a generalized 3-DPC of G_2 joining $\{\bar{x_3}, y_1, y_2\}$ and T'' for some $y_1, y_2 \in Y_2$ with $\{\bar{y_1}, \bar{y_2}\} \neq W$ by Lemma 13; and (iii) a generalized 2-DPC of G_1 joining W and $\{\bar{y_1}, \bar{y_2}\}$.

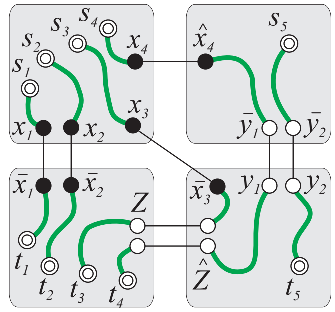
Case 2: $|T'| = 4$ *or* 1. We have $p \geq 4$ by our assumption.

Case 2.1: $|T'| = 4$ *and* $|T''| = 1$. Possibly, there is a critical vertex, $\beta \in V(G_3)$, of H_1. From Lemmas 22 and 23, we have $\bar{x_2} \in V(G_3)$. Moreover, by Lemma 15, there exists a 2-vertex set $Z \subseteq V(G_3)$ that admits a generalized 4-DPC of G_3 joining $\{\bar{x_1}, \bar{x_2}\} \cup Z$ and T'. There are two possibilities. Firstly, assume that β does not exist, or β exists and $\bar{\beta} \in V(G_0)$ (see Fig. 12(b)). We claim that the aforementioned 2-vertex set Z with an additional constraint $\hat{t_5} \notin Z$ exists. The claim is a direct consequence of Lemma 15, if there is no common neighbor of sinks in T' or there is a common neighbor, v, of sinks in T' such that $v \in \{\bar{x_1}, \bar{x_2}\}$. Now, let v exist and $v \notin \{\bar{x_1}, \bar{x_2}\}$. Then, v is not a critical vertex; suppose v is a critical vertex ($v = \beta$ and $\bar{v} \in V(G_0)$), then we have $\bar{v} \in \{x_1, x_2\}$ by Lemma 22, contradicting $v \notin \{\bar{x_1}, \bar{x_2}\}$. It follows $\hat{v} \neq t_5$. There exist at least three such 2-vertex sets Z with $\hat{t_5} \notin Z$, because every Z that admits a generalized 4-DPC of G_3 must contain v, which is proving the claim. As before, it suffices to find the remaining two DPCs and merge the four DPCs of subcomponents: a generalized 3-DPC of G_2 joining $\{\bar{x_3}, y_1, y_2\}$ and $T'' \cup \hat{Z}$ for some $y_1, y_2 \in Y_2$ with $\{\bar{y_1}, \bar{y_2}\} \neq W$, and a generalized 2-DPC of G_1 joining W and $\{\bar{y_1}, \bar{y_2}\}$.

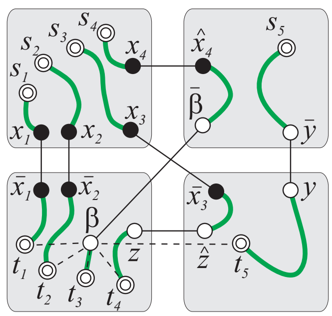
Secondly, assume that β exists and $\bar{\beta} \in V(G_1)$ (see Fig. 12(c)), where $p = 4$. It follows $\beta \notin \{\bar{x_1}, \bar{x_2}\}$ and thus $\beta \in Z$. (Note that β is the common neighbor of sinks in T'.) So, we can assume $Z = \{\beta, z\}$ for some $z \in V(G_3)$ with $\hat{z} \neq t_5$ since we have at least four choices of z by Lemma 15. Pick up an edge $(y, \bar{y})$ from $E_{1,2}$, where $y \in V(G_2)$, such that $y \notin \{t_5, \bar{x_3}, \hat{z}\}$ and $\bar{y} \notin W \cup \{\bar{\beta}\}$, which is possible because $|E_{1,2}| \geq 8$. Then, there exist a generalized 2-DPC of G_1 joining W and $\{\bar{\beta}, \bar{y}\}$, and a generalized 2-DPC of G_2 joining $\{\bar{x_3}, y\}$ and $\{t_5, \hat{z}\}$. We only need to combine the four DPCs of subcomponents into a desired one.



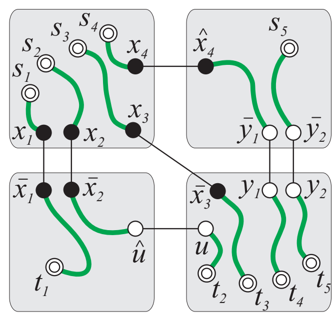
(a) $|T'| = 3$



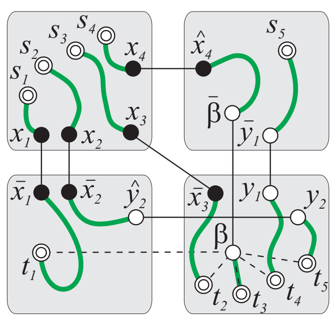
(b) $|T'| = 4$ & (β not exists or $\bar{\beta} \in V(G_0)$)



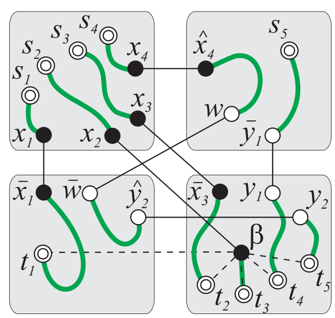
(c) $|T'| = 4$ & $\bar{\beta} \in V(G_1)$



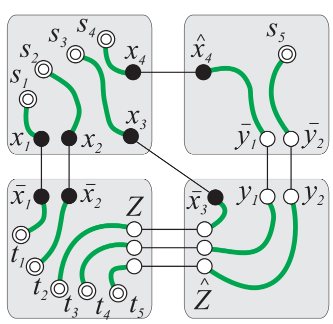
(d) $|T'| = 1$ & (β not exists or $\bar{\beta} = \alpha$)



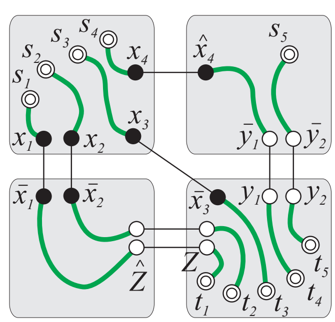
(e) $|T'| = 1$ & $\bar{\beta} \in V(G_1)$



(f) $|T'| = 1$ & $\bar{\beta} \in V(G_0)$ & $\bar{\beta} \neq \alpha$



(g) $|T'| = 5$



(h) $|T'| = 0$

Fig. 12: Illustrations of the proof of Lemma 24.

Case 2.2: $|T'| = 1$ *and* $|T''| = 4$. We have $|Y_2| = |E_{1,2}| \geq 9$ by the assumption A2. If there is a critical vertex, β, of H_1, it will be contained in G_2. Firstly, assume that β does not exist, or β exists and $\bar{\beta} = \alpha$ (see Fig. 12(d)). Then, we have $\bar{x_2} \in V(G_3)$. Let t_i be a sink in T'' with $t_i \notin N_{G_2}(\hat{t_1})$ if β does not exists; let t_i be an arbitrary sink in T'' if otherwise (i.e., β exists and $\bar{\beta} = \alpha$). If we regard t_i as a *virtual* non-sink, then by Lemma 13, there exists a generalized 3-DPC in G_2 joining $\{\bar{x_3}, y_1, y_2\}$ and $T'' \setminus t_i$ for some $y_1, y_2 \in Y_2'$, where $Y_2' = Y_2 \setminus \bar{w}$ for $w = \hat{x_4}$. (Note that $|Y_2'| \geq |Y_2| - 1 \geq 8$.) Then, we have $\hat{x_4} \notin \{\bar{y_1}, \bar{y_2}\}$ and thus $\{\bar{y_1}, \bar{y_2}\} \neq W$. For the immediate successor, u, of t_i in the path that passes through t_i, deleting edge (t_i, u) from the path results in a generalized 4-DPC of G_2 joining $\{\bar{x_3}, y_1, y_2, u\}$ and T''. Here, we have $\hat{u} \neq t_1$ because $t_i \notin N_{G_2}(\hat{t_1})$ or $\bar{x_3} = \bar{\alpha} = \beta = \hat{t_1}$ & $\bar{x_3} \neq u$. Moreover, we can assume $\{t_1, \hat{u}\} \neq \{\bar{x_1}, \bar{x_2}\}$ since there are two choices X and X' by Lemmas 22 and 23. (Also, $\{\bar{y_1}, \bar{y_2}\} \neq W'$ for the set W' derived from X', where $W' = \{\hat{x_4}, s_5\} = W$ if $p = 4$; $W' = \{\hat{x_4}, \hat{x_5'}\}$ if $p = 5$. This is because $\hat{x_4} \in W'$ but $\hat{x_4} \notin \{\bar{y_1}, \bar{y_2}\}$.) It suffices to combine the 4-DPC of G_2 with the following three DPCs of subcomponents other than G_2: a generalized p-DPC of G_0 joining S' and X, a generalized 2-DPC of G_1 joining W and $\{\bar{y_1}, \bar{y_2}\}$, and a generalized 2-DPC of G_3 joining $\{\bar{x_1}, \bar{x_2}\}$ and $\{t_1, \hat{u}\}$.

Secondly, assume that β exists and $\bar{\beta} \in V(G_1)$ (see Fig. 12(e)), where $p = 4$. Then, we have $\bar{x_2} \in V(G_3)$. If we regard t_5 as a *virtual* non-sink vertex, there exists a generalized 3-DPC in G_2 joining $\{\bar{x_3}, y_1, y_2\}$ and $T'' \setminus t_5$ for some $y_1, y_2 \in Y_2 \setminus \beta$ such that $\{\bar{y_1}, \bar{y_2}\} \neq W$ by Lemma 13. Note that $|Y_2 \setminus \beta| \geq 8$ and $\beta \notin \{\bar{x_3}, y_1, y_2\}$. Since β is the common neighbor of sinks in T'', the path in the DPC that passes through t_5 also visits β as an immediate successor of t_5. So, deleting edge (t_5, β) from the path results in a generalized 4-DPC of G_2 joining $\{\bar{x_3}, y_1, y_2, \beta\}$ and T''. We have $\{\bar{\beta}, \bar{y_1}\} \neq W$ or $\{\bar{\beta}, \bar{y_2}\} \neq W$, so we assume w.l.o.g. $\{\bar{\beta}, \bar{y_1}\} \neq W$. In addition, we can assume $\{t_1, \hat{y_2}\} \neq \{\bar{x_1}, \bar{x_2}\}$ since there are two choices X and X' by Lemma 22. (Recall $p = 4$ and $\{x_3, x_4\} \subseteq X' \cap X$.) It remains to combine the 4-DPC of G_2 with the following three DPCs of subcomponents other than G_2: a generalized 4-DPC of G_0 joining S' and X, a generalized 2-DPC of G_1 joining W and $\{\bar{\beta}, \bar{y_1}\}$, and a generalized 2-DPC of G_3 joining $\{\bar{x_1}, \bar{x_2}\}$ and $\{t_1, \hat{y_2}\}$.

Thirdly, assume that β exists and moreover, $\bar{\beta} \in V(G_0)$ and $\bar{\beta} \neq \alpha$ (see Fig. 12(f)), where $p = 4$. Then, we have $\bar{x_2} \in V(G_2)$ by Lemma 22. Assume w.l.o.g. that $x_2 = \bar{\beta}$ and $x_3 = \alpha$. If we regard t_5 as a *virtual* non-sink, there exists a generalized 3-DPC in G_2 joining $\{\bar{x_3}, y_1, y_2\}$ and $T'' \setminus t_5$ for some vertices $y_1, y_2 \in Y_2$ such that $\{\bar{y_1}, \bar{y_2}\} \neq W$ by Lemma 13. We have $y_1, y_2 \neq \beta$ since $\beta \notin Y_2$. The path in the DPC that passes through t_5 will also pass through β, where β is the immediate successor of t_5, since β is the common neighbor of sinks in T''. Deleting edge (t_5, β) from the path results in a generalized 4-DPC joining $\{\bar{x_3}, y_1, y_2, \beta\}$ and T''. Moreover, one of $\bar{y_1}$ and $\bar{y_2}$, say $\bar{y_1}$, satisfies $\bar{y_1} \notin W$ since $\{\bar{y_1}, \bar{y_2}\} \neq W$. Pick up an edge $(w, \bar{w}) \in E_{1,3}$, where $w \in V(G_1)$, such that $\{\bar{x_1}, \bar{w}\} \neq \{t_1, \hat{y_2}\}$, which exists because $|E_{1,3}| = |E_{0,2}| \geq 2$. It suffices to find the following two DPCs and merge the DPCs of subcomponents into an unpaired 5-DPC: a generalized 2-DPC of G_1 joining W and $\{\bar{y_1}, w\}$, and a generalized 2-DPC of G_3 joining $\{\bar{x_1}, \bar{w}\}$ and $\{t_1, \hat{y_2}\}$.

Case 3: $|T'| = 5$ *or* 0. There is no critical vertex of H_1, so $\bar{x_2} \in V(G_3)$. We have $p \geq 4$ by our assumption. Firstly, assume that $|T'| = 5$ and $|T''| = 0$ (see Fig. 12(g)). There exists a generalized 2-DPC in G_3 joining $\{\bar{x_1}, \bar{x_2}\}$ and $\{t_i, t_j\}$ for some $t_i, t_j \in T'$ such that $\{t_i, t_j\} \neq \{\bar{x_1}, \bar{x_2}\}$. From this we can extract a generalized 5-DPC of G_3 joining $\{\bar{x_1}, \bar{x_2}\} \cup Z$ and T' for some 3-vertex set $Z \subseteq V(G_3)$, which is similar to the proof of Lemma 6. It remains to find the following two DPCs and merge the DPCs of subcomponents: a generalized 3-DPC of G_2 joining $\{\bar{x_3}, y_1, y_2\}$ and $\hat{Z}$ for some $y_1, y_2 \in Y_2$ with $\{\bar{y_1}, \bar{y_2}\} \neq W$, and a generalized 2-DPC of G_1 joining W and $\{\bar{y_1}, \bar{y_2}\}$.

Secondly, assume that $|T'| = 0$ and $|T''| = 5$ (see Fig. 12(h)). If we regard t_4 and t_5 as *virtual* non-sink vertices, then by Lemma 13, there exists a generalized 3-DPC in G_2 joining $\{\bar{x_3}, y_1, y_2\}$ and $T'' \setminus \{t_4, t_5\}$ for some $y_1, y_2 \in Y_2'$, where $Y_2' = Y_2 \setminus \bar{w}$ for $w = \hat{x_4}$. From this we can extract a generalized 5-DPC joining $\{\bar{x_3}, y_1, y_2\} \cup Z$ and T'' for some 2-vertex set $Z \in V(G_2)$. Note that $|Y_2'| \geq 8$ since $|Y_2| \geq 9$ by the assumption A2. Moreover, we can assume $\hat{Z} \neq \{\bar{x_1}, \bar{x_2}\}$ since there are two choices X and X' by Lemmas 22 and 23. (The set W' derived from X' is not equal to the set $\{\bar{y_1}, \bar{y_2}\}$ since $\hat{x_4} \in W'$ and $\hat{x_4} \notin \{\bar{y_1}, \bar{y_2}\}$.) It remains to find the following three DPCs and merge the DPCs of subcomponents: a generalized p-DPC of G_0 joining S' and X, a generalized 2-DPC of G_1 joining W and $\{\bar{y_1}, \bar{y_2}\}$, and a generalized 2-DPC of G_3 joining $\{\bar{x_1}, \bar{x_2}\}$ and $\hat{Z}$. This completes the entire proof. □

5. Conclusion

In this paper, we have proven that every m-dimensional RHL graph, $m \geq 5$, is f-fault unpaired k-disjoint path coverable for any $f \geq 0$ and $k \geq 2$ subject to $f + k \leq m - 1$, settling one of the two conjectures posed in [18]. The bound $m - 1$ on $f + k$ is best possible for the m-dimensional RHL graph to be f-fault unpaired k-disjoint path coverable for $f \geq 0$ and $k \geq 2$. The findings with respect to the unpaired disjoint path covers may play some roles in resolving the remaining conjecture of [18] asserting that every m-dimensional RHL graph, $m \geq 5$, is f-fault *paired* k-disjoint path coverable for any f and $k \geq 2$ subject to $f + 2k \leq m + 1$.

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